Plasmas as Fluids:

Developing Models of Plasmas

Derek Thuecks, Washington College, Chestertown, MD
dthuecks2@washcoll.edu

2025 Introduction to Plasma and Fusion Course, PPPL, 03 June 2025

Who am I and what was my path?

Lawrence University, B.A,. Physics

- First-generation college student
- Research with toroidal confinement of pure-electron plasmas

University of Iowa, Ph.D, Physics

- Experiments conducted using LaPD.
- Shear Alfvén wave dispersion/damping
- Whistler wave absorption diagnostic to measure electron velocity distribution function

University of Wisconsin – Madison

- Worked on Madison Symmetric Torus, a reversed field pinch (RFP) device
- Properties of turbulent cascade
- Current and energy transport caused by coherent electric and magnetic field fluctuations

Ripon College

- Visiting assistant professor of physics
- Continued collaboration on MST research

Washington College

- Faculty member, department chair
- Continued collaboration on MST research
- Began development of local plasma lab for the study of basic plasma waves and turbulence

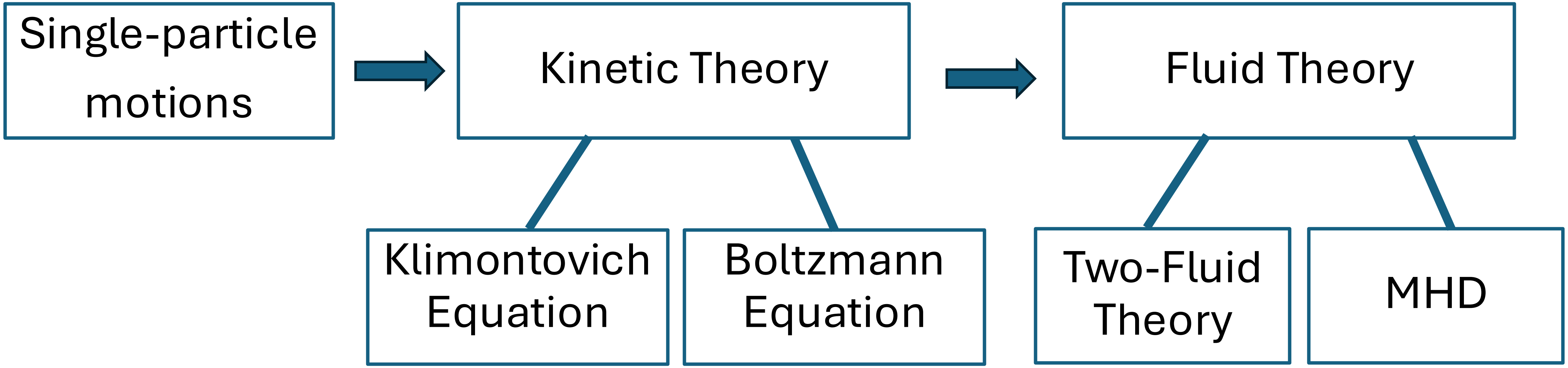
Hobbies

- Biking
- Board games
- TTRPGs

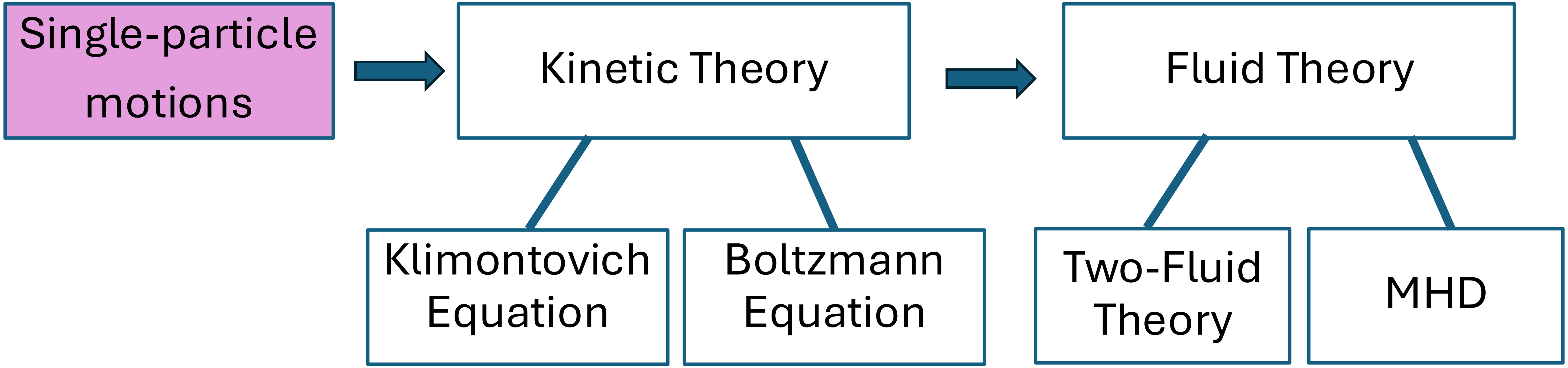
Developing mathematical models

- What do I mean by a ‘model’?
  - A model is a mathematical and conceptual framework used to approximate a physical phenomena, containing enough detail to capture the phenomenon’s essence while being small and simple enough to understand or analyze.
- Goals of developing models
  - Balance accuracy and complexity
  - Discuss role of approximations
  - Find the simplest model that is still able to accurately describe phenomena of interest. Ignore details that are unimportant to your investigation.
- To determine best model for your purposes, it is critical to understand the approximations underpinning a given model

