Single Particle Motion in Collisionless Plasmas

SULI Introductory Course in Plasma Physics, PPPL

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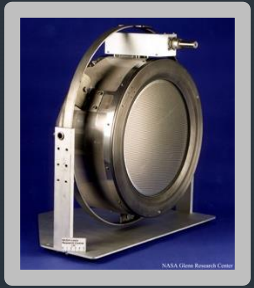
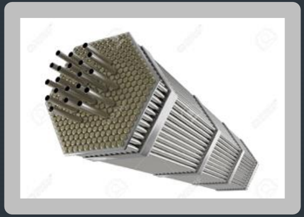
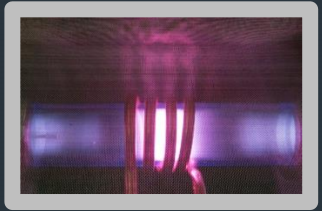
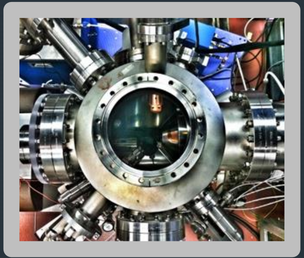
My Path in Science







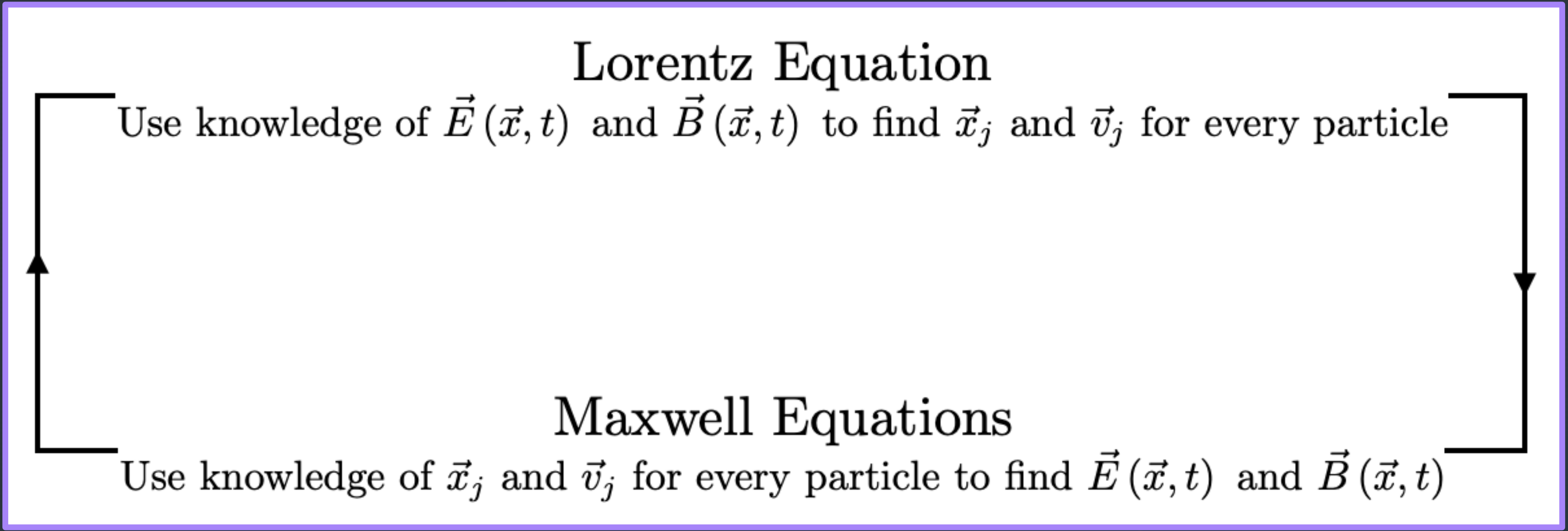


BETTIS LABORATORY

PPPL

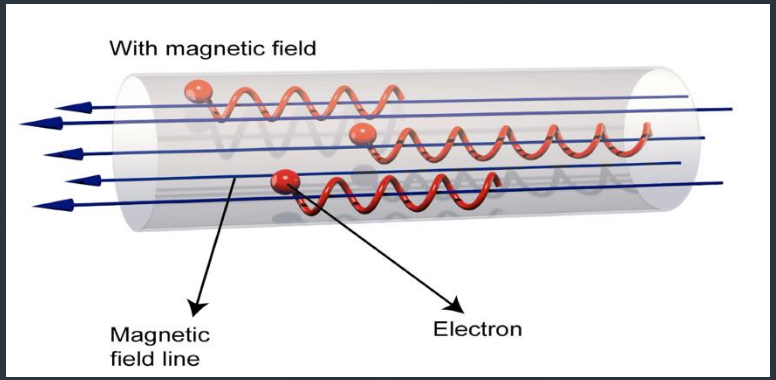
How do we describe the behavior of a plasma?

- The dynamics of a plasma is determined by the self-consistent interaction between electromagnetic fields and a statistically large number of particles.



- Conceptually, this is simple but in practice quite hard

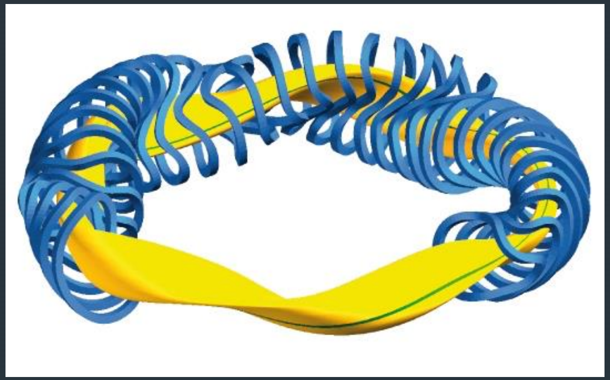
Hierarchy of Models



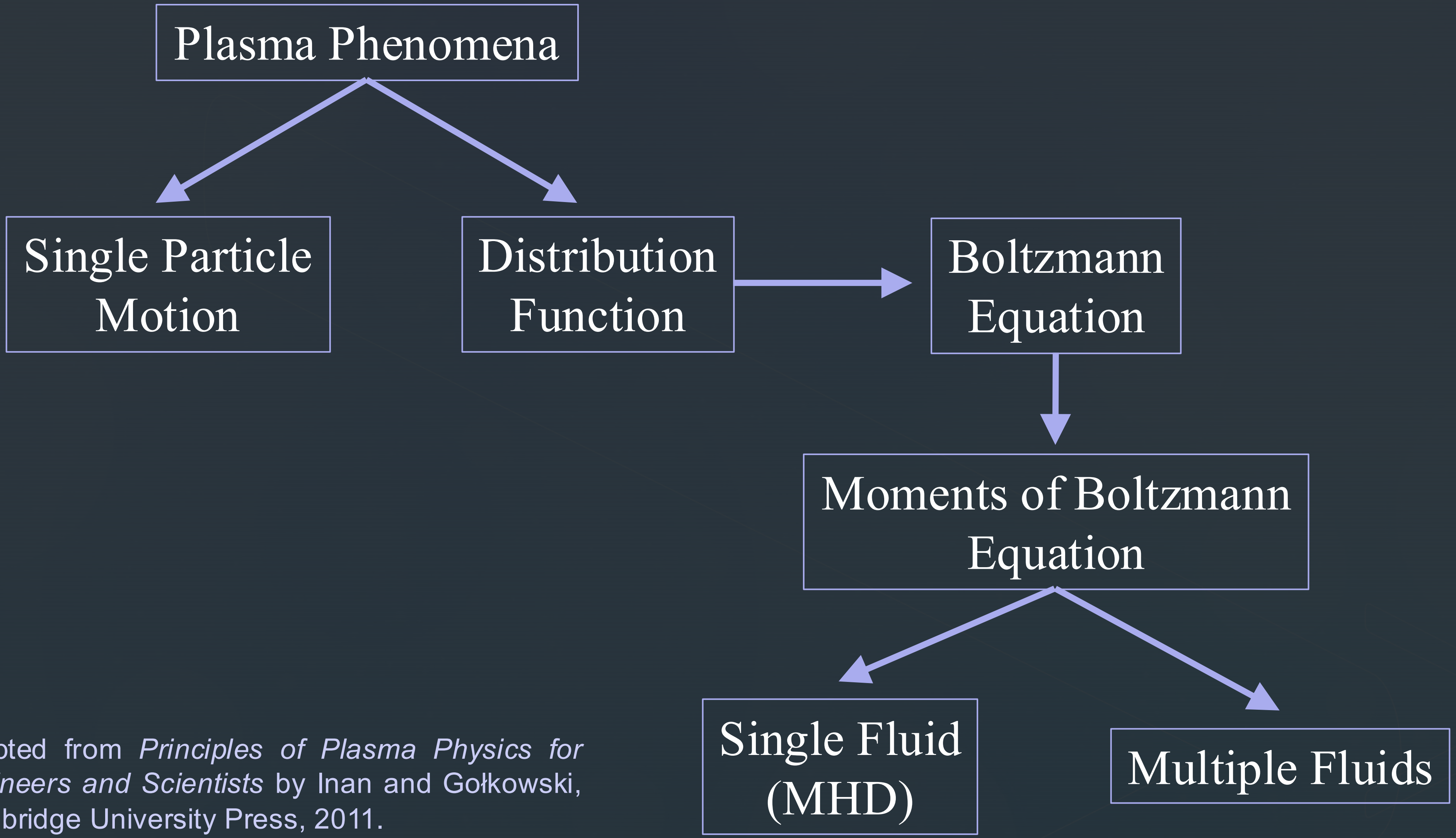
Single particle motion (particle pushing)

Kinetic Theory (distribution functions)

Fluid Model (moments, conservation eq.)

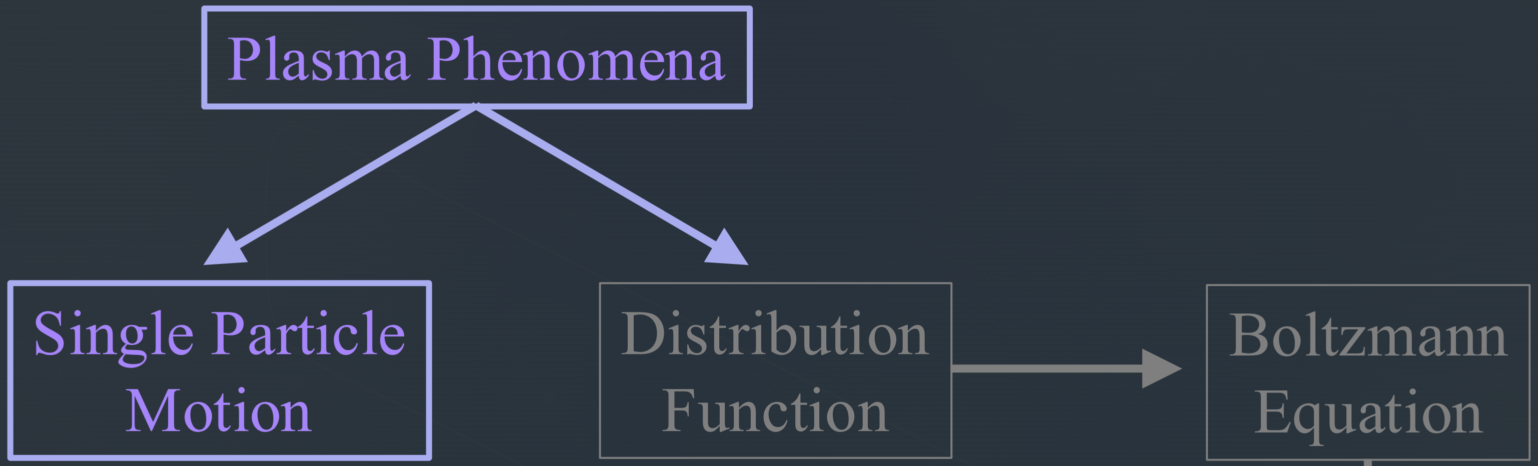


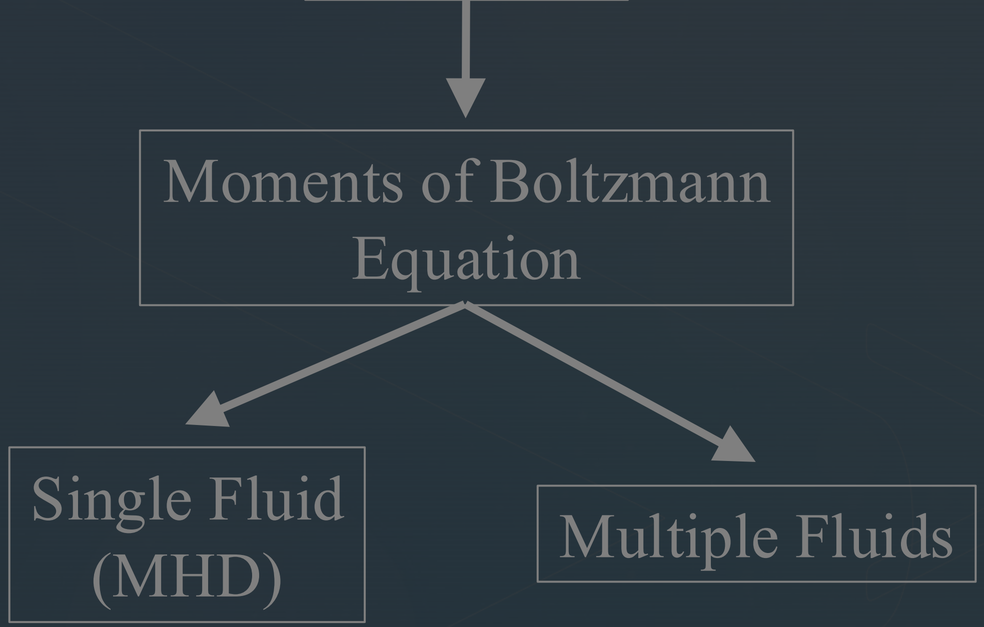
Hierarchy of Models



Adapted from *Principles of Plasma Physics for Engineers and Scientists* by Inan and Gołkowski, Cambridge University Press, 2011.

Single Particle Motion





Simplifying assumptions:

- Charged particles do not interact with one another and their motions do not constitute a large enough current to significantly affect the EM fields.
- Motion is non-relativistic and collisions can be ignored

Single Particle Motion

Consider an ion in a fusion reactor

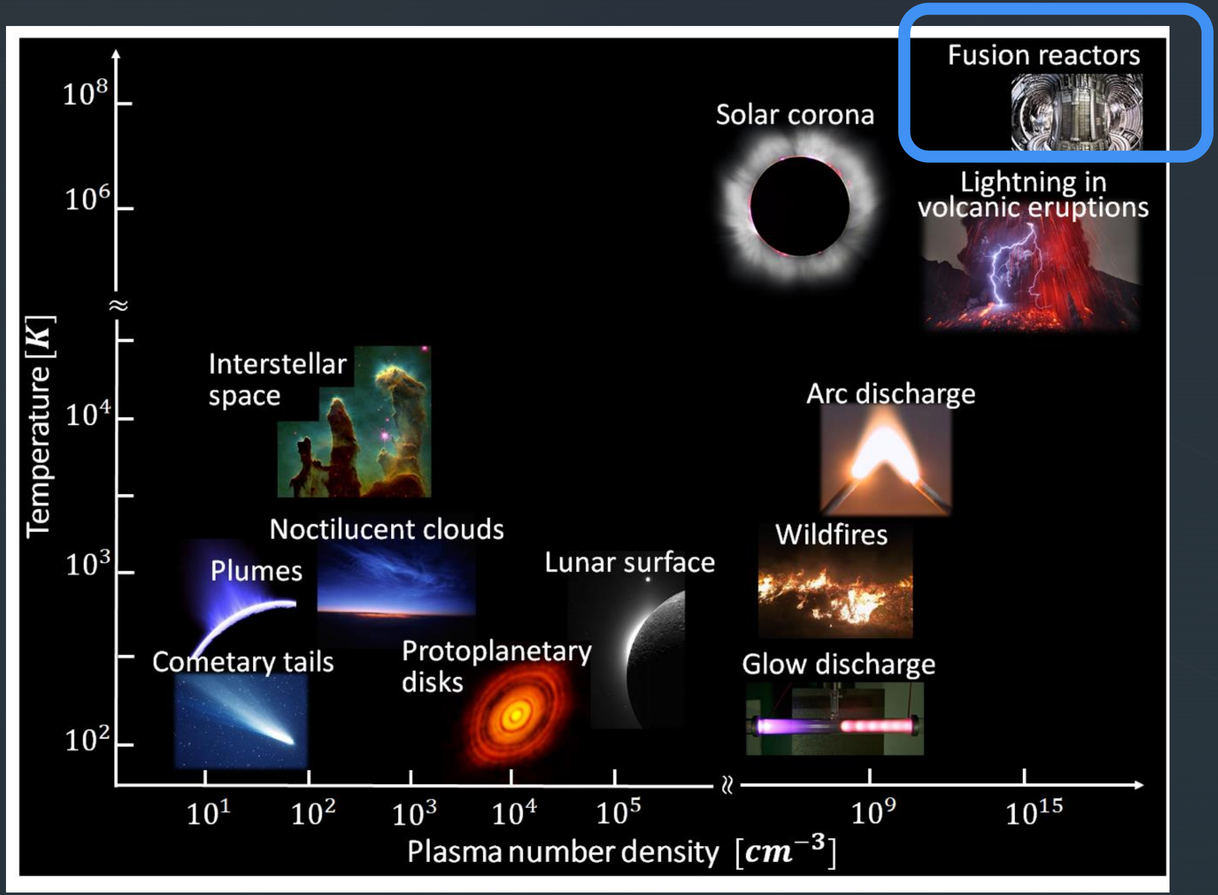


Image from L. Matthews 2021 SULI Talk, *Introduction to Plasma Physics*

$$n_i \sim 10^{20} \frac{\text{ions}}{\text{m}^3},\ T_i = 10^8\ \text{K}$$

$$k_B T = \frac{1}{2} m v_{th}^2$$

$$v_{th} \sim 6 \times 10^5 \frac{\text{m}}{\text{s}}$$

- ion would travel ~10 km before colliding with another ion

The Basics

(1)

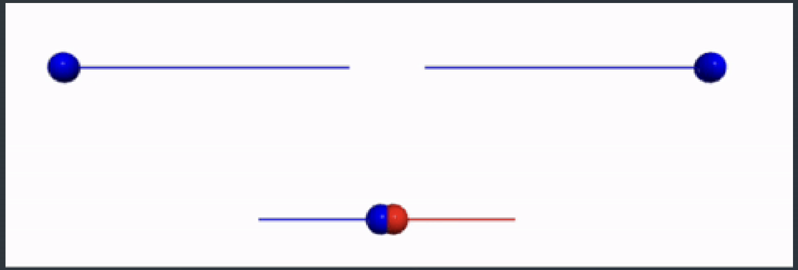
Like charges repel

Opposite charges attract

Animations made in VPython

The Basics

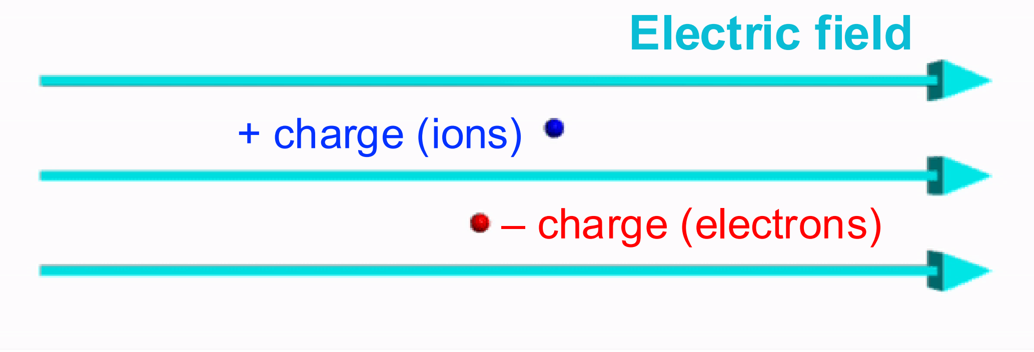
(1)



Like charges repel

Opposite charges attract

(2)



Ions move **with** the electric field

Electrons move **against** the electric field

Animations made in VPython

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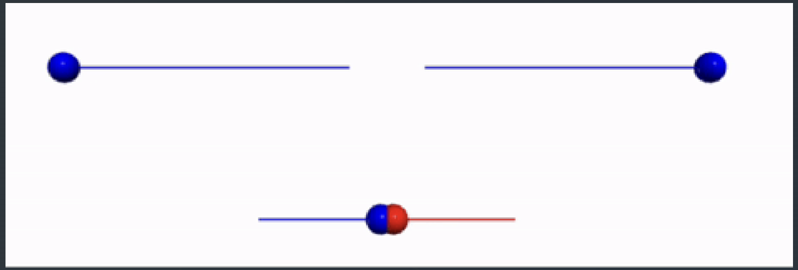

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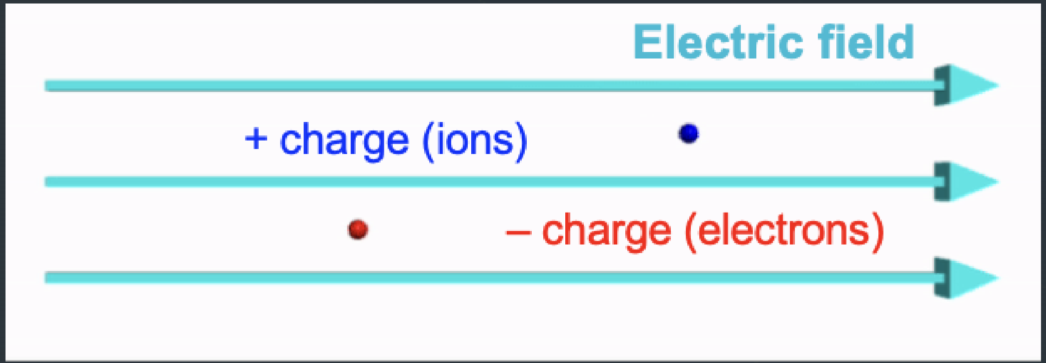

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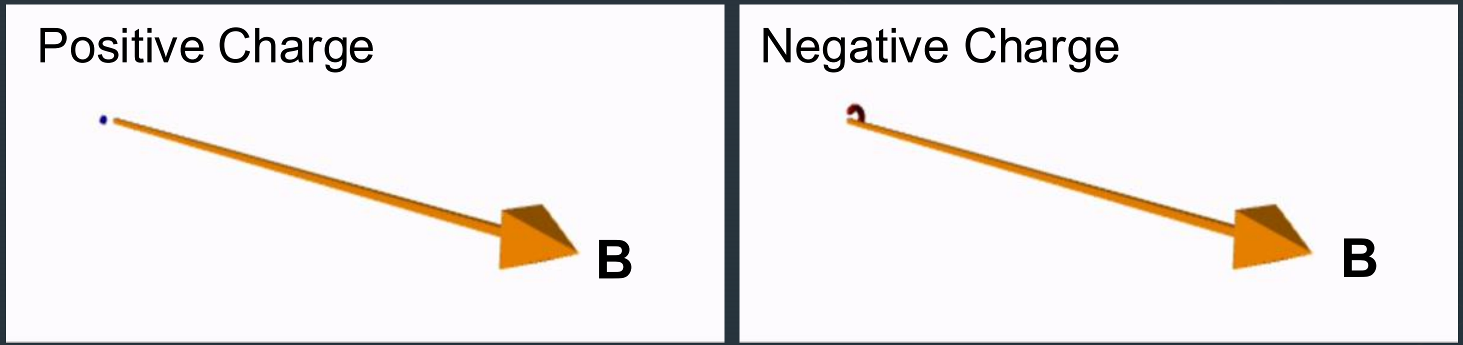


Charged particles create helixes around magnetic field lines.

Radius decreases as magnetic field increases.

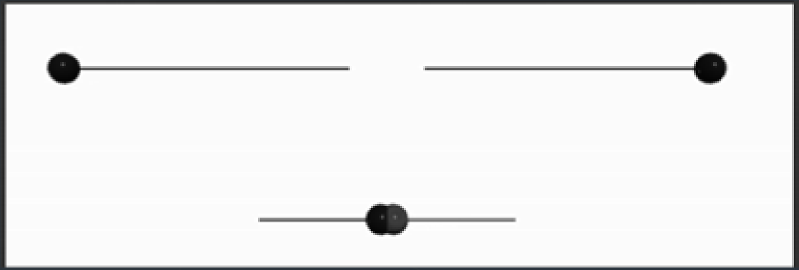
Animations made in VPython

The Basics

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Lorentz Force Equation

(1)



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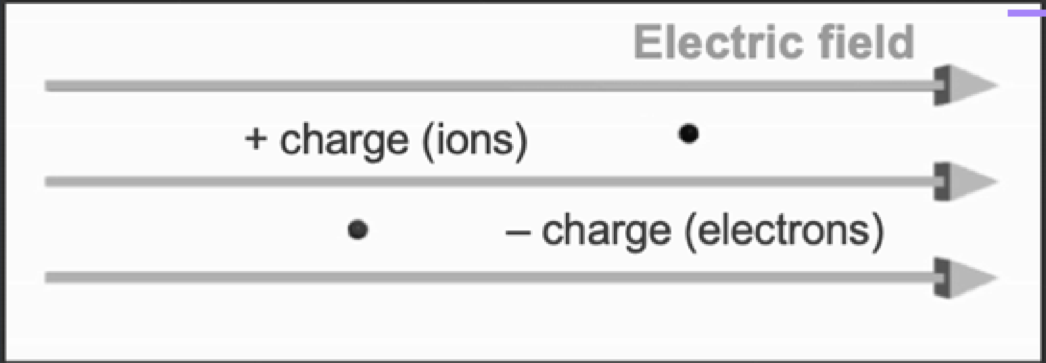