Plasma models

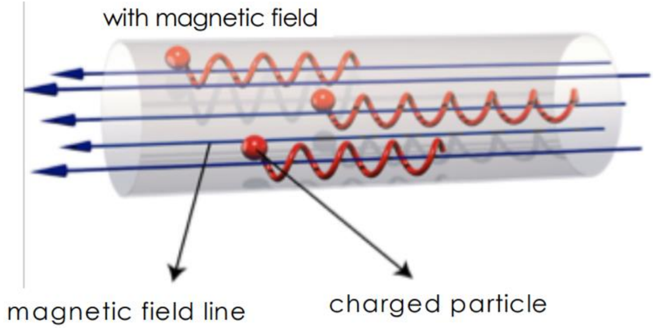


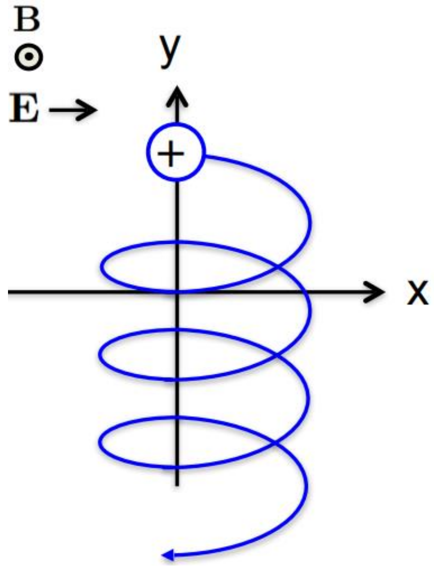
Plasma models

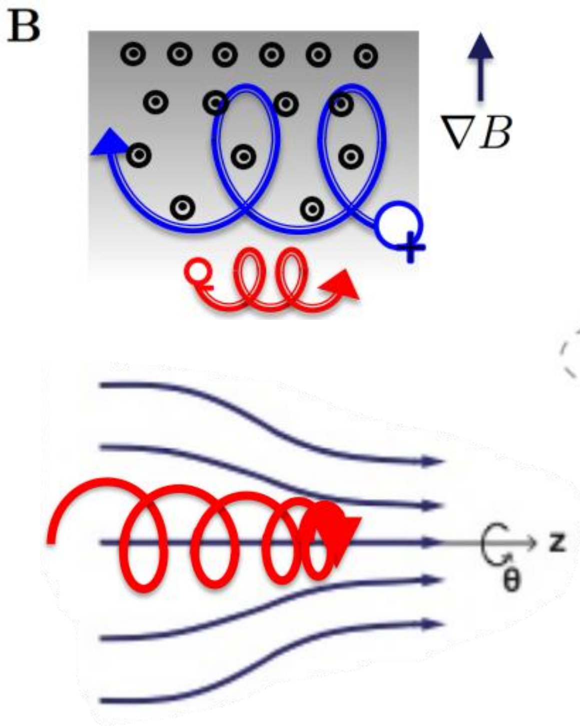


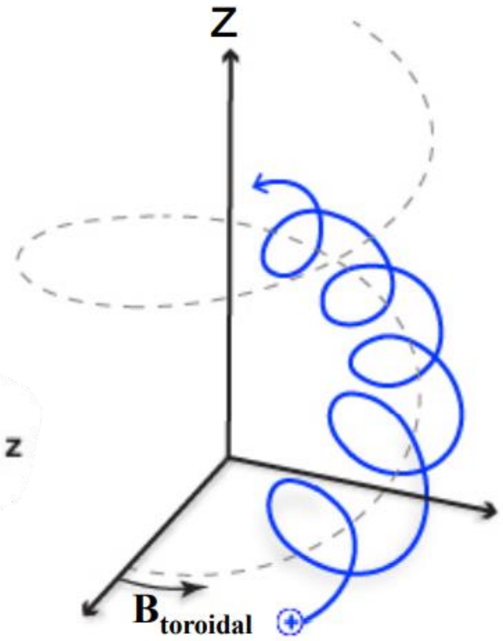
Single particle motions: Results

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$









(Images sourced from single particle motion presentations from summer course in prior years)

How do we model the behavior of plasma containing many particles?

How do we model the behavior of plasma containing many particles?

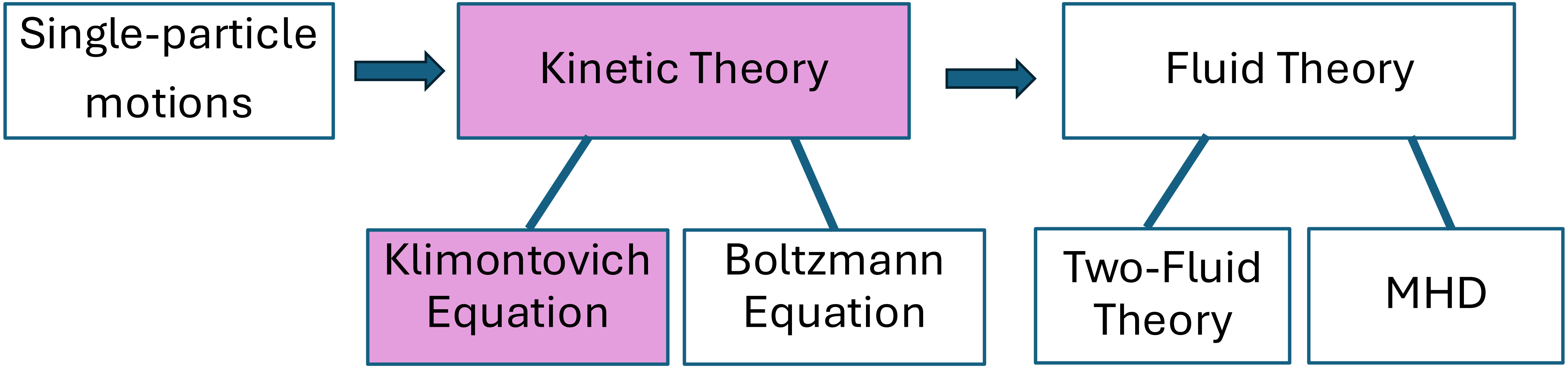
Treat as a conducting fluid!

How do we model the behavior of plasma containing many particles?

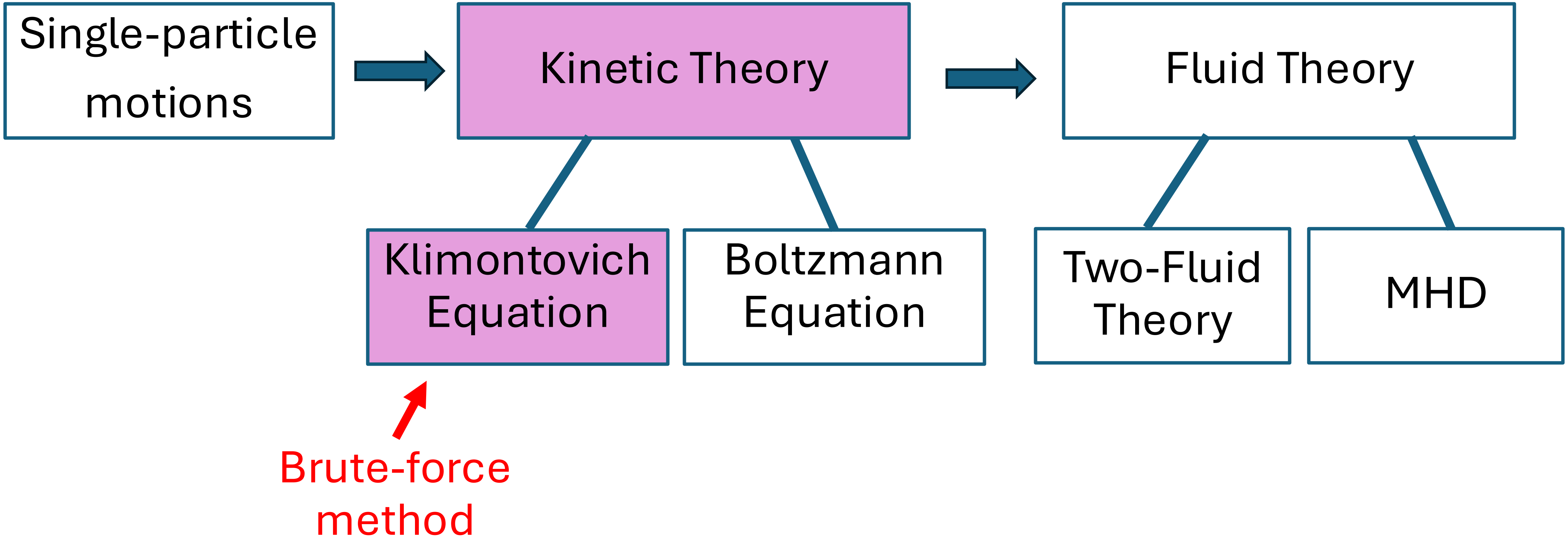
Treat as a conducting fluid!

But, let’s detour first.