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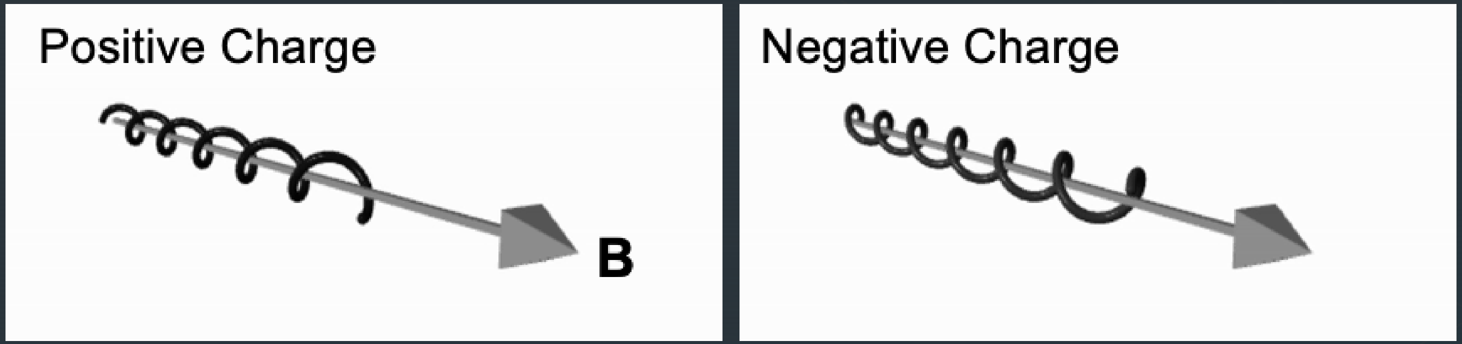
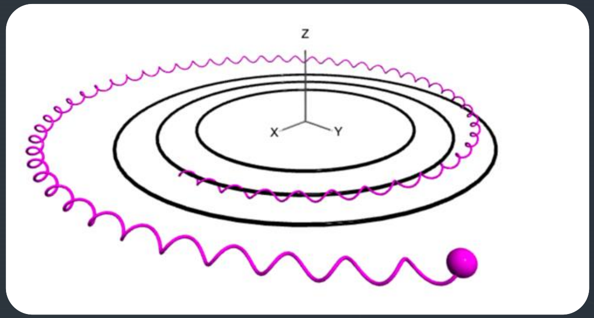
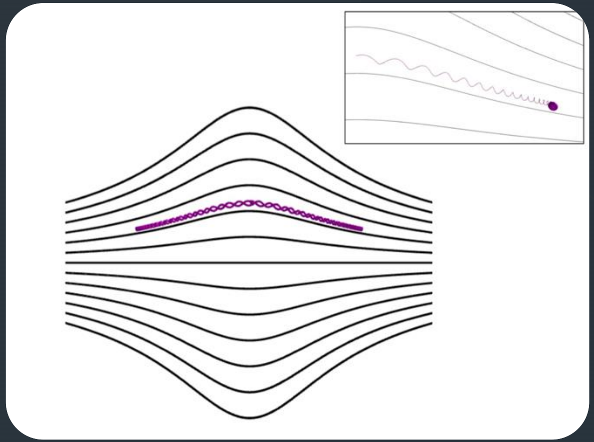
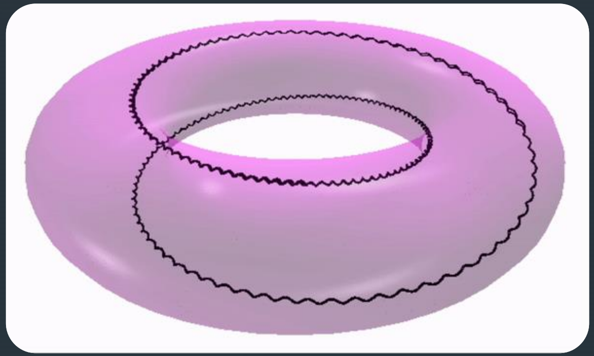


Animations made in VPython

The Lorentz Force Equation







$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Lorentz Force Equation

Start with simple cases

→ Spatially varying fields

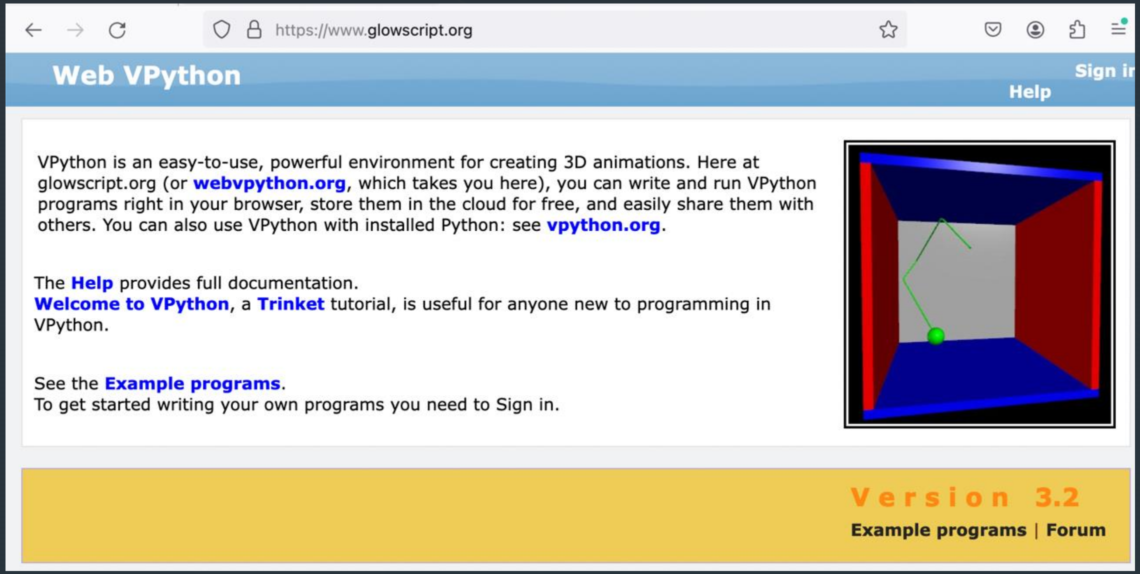
→ Time-varying fields

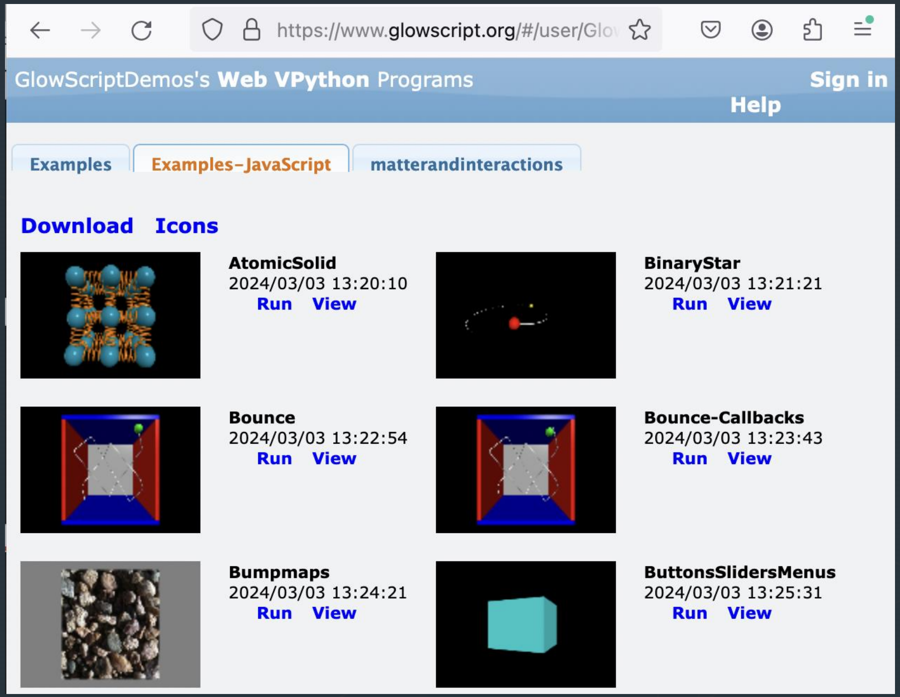
→ Tokamaks

A Note About the Simulations

- Simulations were developed in Vpython (glowscript.org)

→ All simulations will be available in a single Jupyter notebook (forthcoming)





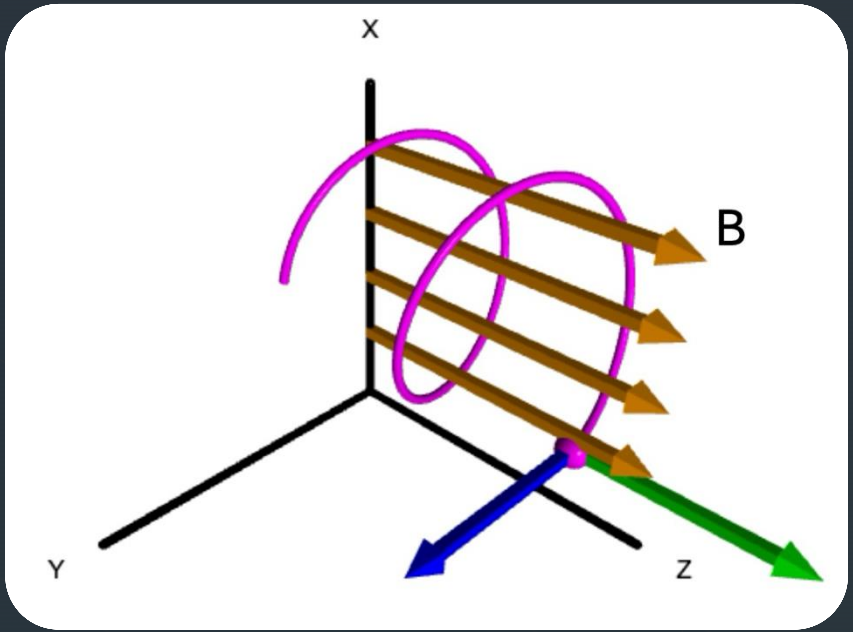
A Note About the Simulations

- The equation of motion

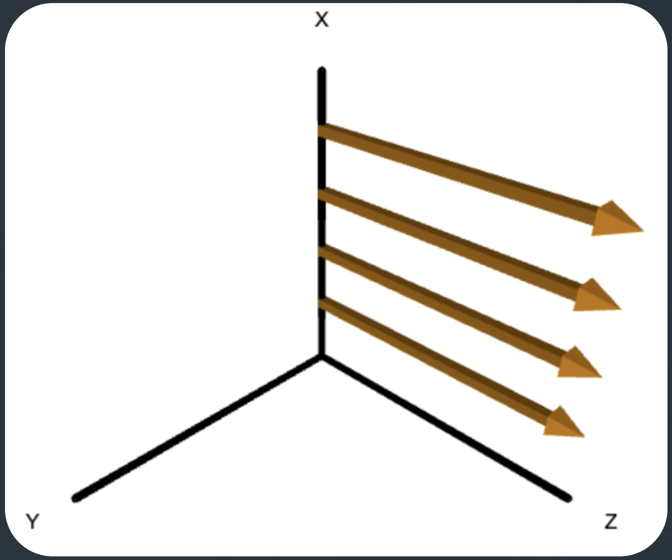
$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} = q(\vec{E} + \vec{v} \times \vec{B})$$

is solved numerically using a fourth-order Runge-Kutta method

- At each time step, we update:
  - Fields (when varying)
  - Particle velocity
  - Particle position
  - Time
- Particle trajectories are visualized as a function of time

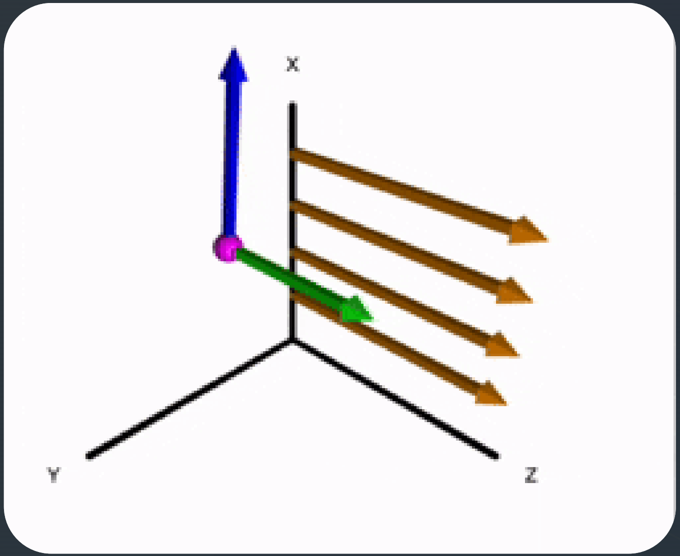


Case 1: Uniform Magnetic Field



Consider a charged particle (proton) in a constant, uniform magnetic field: $\vec{E} = 0, \ \ \vec{B} = B_0 \, \hat{z}$

Case 1: Uniform Magnetic Field



Consider a charged particle (proton) in a constant, uniform magnetic field: $\vec{E} = 0, \ \vec{B} = B_0 \, \hat{z}$

- We will get circular motion about the magnetic field
- Simulation shown for a proton with an initial velocity having components in the +*x* (⊥ to B) and +*z* (∥ to B) directions.
- Trajectory traces out a helix

Case 1: Uniform Magnetic Field

Proton in a constant, uniform magnetic field: $\vec{E} = 0, \;\; \vec{B} = B_0 \, \hat{z}, \;\; \vec{v} = < v_x, v_y, v_z >$

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Lorentz Force:

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$$m < \ddot{x}, \ddot{y}, \ddot{z} > = < qv_yB_0, -qv_xB_0, 0 >$$

Dots represent time derivatives

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Dots represent time derivatives

Breaking the force into components:

$$\ddot{x} = \frac{qB_0}{m}v_y = \frac{q\,B_0}{m}\dot{y}$$

$$\ddot{y} = -\frac{qB_0}{m}v_x = -\frac{qB_0}{m}\dot{x}$$

$$\ddot{z} = 0$$

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$$\ddot{z} = 0$$

v_z = constant

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Motion in the x & y directions is coupled:

$$\ddot{x} = \omega_c \dot{y}$$
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$$\ddot{z} = 0$$