Plasma models



Plasma models



Kinetic theory: 6D Phase Space Density

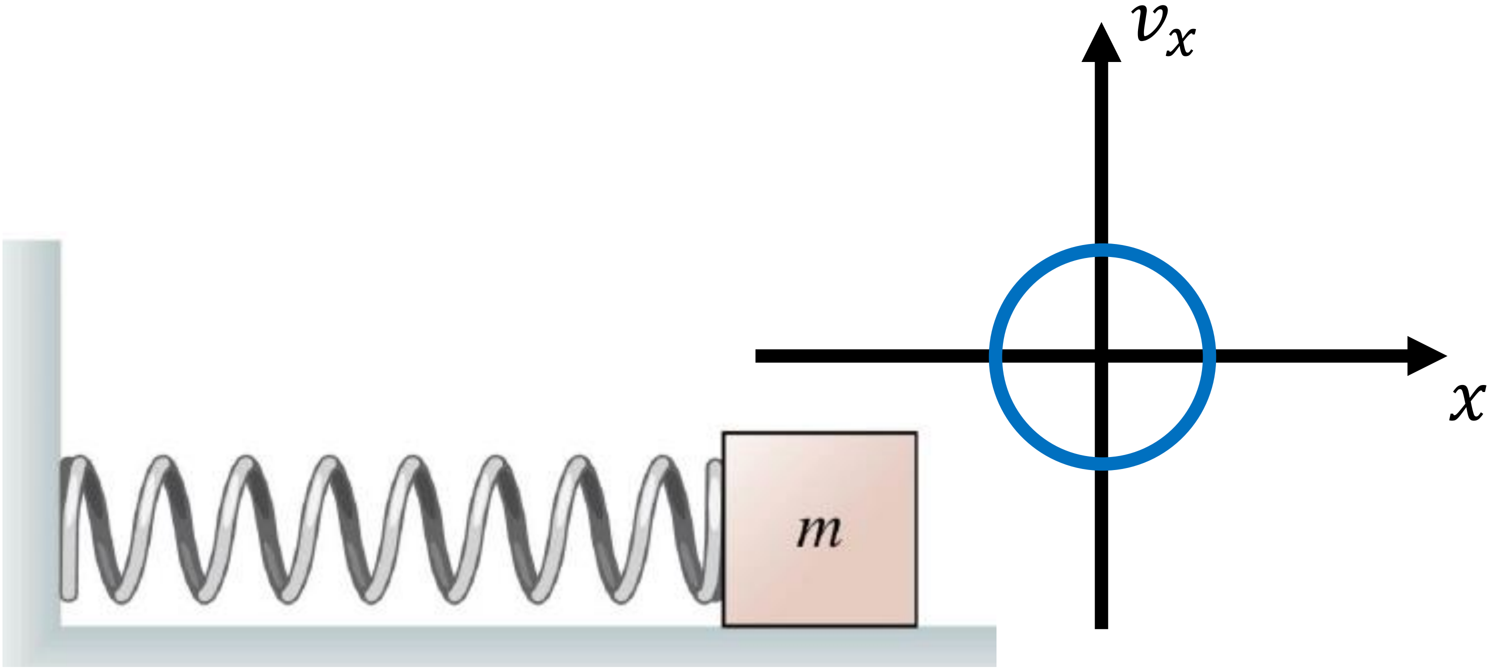
Particle density in 6D phase space for a single particle:

$$N(\vec{x}, \vec{v}, t) = \delta[\vec{x} - \vec{X}_1(t)]\delta[\vec{v} - \vec{V}_1(t)]$$

Kinetic theory: 6D Phase Space Density

Particle density in 6D phase space for a single particle:

$$N(\vec{x}, \vec{v}, t) = \delta[\vec{x} - \vec{X}_1(t)]\delta[\vec{v} - \vec{V}_1(t)]$$

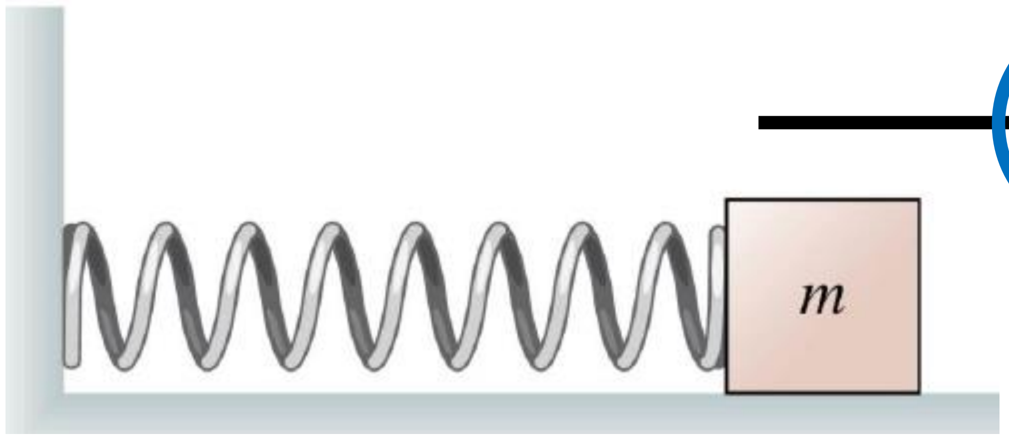


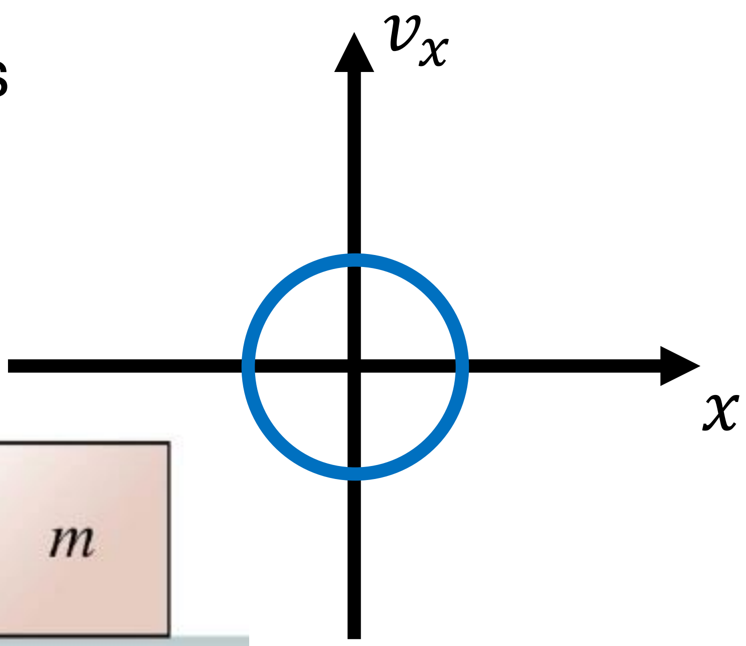
Kinetic theory: 6D Phase Space Density

Particle density in 6D phase space for a single particle:

$$N(\vec{x}, \vec{v}, t) = \delta[\vec{x} - \vec{X}_1(t)]\delta[\vec{v} - \vec{V}_1(t)]$$