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Motion in the x & y directions is coupled:

$$\ddot{x} = \omega_c \dot{y}$$
$$\ddot{y} = -\omega_c \dot{x}$$

These can be decoupled by taking a time derivative:

$$\dddot{x} = \omega_c\, \ddot{y} = -\omega_c^2\, \dot{x}$$

$$\dddot{y} = -\omega_c\, \ddot{x} = -\omega_c^2\, \dot{y}$$

Case 1: Uniform Magnetic Field

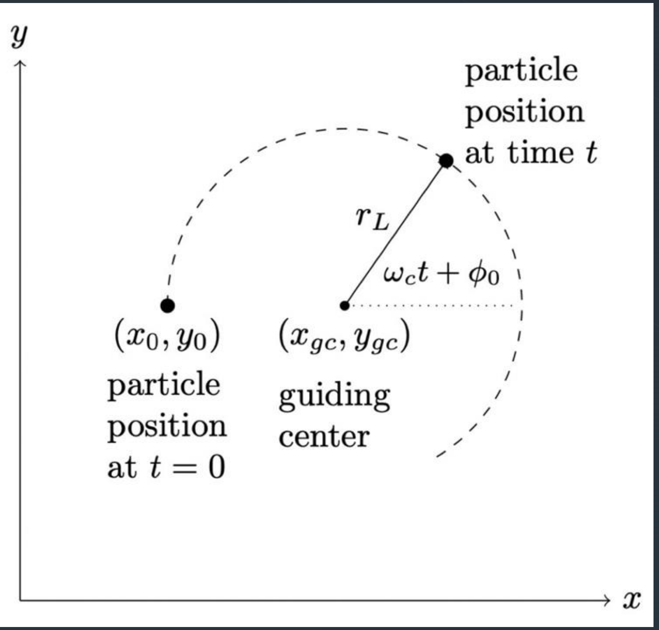
Solutions are:

$$x(t) = r_L \sin(\omega_c t + \phi_0) + x_{gc}$$
$$y(t) = r_L \cos(\omega_c t + \phi_0) + y_{gc}$$

Larmor radius $r_L = \dfrac{mv_\perp}{|q|B}$

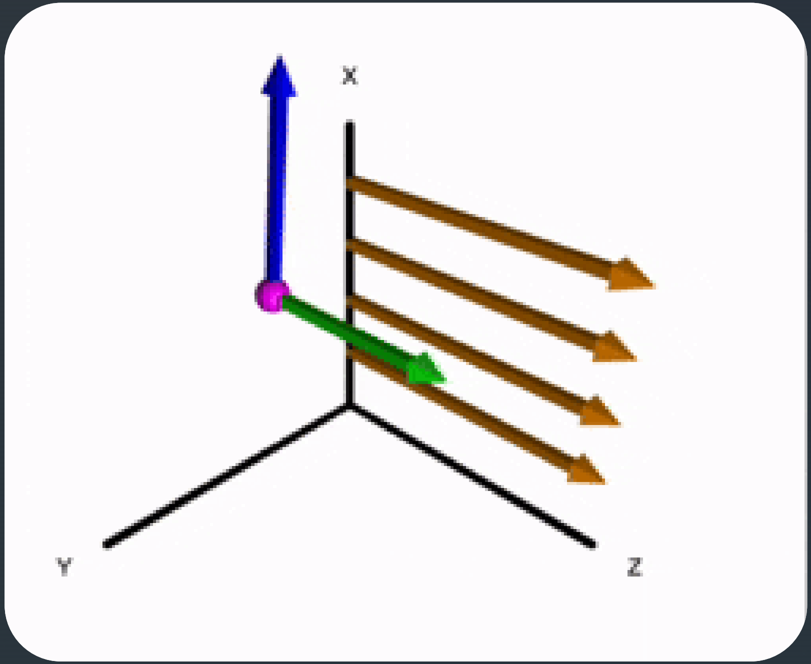
Phase constant to match the initial velocity

Coordinates of the circle's center



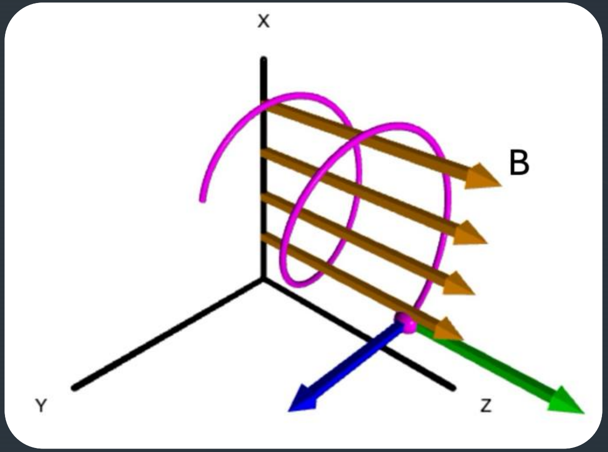
Particle traces out a circle of radius r_L with a frequency of ω_c

Velocity Components



- Blue arrow = $v_\perp$ (velocity perpendicular to B)
- Green arrow = $v_\parallel$ (velocity parallel to B)

Velocity Components



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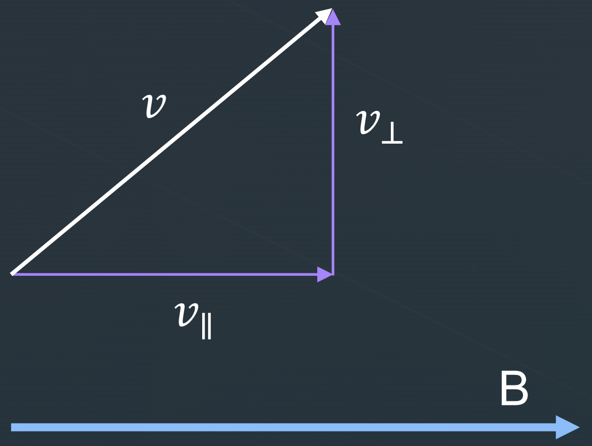
It is often useful to decompose the velocity of the particle into components parallel and perpendicular to B.



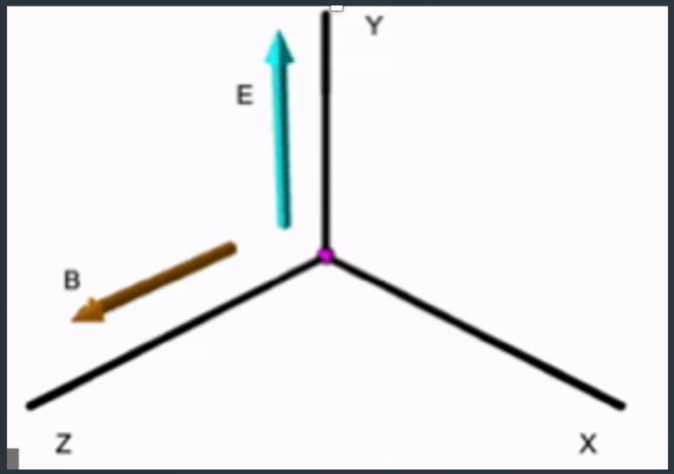
$$\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$$

$$v^2 = v_\perp^2 + v_\parallel^2$$

Case 2: Uniform E and B Fields

What if there is an electric field?

$$\vec{E} = E_0\,\hat{y}, \qquad \vec{B} = B_0\,\hat{z}$$



From the Lorentz Force:

$$\ddot{x} = \omega_c \dot{y}$$

$$\ddot{y} = -\omega_c \dot{x} + \frac{qE_0}{m}$$

$$\ddot{z} = 0$$

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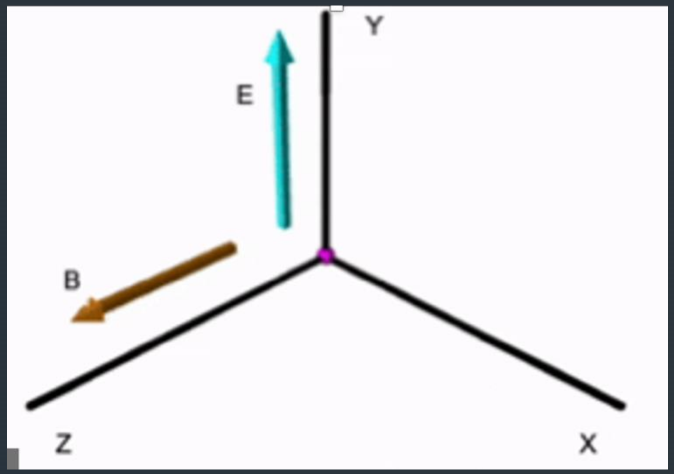

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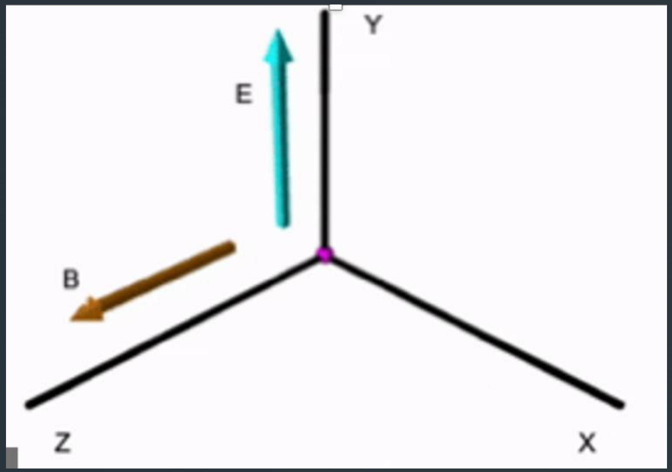

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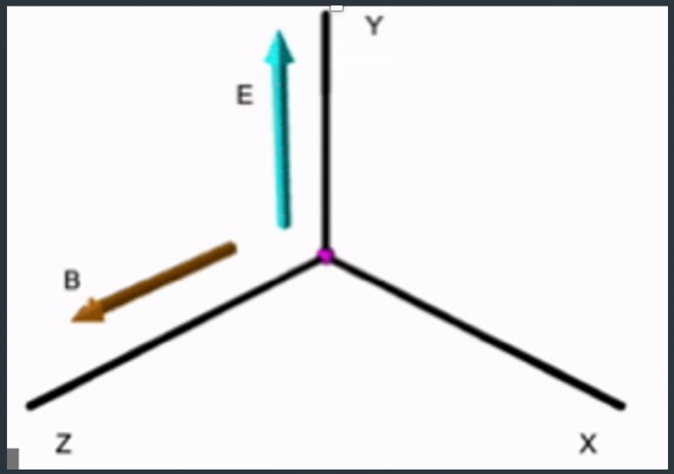

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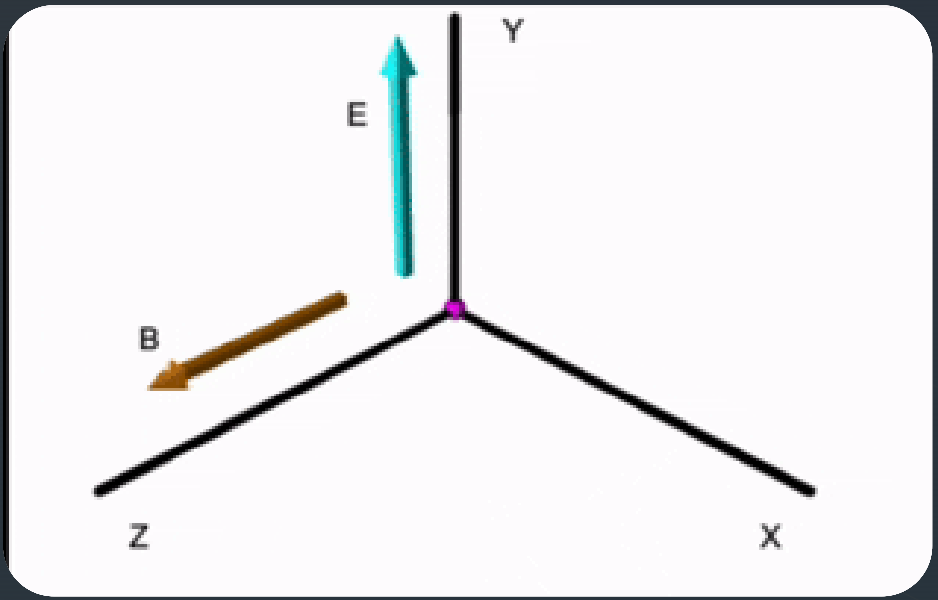
Again, differentiate to decouple the equations:

$$\dddot{x} = \omega_c\,\ddot{y} = -\omega_c^2 \left(\dot{x} + \frac{E_0}{B_0} \right)$$

$$\dddot{y} = -\omega_c\,\ddot{x} = -\omega_c^2\,\dot{y}$$

Cycloidal Motion



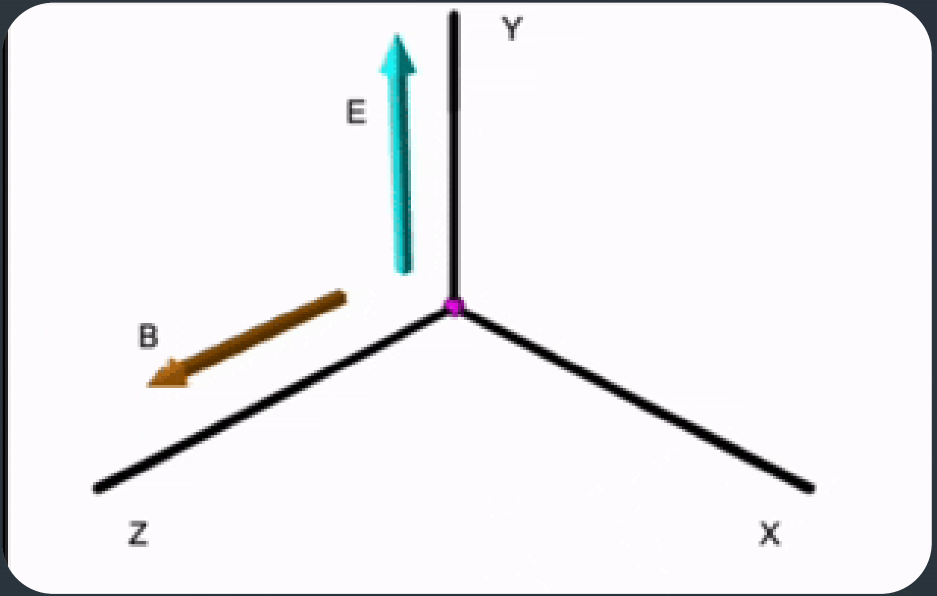
Solutions will be of the form:

$$x(t) = r_L \sin(\omega_c t + \phi_0) + \frac{E_0}{B_0} t$$

$$y(t) = r_L \cos(\omega_c t + \phi_0)$$

$$z(t) = 0$$

Cycloidal Motion



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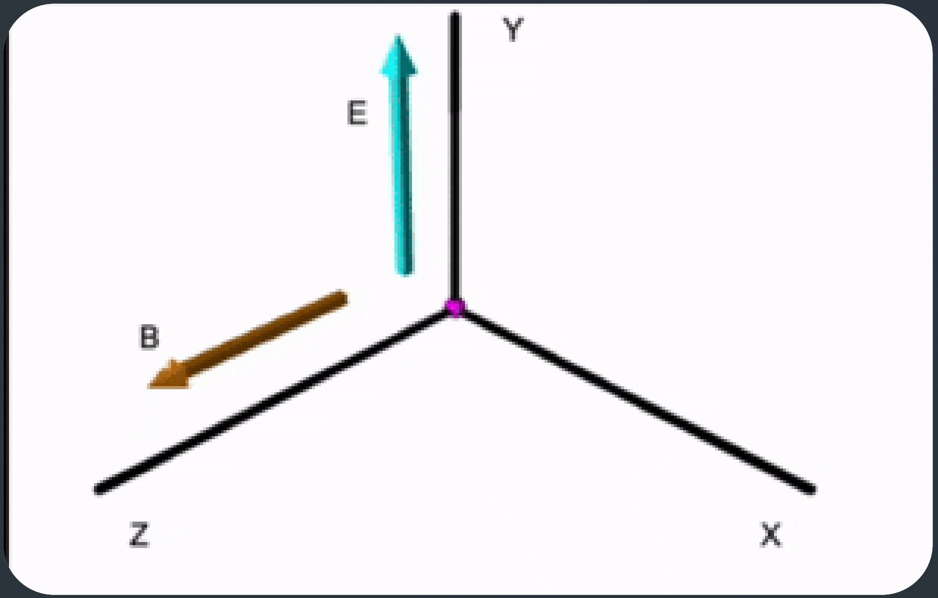
$$z(t) = 0$$

- Particle moves initially in the +y direction because of the electric force
- Velocity is then perpendicular to B, which causes it to deflect
- Magnetic force causes the particle to change direction and move opposite to E
- Particle slows down, stops and then begins moving in the +y direction again

Cycloidal Motion



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$$y(t) = r_L \cos(\omega_c t + \phi_0)$$

$$z(t) = 0$$

Represents linear motion in the x direction!

This is the drift!

- *x(t)* contains an oscillatory part + linear drift
- The drift velocity is: $v_d = \frac{E_0}{B_0}$

The $E \times B$ Drift

- More generally, this is called the $E \times B$ drift, given by:

$$\vec{v}_{E\times B} = \frac{\vec{E} \times \vec{B}}{B^2}$$

- For any constant force (e.g., gravitational), replace E with $\frac{F}{q}$
- Any external force will result in a drift given by:

$$\vec{v}_d = \frac{\vec{F} \times \vec{B}}{qB^2}$$

The Guiding Center

- The particle's motion can be separated into two components:

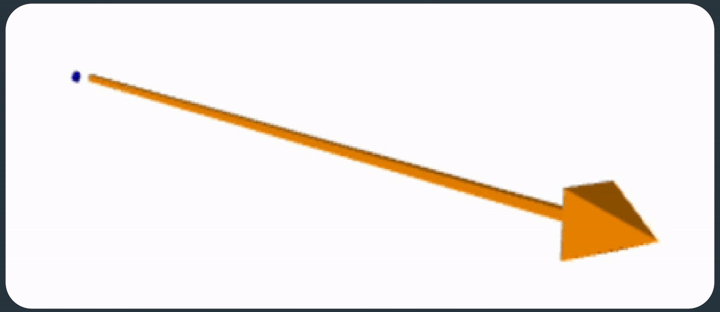
1. Fast cyclotron motion (gyromotion): rapid circular motion *about* the guiding center

   → described by $\vec{x}_L$, $\vec{v}_L$

2. Slow guiding-center motion: drift of the guiding center described by

   → described by $\vec{x}_{gc}$, $\vec{v}_{gc}$

   → due to field gradients, curvature, and electric fields



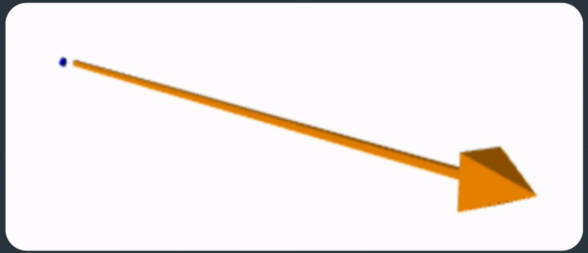
The Guiding Center

- The resulting trajectory = net drift motion of the guiding center superimposed on the gyromotion *about* the guiding center

$$\vec{x} = \vec{x}_{gc} + \vec{x}_L$$
$$\vec{v} = \vec{v}_{gc} + \vec{v}_L$$

$\vec{x}_{gc}, \vec{v}_{gc}$: position & velocity of guiding center *relative to the origin*

$\vec{x}_L, \vec{v}_L$: position & velocity of particle *relative to guiding center*



Guiding-center theory:

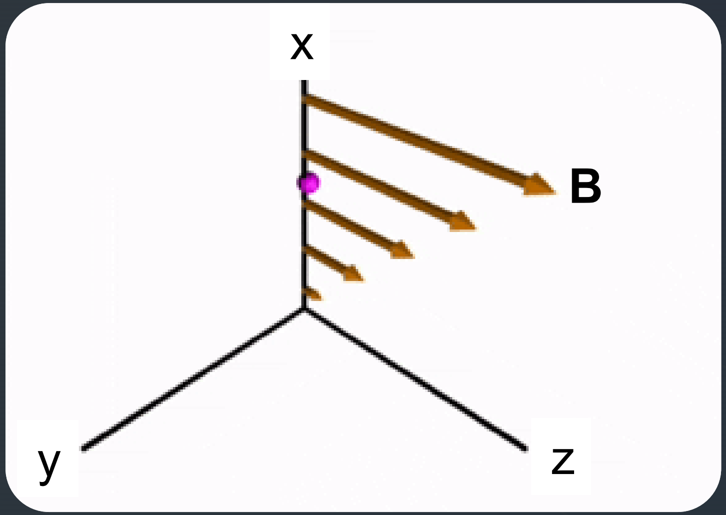
- We can analyze particle motion in more complex fields using this framework
- Average over the fast gyromotion to better understand the slower drift motion

Key Takeaways

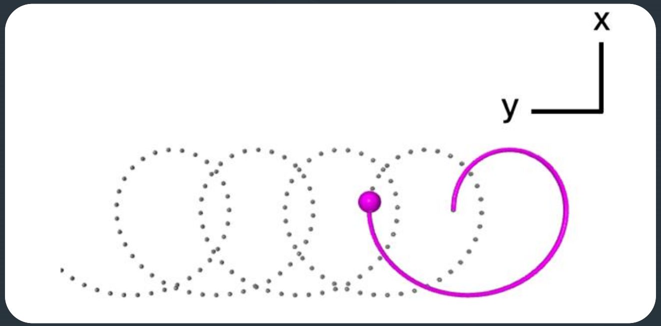
Particle motion can be broken into two types of motion

- Gyration
  - Particles orbit the applied magnetic field
  - Radius of this orbit is the Larmor radius, $r_L = \frac{mv_\perp}{|q|B}$
  - Frequency of the orbit is the cyclotron frequency, $\omega_c = \frac{|q|B}{m}$
- Guiding center (what is left when you average over the gyration)

Nonuniform Fields: ∇B Drift

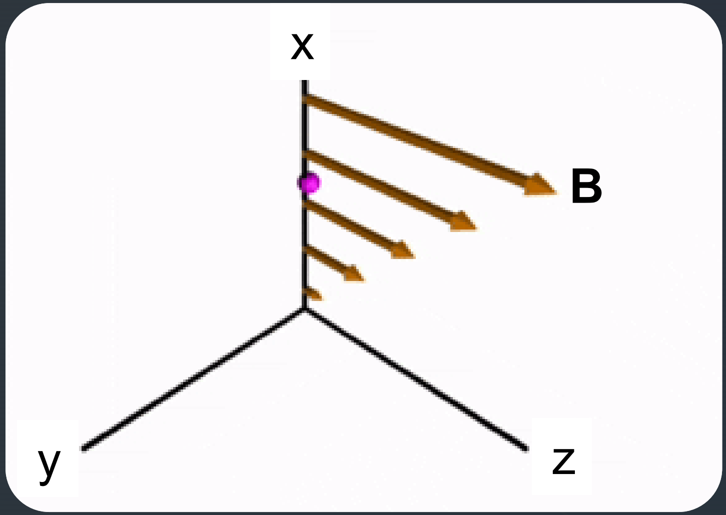


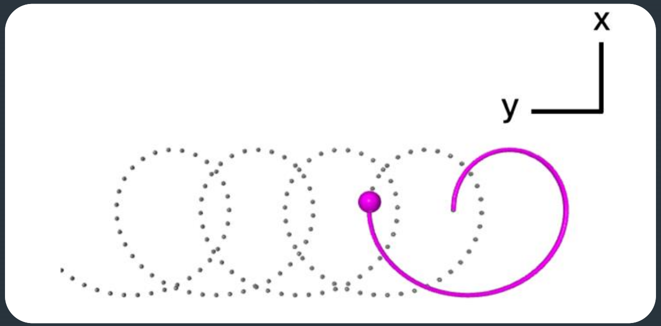
- Variations in magnetic field strength change the size of the gyroradius
  - Large radius in regions of weaker B
  - Small radius in regions of stronger B
  - Asymmetry means the particle travels farther and spends more time regions of weak B



- Simulation of a positive charge in a linearly varying B field. Initial velocity in the +x direction.
- Particles drift in the B × ∇B direction (+y direction)
- Opposite charges drift in opposite directions

Nonuniform Fields: ∇B Drift





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How does the gradient in magnetic field influence particle motion?

What is the drift velocity?

Nonuniform Fields: ∇B Drift

1. Examine how the B field changes across a single cyclotron orbit
   - Assume that the gradient is small, so we can approximate the magnetic field near the guiding center
   - Expand the magnetic field in a Taylor series about $\vec{x}_{gc}$. First two terms give:

$$\vec{B}(\vec{x}) \approx \vec{B}\left(\vec{x}_{gc}\right) + (\vec{r}_L \cdot \nabla)\,\vec{B}$$

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2. Using this and the guiding center framework, $\vec{v} = \vec{v}_{gc} + \vec{v}_L$, the Lorentz force becomes:

$$\vec{F} = m\left[\frac{d\vec{v}_{gc}}{dt} + \frac{d\vec{v}_L}{dt}\right] = q(\vec{v}_{gc} + \vec{v}_L) \times [\vec{B}(\vec{x}_{gc}) + (\vec{r}_L \cdot \nabla)B]$$

Nonuniform Fields: ∇B Drift

3. Substitute in the form of the solution…
   - $\vec{r}_L$ and $\vec{v}_L$ will have solutions of the form:

$$\vec{r}_L = r_L < \sin\omega_c t, \cos\omega_c t, 0 >$$

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4. …and average over one cyclotron orbit
   - The average of $\sin\omega_c t$ and $\cos\omega_c t$ is 0; average of $\cos^2\omega_c t$ and $\sin^2\omega_c t$ is ½

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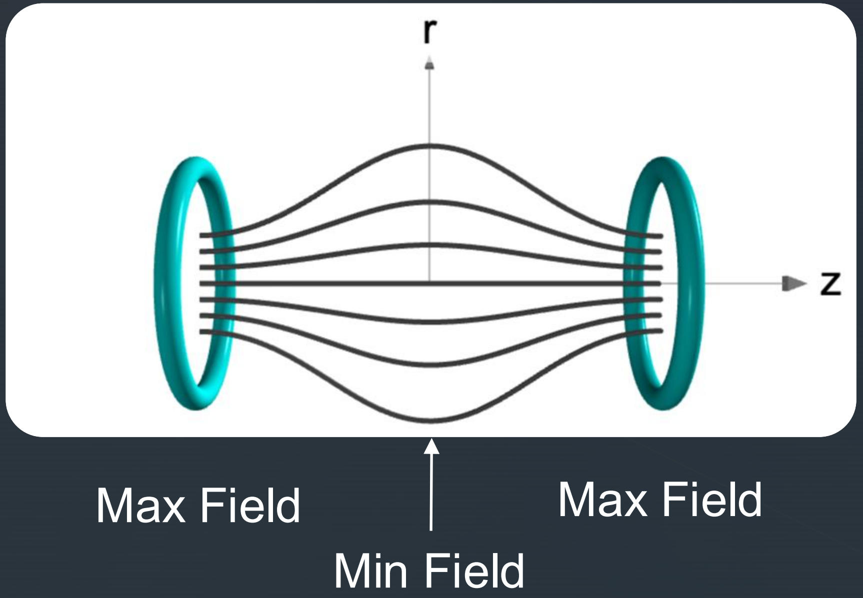
5. The force associated with the gradient in magnetic field is: $\vec{F} = -\frac{mv_\perp^2}{2B}\nabla B$
6. The drift velocity is then…

$$\vec{v}_d = \frac{\vec{F} \times \vec{B}}{qB^2} = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\vec{B} \times \nabla B}{B^2}$$

✓ Result is consistent with our simulation.