Particle phase-space trajectories





Kinetic theory: 6D Phase Space Density

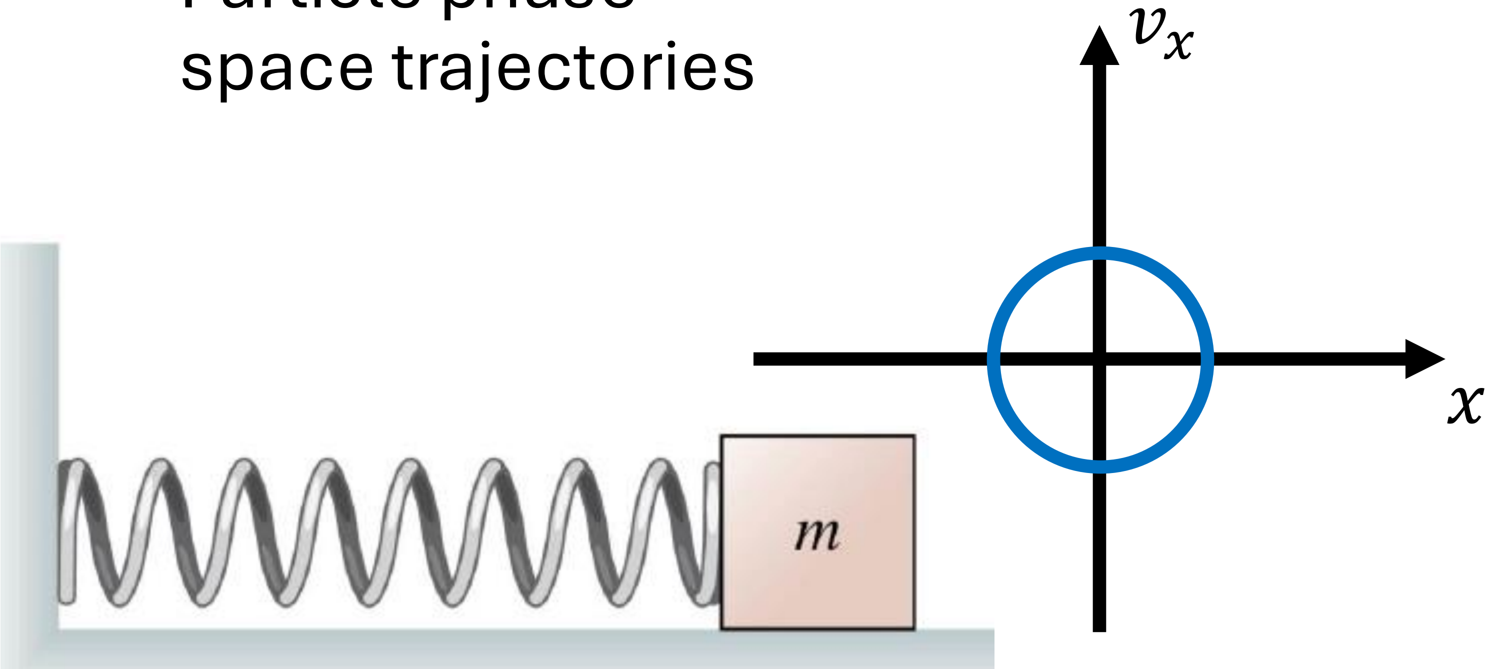
Particle density in 6D phase space for a single particle:

$$N(\vec{x}, \vec{v}, t) = \delta[\vec{x} - \vec{X}_1(t)]\delta[\vec{v} - \vec{V}_1(t)]$$

Particle phase-space trajectories

$$\delta(x) = \begin{cases} \infty & (x = 0) \\ 0 & \text{otherwise} \end{cases}$$

Integral over x results in a value equal to 1.



Kinetic theory: Track individual particles

$$N(\vec{x}, \vec{v}, t) = \delta[\vec{x} - \vec{X}_1(t)]\delta[\vec{v} - \vec{V}_1(t)]$$ (single particle)

Kinetic theory: Track individual particles

$$N(\vec{x},\vec{v},t) = \delta[\vec{x}-\vec{X}_1(t)]\delta[\vec{v}-\vec{V}_1(t)]$$ (single particle)

$$N(\vec{x},\vec{v},t) = \sum_{j=1}^{N_0} \delta[\vec{x}-\vec{X}_j(t)]\delta[\vec{v}-\vec{V}_j(t)]$$ (many particles)

Kinetic theory: Track individual particles

$$N(\vec{x}, \vec{v}, t) = \delta[\vec{x} - \vec{X}_1(t)]\delta[\vec{v} - \vec{V}_1(t)]$$