Magnetic Mirrors

Field lines between two current-carrying coils



- Magnetic field is curved
- Field is highest at the coil centers
- Gradient is parallel to the field lines
- B field has both radial and axial components
- B is axisymmetric about the z axis

Magnetic Mirrors

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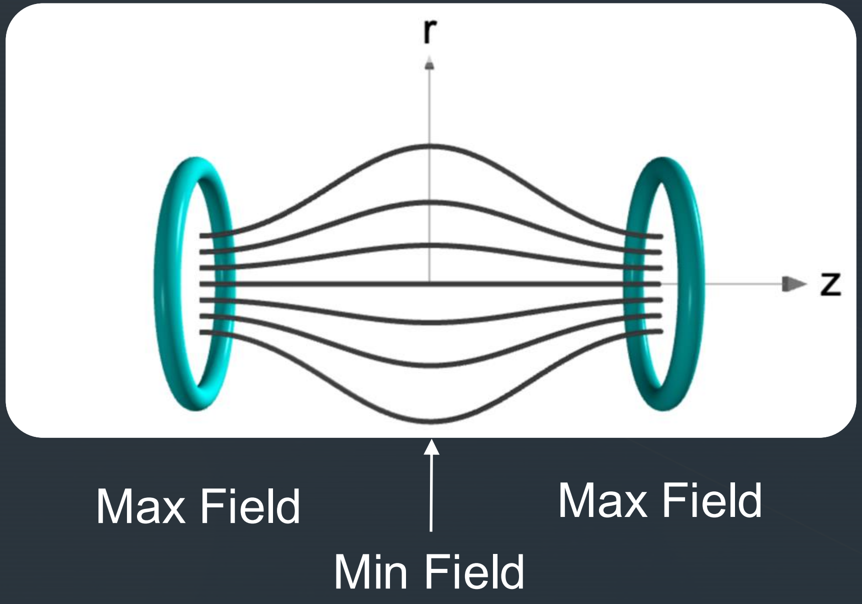

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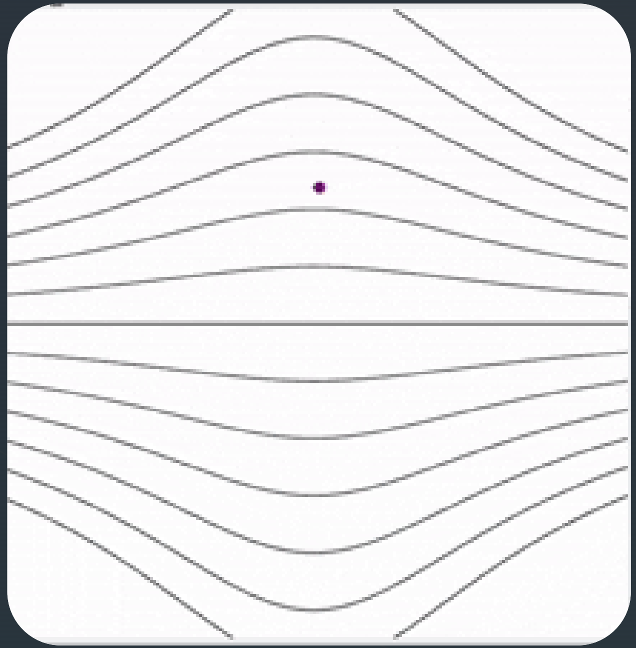
Proton trapped in a magnetic mirror



- Proton spirals around the field lines
- Reflects at a point in the region z>0
- Reflects again at a point z<0
- Bounces back and forth between mirror points

Magnetic Mirrors

What is going on?

- The particle must experience a force that slows it down, F_z

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Analytic expression for B: $\vec{B} = B_0\left[-\frac{rz}{L^2}\,\hat{r} + \left(1 + \frac{z^2}{L^2}\right)\hat{z}\right]$

- z = the axial distance
- r = the radial distance from the z axis
- L = is a characteristic length of the system
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From the Lorentz Force:

- $\vec{F} = q\vec{v} \times \vec{B}$ → $F_z = -qv_\theta B_r$
- where v_θ is the velocity component about the z-axis

The First Adiabatic Invariant

- Mirroring can be explained by invariance of the magnetic moment, μ.
  - μ is the first adiabatic invariant
- The particle moves in circular motion around the field lines, so its magnetic moment is:
  - $\mu = IA = \frac{q}{T}\pi r^2 = \left(\frac{q\omega_c}{2\pi}\right)\frac{\pi m^2 v_\perp^2}{q^2 B^2} = \frac{m v_\perp^2}{2B}$
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→ but what about $v_\parallel$?

Magnetic Mirrors

- Total kinetic energy is conserved (assuming *E = 0*)

Conservation of Kinetic Energy:

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2$$

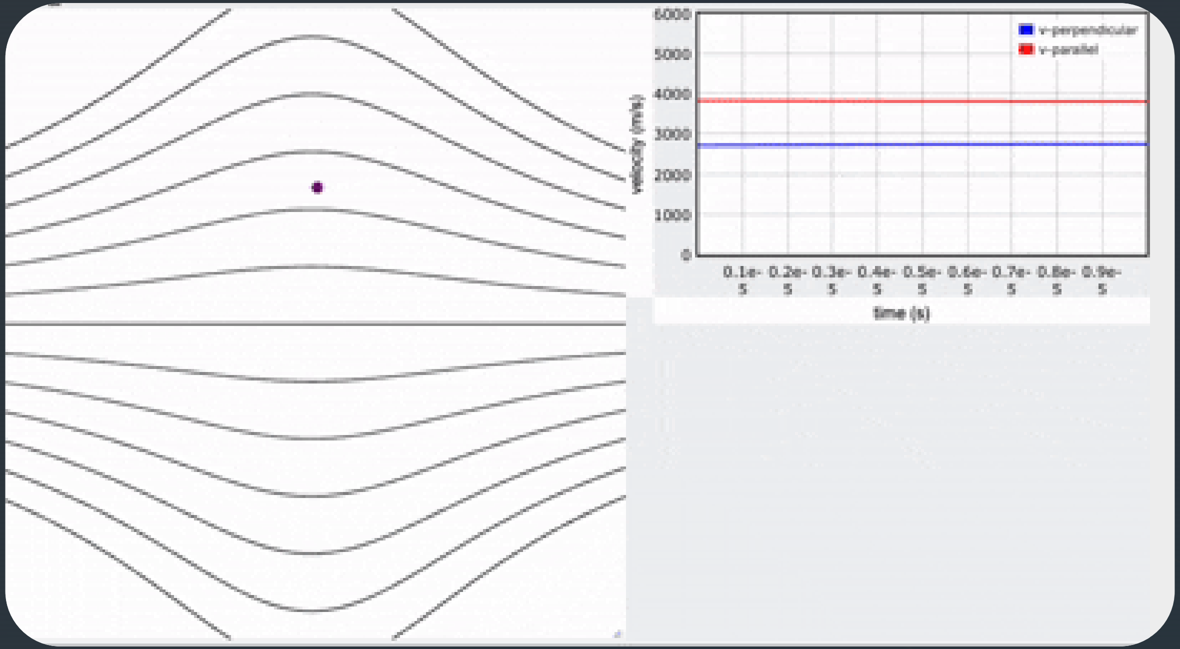
Recall that: $v^2 = v_\perp^2 + v_\parallel^2$

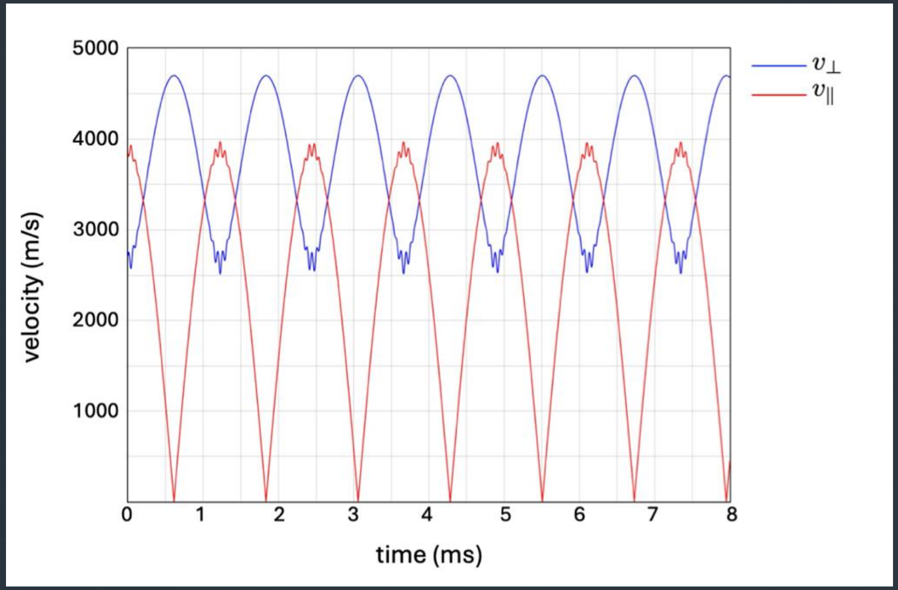
$$v_{\perp i}^2 + v_{\parallel i}^2 = v_{\perp f}^2 + v_{\parallel f}^2$$

Stronger B → Larger $v_\perp$ & Smaller $v_\parallel$

- An increase in $v_\perp$ comes with a decrease in $v_\parallel$.
- When $v_\parallel = 0$, the particle reflects and reverses direction along the field line.

Magnetic Mirrors





- The initial conditions determine whether reflection will occur and at what location
- The mirror effect occurs for particles when:

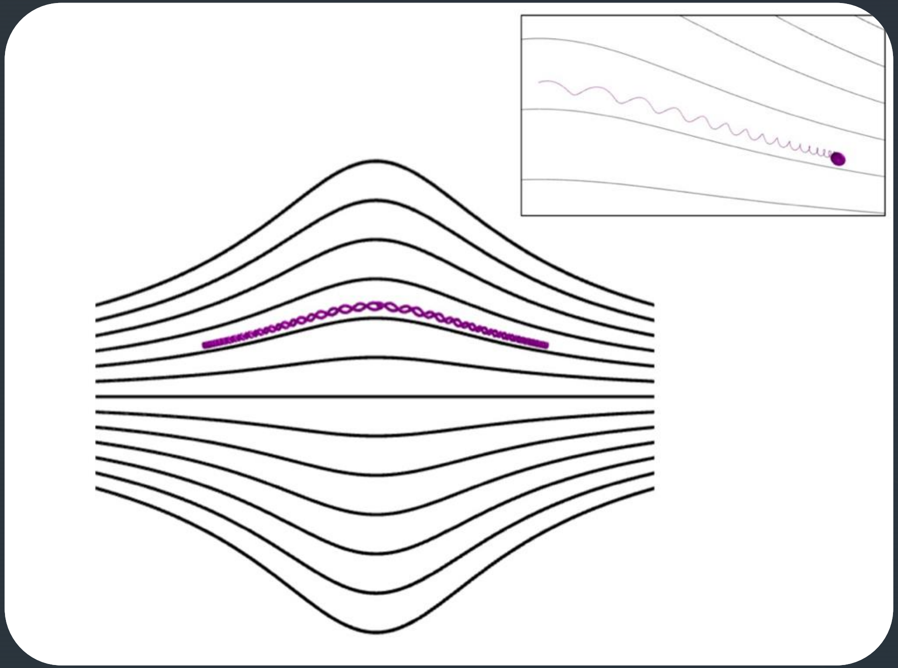
$$\left(\frac{v_{\parallel_0}}{v_{\perp_0}}\right)^2 \leq \frac{B_{max}}{B_i} - 1$$

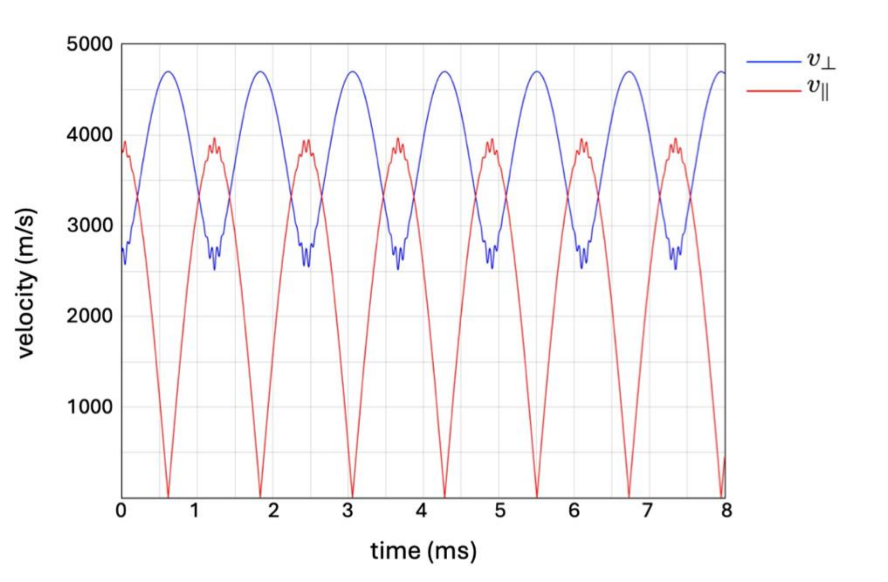
At the mirror points:

$v_\parallel$ goes to zero

$v_\perp$ reaches a max

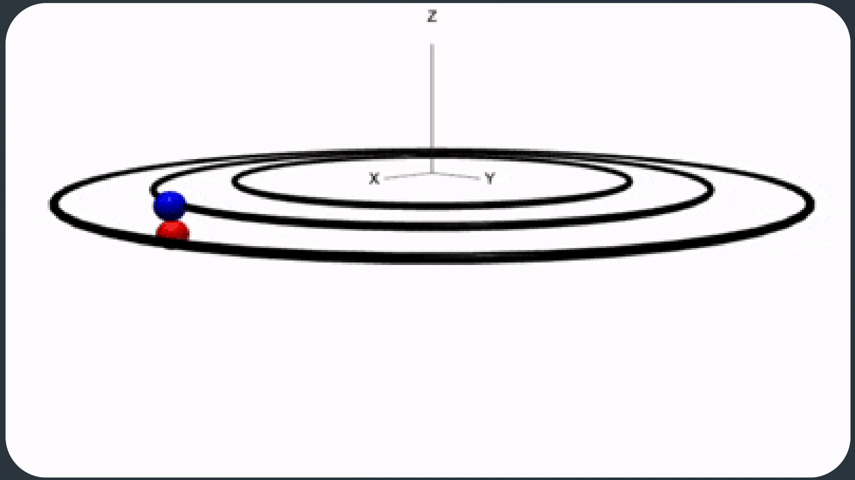
Magnetic Mirrors





- The particles experience a force from the radial component of the magnetic field (B_r
- Force points in the direction of weaker magnetic field
- Magnetic mirroring occurs in Earth's magnetic field, reflecting charged particles between the poles

Nonuniform Fields: Curvature Drift

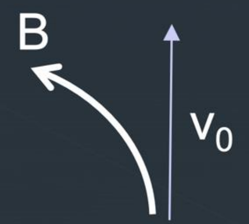


Positive (blue) & negative (red) charges in a constant, curved magnetic field.

Particles drift in the $R_c \times B$ direction.

Opposite charges drift in opposite directions (± z, depending on the sign).

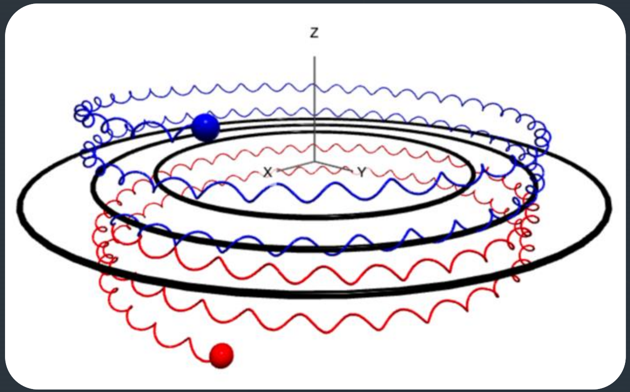
- Initial velocity is parallel to the magnetic field.
- As the particles moves, the B field curves away from the particle, creating velocity component ⊥ to the B field.



- This causes the particles to gyrate around the field line.
- For +q particle: gyroradius is larger on the outside of the curve, smaller when on the inside (closer to the center)
- This difference in radius causes an upward drift
- For –q particle: gyration around the field lines is in the opposite direction, which causes a drift downwards.

Nonuniform Fields: Curvature Drift

What is the drift velocity?



- Assume that *B* points in the $\hat{\phi}$ direction and is curved with a constant radius of curvature R_c.
- As particle gyrates around the magnetic field lines, its guiding center will travel in a circle of radius R_c.
- Averaging over a gyroperiod, the particle experiences a force in the radial direction:

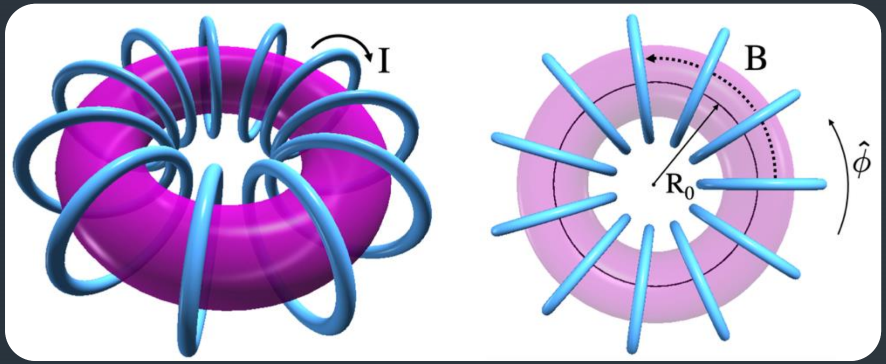
$$< \vec{F} > = \frac{mv_{\parallel}^2}{R_c} \hat{R}_c$$

- So we get:

$$\vec{v}_d = \frac{\vec{F} \times \vec{B}}{qB^2} = \frac{mv_{\parallel}^2}{qB^2} \frac{\vec{R}_c \times \vec{B}}{R_c^2}$$

Nonuniform Fields: Toroidal Drift

- Gradient and curvature drifts typically occur together
  - Example → Tokamaks
  - B_T created by a set of current-carrying coils surrounding the tokamak



Using Ampere's Law, the toroidal magnetic field is:

$$\vec{B}_T = \frac{\mu_0 N I}{2\pi R}\hat{\boldsymbol{\phi}} = \frac{B_0 R_0}{R}\hat{\boldsymbol{\phi}} = \frac{B_0 R_0}{(x^2+y^2)}(-y\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}})$$

N = the number of coils

I = the current in each coil

R = the distance from the center of the torus to a point inside the torus (the major radius)

$\hat{\boldsymbol{\phi}}$ = the direction around the torus

B_0 = magnetic field at R_0

Nonuniform Fields: Toroidal Drift

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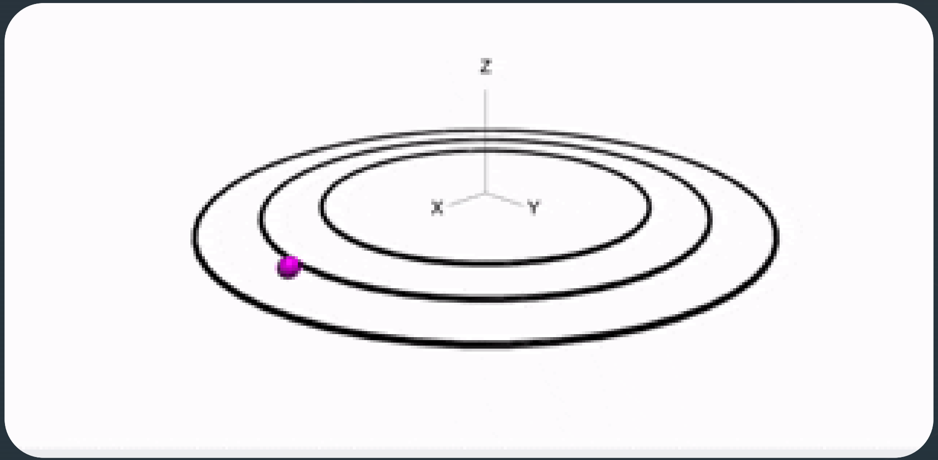
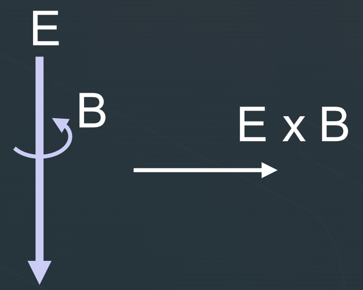
- Toroidal magnetic field has both spatial gradient and curvature
  - The drifts add: positive charges drift up, negative drift down
- Charge separation sets up an E field (in –z direction)
- All particles (independent of charge) experience an E×B drift radially outward





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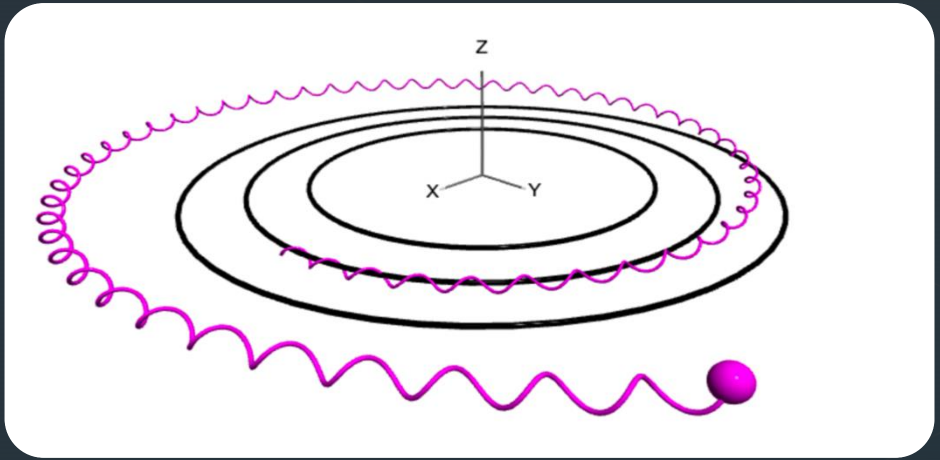
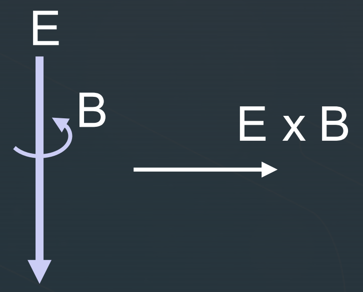
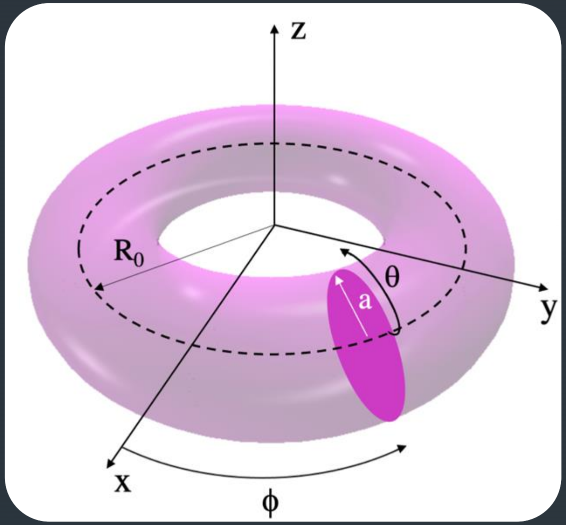
To account for this drift, tokamaks include a **poloidal magnetic field**, which improves confinement.



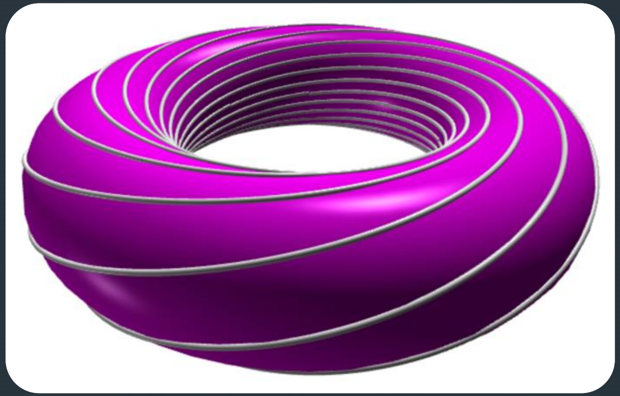


Tokamaks

- Fusion occurs when hydrogen atoms are fused together, releasing energy in the process.
- H is ionized, resulting in a plasma.
- To account for the drift outward, tokamaks include an poloidal magnetic field to improve confinement.

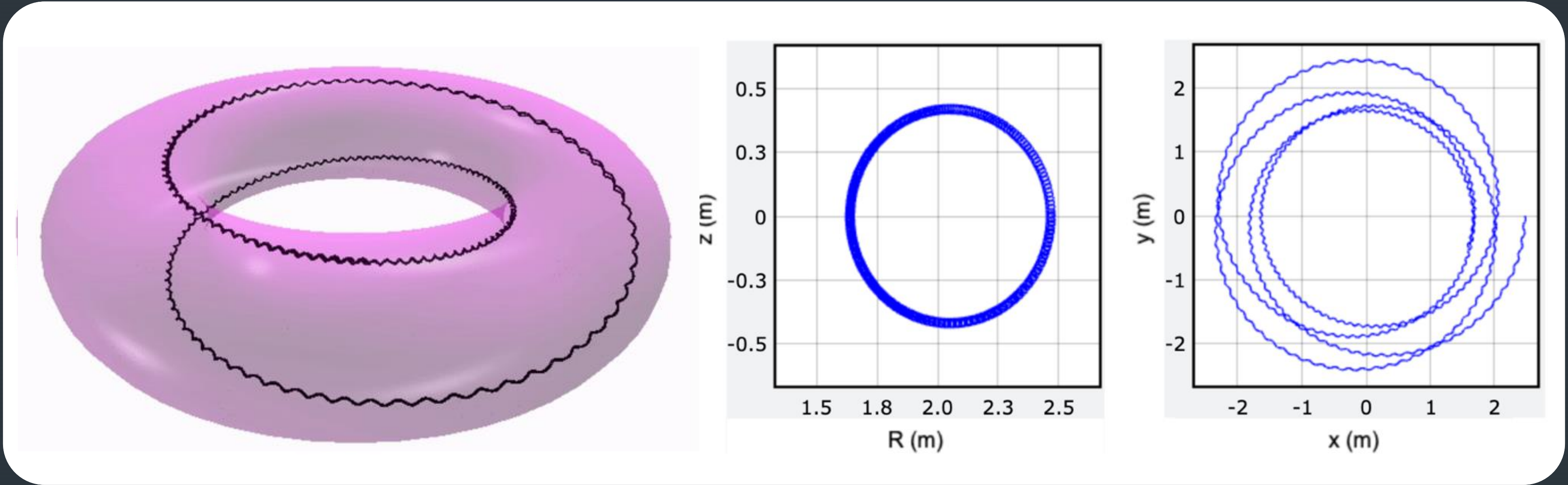


$$B_T = \frac{\mu_0 NI}{2\pi \mathrm{R}}\,\hat{\boldsymbol{\phi}},\ B_p = \frac{b_0 r}{a}\,\hat{\boldsymbol{\theta}}$$



- The toroidal and poloidal magnetic fields combine to create a net field that twists around the device.
- A set of current-carrying coils generates a toroidal magnetic field that wraps around the tokamak.