(single particle)

$$N(\vec{x}, \vec{v}, t) = \sum_{j=1}^{N_0} \delta[\vec{x} - \vec{X}_j(t)]\delta[\vec{v} - \vec{V}_j(t)]$$

(many particles)

$$N_s(\vec{x}, \vec{v}, t) = \sum_{j=1}^{N_0} \delta[\vec{x} - \vec{X}_{sj}(t)]\delta[\vec{v} - \vec{V}_{sj}(t)]$$

(many particles, species s)

Kinetic theory: Track individual particles

$$N(\vec{x},\vec{v},t) = \delta[\vec{x}-\vec{X}_1(t)]\delta[\vec{v}-\vec{V}_1(t)]$$

(single particle)

$$N(\vec{x},\vec{v},t) = \sum_{j=1}^{N_0} \delta[\vec{x}-\vec{X}_j(t)]\delta[\vec{v}-\vec{V}_j(t)]$$

(many particles)

$$N_s(\vec{x},\vec{v},t) = \sum_{j=1}^{N_0} \delta[\vec{x}-\vec{X}_{sj}(t)]\delta[\vec{v}-\vec{V}_{sj}(t)]$$

(many particles, species s)

$$N(\vec{x},\vec{v},t) = \sum_{s=e,i} N_s(\vec{x},\vec{v},t)$$

(many particles, electrons and ions)

Kinetic theory: Plasma evolution over time

Each particle is going to experience a force according to the Lorentz force law based on its location at a given time. Thus, we'll examine a time derivative:

$$\frac{d}{dt}N_s(\vec{x}, \vec{v}, t) = 0$$

Statement of particle incompressibility

Kinetic theory: Plasma evolution over time

Each particle is going to experience a force according to the Lorentz force law based on its location at a given time. Thus, we'll examine a time derivative:

$$\frac{d}{dt} N_s(\vec{x}, \vec{v}, t) = 0$$

Statement of particle incompressibility

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{d\vec{x}}{dt} \cdot \nabla_x + \frac{d\vec{v}}{dt} \cdot \nabla_v$$

Convective derivative

Kinetic theory: Plasma evolution over time

Each particle is going to experience a force according to the Lorentz force law based on its location at a given time. Thus, we'll examine a time derivative:

$$\frac{d}{dt} N_s(\vec{x}, \vec{v}, t) = 0$$

Statement of particle incompressibility

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{d\vec{x}}{dt} \cdot \nabla_x + \frac{d\vec{v}}{dt} \cdot \nabla_v$$

Convective derivative

resulting in

$$\frac{\partial N_s(\vec{x}, \vec{v}, t)}{\partial t} + \vec{v} \cdot \nabla_x N_s + \frac{q_s}{m_s} \left(\vec{E}^m + \vec{v} \times \vec{B}^m \right) \cdot \nabla_v N_s = 0$$

Kinetic theory: Klimontovich equation

$$\frac{\partial N_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x N_s + \frac{q_s}{m_s}\left(\vec{E}^m + \vec{v}\times\vec{B}^m\right)\cdot\nabla_v N_s = 0$$

$$\nabla\cdot\vec{E}^m(\vec{x},t) = \frac{\rho^m(\vec{x},t)}{\varepsilon_0}$$

$$\nabla\cdot\vec{B}^m(\vec{x},t) = 0$$

$$\nabla\times\vec{E}^m(\vec{x},t) = -\frac{\partial\vec{B}^m(\vec{x},t)}{\partial t}$$

$$\nabla\times\vec{B}^m(\vec{x},t) = \mu_0\vec{J}^m(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}^m(\vec{x},t)}{\partial t}$$

$$\rho^m(\vec{x},t) = \sum_{s=e,i} q_s\int N_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

$$\vec{J}^m(\vec{x},t) = \sum_{s=e,i} q_s\int \vec{v}N_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

Klimontovich + Maxwell’s equations provide an exact description of plasma and electromagnetic field evolution.

Mission accomplished!

Kinetic theory: Klimontovich equation

$$\frac{\partial N_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x N_s + \frac{q_s}{m_s}\left(\vec{E}^m + \vec{v}\times\vec{B}^m\right)\cdot\nabla_v N_s = 0$$

$$\nabla\cdot\vec{E}^m(\vec{x},t) = \frac{\rho^m(\vec{x},t)}{\varepsilon_0}$$

$$\nabla\cdot\vec{B}^m(\vec{x},t) = 0$$

$$\nabla\times\vec{E}^m(\vec{x},t) = -\frac{\partial\vec{B}^m(\vec{x},t)}{\partial t}$$

$$\nabla\times\vec{B}^m(\vec{x},t) = \mu_0\vec{J}^m(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}^m(\vec{x},t)}{\partial t}$$

$$\rho^m(\vec{x},t) = \sum_{s=e,i} q_s\int N_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

$$\vec{J}^m(\vec{x},t) = \sum_{s=e,i} q_s\int \vec{v}N_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

Klimontovich + Maxwell's equations provide an exact description of plasma and electromagnetic field evolution.