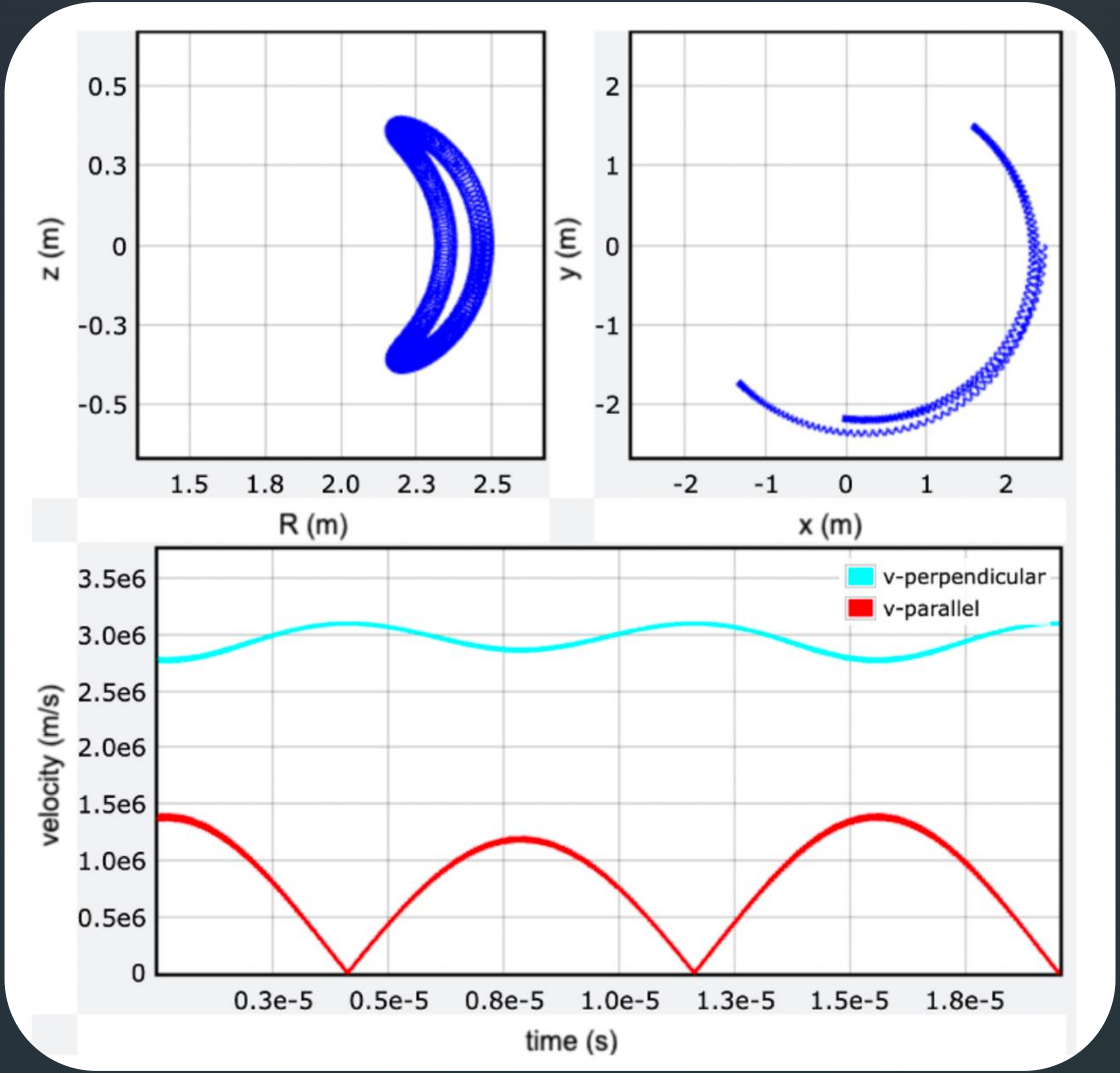
Tokamaks: Passing Trajectories



- Simulation shown for 100 keV deuterium ion in tokamak with major radius R_0 = 2 m, minor radius a = 0.667 m.
- Initial position: (R0 + 0.7a, 0, 0); Initial velocity: $v_{\|,0}$ / $v_{\perp,0}$ = 2
- D+ particle remains within the tokamak, makes several orbits around the device, traces out a circle in the poloidal plane.

Tokamaks: Trapped Trajectories

Simulation shown for $v_{\parallel,0} / v_{\perp,0} = 0.5$



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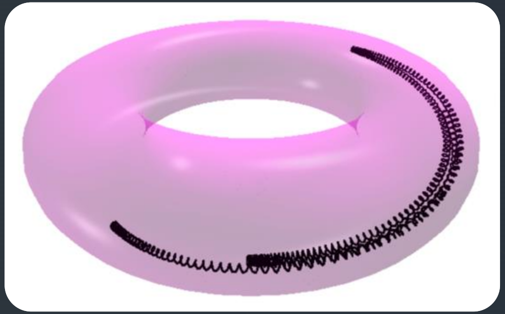
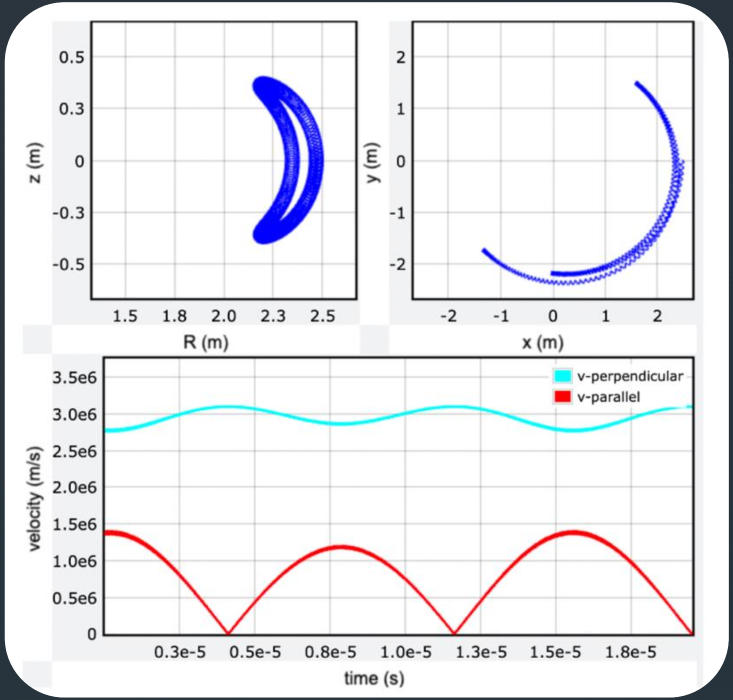
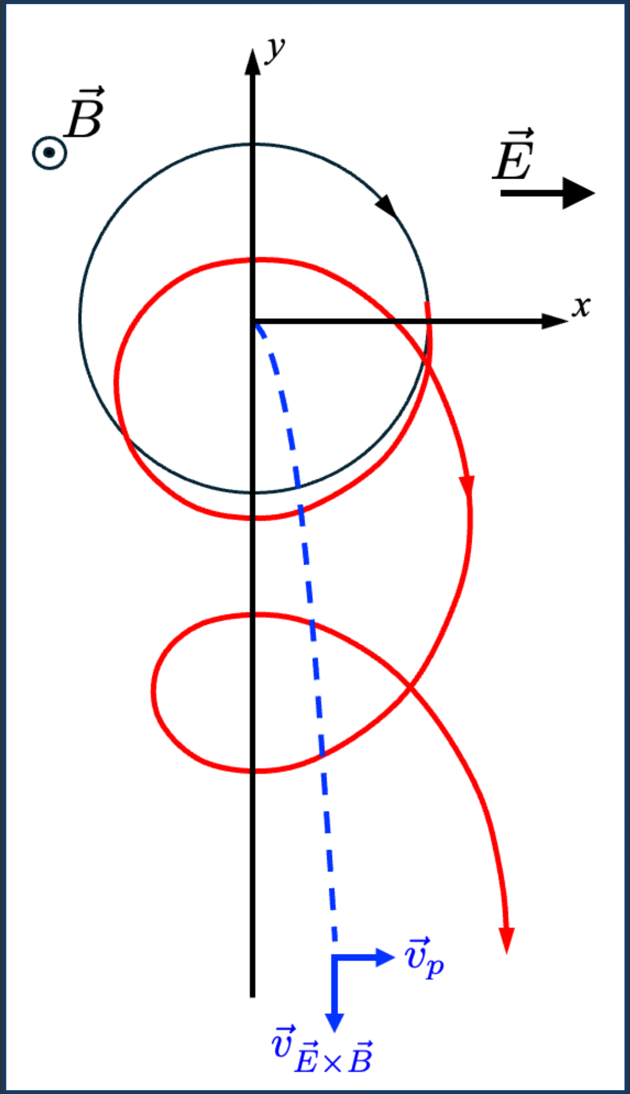




- Particles become trapped when : $v_{\parallel,0}$ / $v_{\perp,0}$ is lowered.
- Particle does not make a complete orbit around the tokamak. It reflects between mirror points (when $v_{\parallel} \rightarrow 0$).
- By conservation of energy, when $v_{\parallel} \rightarrow 0$, $v_{\perp}$ at a max.
- Trapped particles start in the low field region near the outer wall and are reflected from regions of high magnetic field.
- Trapped particle traces out a banana-shaped path in the poloidal plane.

Condition for mirroring: $$\left(\frac{v_{\parallel_0}}{v_{\perp_0}}\right)^2 \leq \frac{B_{max}}{B_i} - 1$$

Time-Varying Fields: Polarization Drift



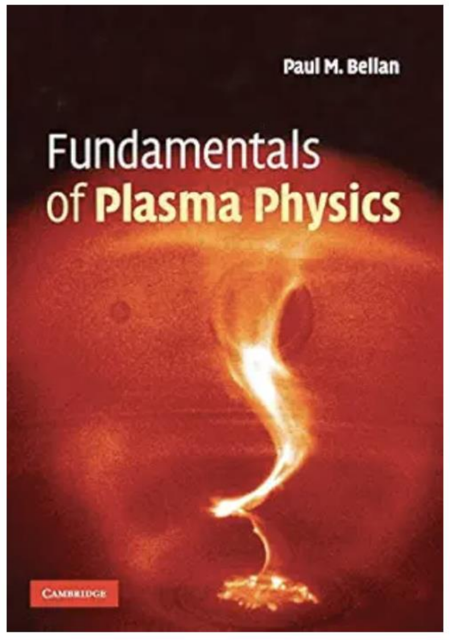
- Time-varying E field, constant B field
- Will have the typical E x B drift + a polarization drift:

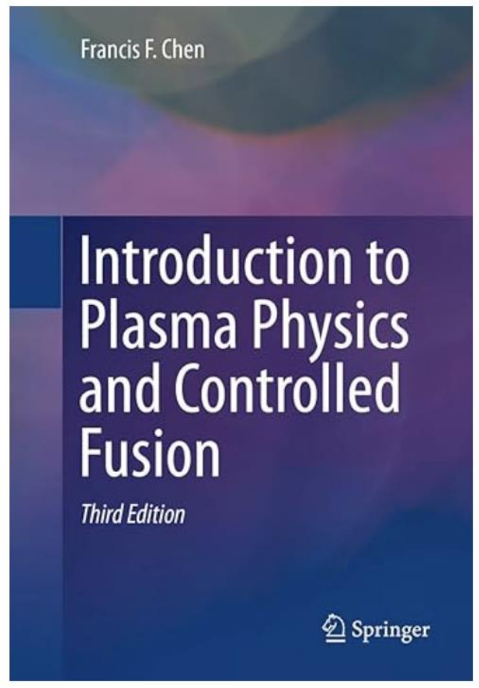
$$v_p = \pm \frac{1}{\omega_c B}\frac{dE}{dt}$$

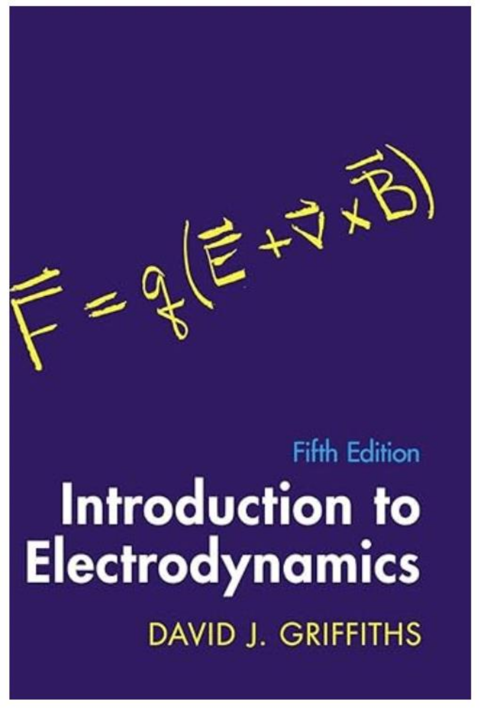
$\pm$ refers to pos/neg charge

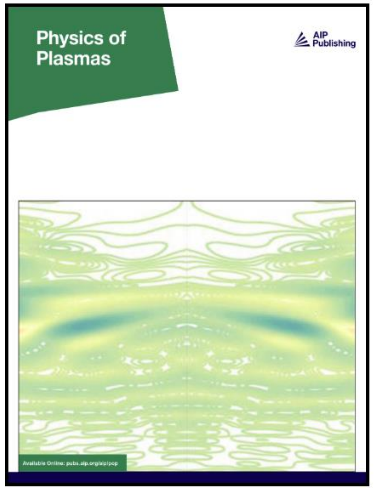
- The drift causes ions and electrons to move in opposite directions, which results in a net current
- $v_d \propto m$, so this drift is negligible for electrons: $\frac{m_i}{m_e} \sim 2000$
- Plot shown for linearly varying electric field
  - particles will experience an ExB drift
  - the guiding center will follow a parabolic trajectory (blue) which drives a polarization current

For More Information









Physics of Plasmas

Acknowledgements

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