Mission accomplished?

$$n_i \sim 10^{20}\,\frac{\text{ions}}{m^3}$$

Kinetic theory: Klimontovich equation

$$\frac{\partial N_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x N_s + \frac{q_s}{m_s}\left(\vec{E}^m + \vec{v}\times\vec{B}^m\right)\cdot\nabla_v N_s = 0$$

$$\nabla\cdot\vec{E}^m(\vec{x},t) = \frac{\rho^m(\vec{x},t)}{\varepsilon_0}$$

$$\nabla\cdot\vec{B}^m(\vec{x},t) = 0$$

$$\nabla\times\vec{E}^m(\vec{x},t) = -\frac{\partial\vec{B}^m(\vec{x},t)}{\partial t}$$

$$\nabla\times\vec{B}^m(\vec{x},t) = \mu_0\vec{J}^m(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}^m(\vec{x},t)}{\partial t}$$

$$\rho^m(\vec{x},t) = \sum_{s=e,i} q_s\int N_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

$$\vec{J}^m(\vec{x},t) = \sum_{s=e,i} q_s\int \vec{v}N_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

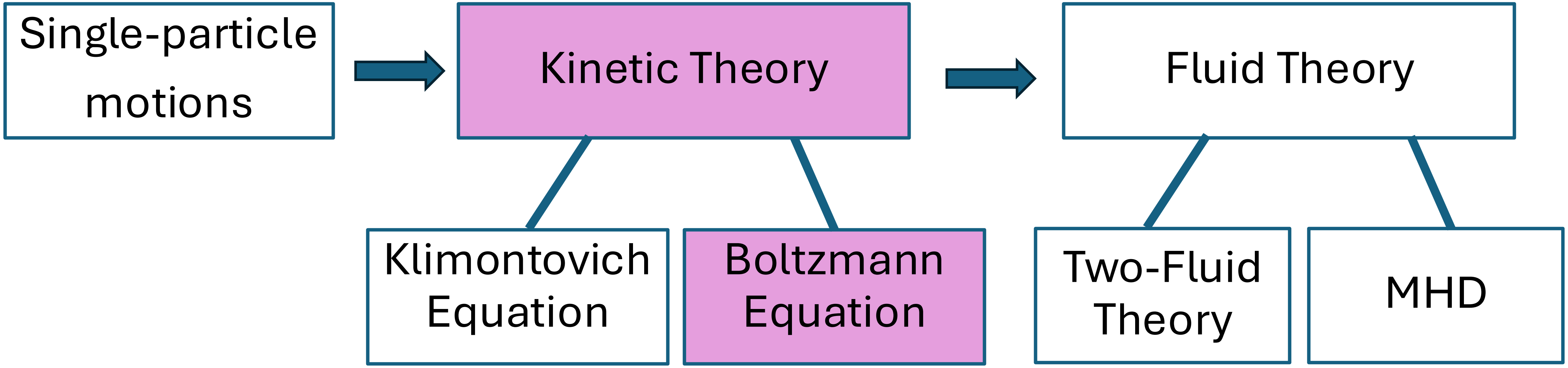
Klimontovich + Maxwell's equations provide an exact description of plasma and electromagnetic field evolution.

~~Mission accomplished~~?

$$n_i \sim 10^{20}\,\frac{\text{ions}}{m^3} \Rightarrow$$

Problem not tractable

Plasma models



Must we really track every individual particle?

Kinetic theory: Smoothing approximation

Let's instead try to track a smooth distribution function

$$f(\vec{x}, \vec{v}, t)\mathrm{d}\vec{x}\mathrm{d}\vec{v} = \#\text{ of particles in 6D phase-space volume }\mathrm{d}\vec{x}\ \mathrm{d}\vec{v}$$

Kinetic theory: Smoothing approximation

Let's instead try to track a smooth distribution function

$$f(\vec{x}, \vec{v}, t)\mathrm{d}\vec{x}\mathrm{d}\vec{v} = \#\text{ of particles in 6D phase-space volume d}\vec{x}\text{ d}\vec{v}$$

We can relate this to our work thus far with the following:

$$N_s(\vec{x}, \vec{v}, t) = f_s(\vec{x}, \vec{v}, t) + \delta N_s(\vec{x}, \vec{v}, t)$$

Kinetic theory: Smoothing approximation

Let's instead try to track a smooth distribution function

$$f(\vec{x},\vec{v},t)\mathrm{d}\vec{x}\mathrm{d}\vec{v} = \#\ \text{of particles in 6D phase-space volume}\ \mathrm{d}\vec{x}\ \mathrm{d}\vec{v}$$

We can relate this to our work thus far with the following:

$$\begin{aligned} N_s(\vec{x},\vec{v},t) &= f_s(\vec{x},\vec{v},t) + \delta N_s(\vec{x},\vec{v},t) \\ \vec{E}^m(\vec{x},\vec{v},t) &= \vec{E}(\vec{x},\vec{v},t) + \delta\vec{E}(\vec{x},\vec{v},t) \\ \vec{B}^m(\vec{x},\vec{v},t) &= \vec{B}(\vec{x},\vec{v},t) + \delta\vec{B}(\vec{x},\vec{v},t) \end{aligned}$$

Plug into Boltzmann equation

Kinetic theory: Boltzmann equation

$$\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\nabla_v f_s = -\frac{q_s}{m_s}\langle(\delta\vec{E} + \vec{v}\times\delta\vec{B})\cdot\nabla_v\delta N_s\rangle$$

Kinetic theory: Boltzmann equation

$$\underbrace{\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s}_{\text{Collective behaviors}} = -\frac{q_s}{m_s}\langle(\delta\vec{E}+\vec{v}\times\delta\vec{B})\cdot\nabla_v\delta N_s\rangle$$

Kinetic theory: Boltzmann equation

$$\underbrace{\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s}_{\text{Collective behaviors}} = \underbrace{-\frac{q_s}{m_s}\langle(\delta\vec{E}+\vec{v}\times\delta\vec{B})\cdot\nabla_v \delta N_s\rangle}_{\substack{\text{Collisions} \\ \text{(Discrete particle nature)}}}$$

Kinetic theory: Boltzmann equation

$$\underbrace{\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s}_{\text{Collective behaviors}} = \underbrace{-\frac{q_s}{m_s}\langle(\delta\vec{E}+\vec{v}\times\delta\vec{B})\cdot\nabla_v\delta N_s\rangle}_{\text{Collisions (Discrete particle nature)}}$$

Still too much info, as we need to track individual particles to precisely compute the RHS.

Kinetic theory: Boltzmann equation

$$\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\nabla_v f_s = \left(\frac{\partial f_s}{\partial t}\right)_c$$

$$\nabla\cdot\vec{E}(\vec{x},t) = \frac{\rho(\vec{x},t)}{\varepsilon_0}$$

$$\nabla\cdot\vec{B}(\vec{x},t) = 0$$

$$\nabla\times\vec{E}(\vec{x},t) = -\frac{\partial\vec{B}(\vec{x},t)}{\partial t}$$

$$\nabla\times\vec{B}(\vec{x},t) = \mu_0\vec{J}(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}(\vec{x},t)}{\partial t}$$

Approximations are made to the collisional term based on the specific problem being addressed.

Vlasov

Kinetic theory: ~~Boltzmann~~ equation

$$\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s = \cancelto{0}{\left(\frac{\partial f_s}{\partial t}\right)_c}$$

$$\nabla\cdot\vec{E}(\vec{x},t) = \frac{\rho(\vec{x},t)}{\varepsilon_0}$$

$$\nabla\cdot\vec{B}(\vec{x},t) = 0$$

$$\nabla\times\vec{E}(\vec{x},t) = -\frac{\partial\vec{B}(\vec{x},t)}{\partial t}$$

$$\nabla\times\vec{B}(\vec{x},t) = \mu_0\vec{J}(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}(\vec{x},t)}{\partial t}$$

Approximations are made to the collisional term based on the specific problem being addressed.

Kinetic theory: Boltzmann equation

$$\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s = \left(\frac{\partial f_s}{\partial t}\right)_c$$

$$\nabla\cdot\vec{E}(\vec{x},t) = \frac{\rho(\vec{x},t)}{\varepsilon_0}$$

$$\nabla\cdot\vec{B}(\vec{x},t) = 0$$

$$\nabla\times\vec{E}(\vec{x},t) = -\frac{\partial\vec{B}(\vec{x},t)}{\partial t}$$

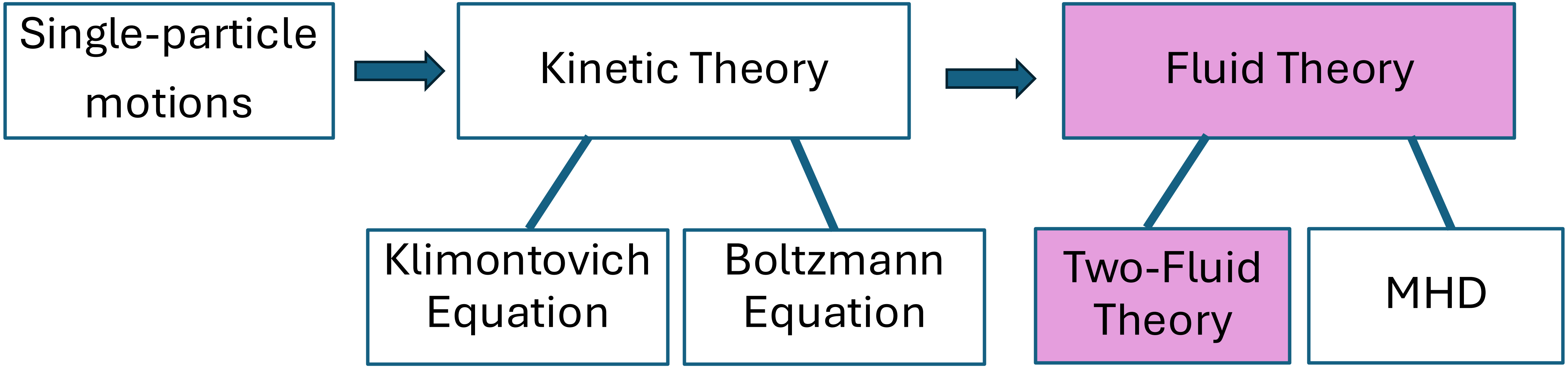
$$\nabla\times\vec{B}(\vec{x},t) = \mu_0\vec{J}(\vec{x},t) + \mu_0\varepsilon_0\frac{\partial\vec{E}(\vec{x},t)}{\partial t}$$

Approximations are made to the collisional term based on the specific problem being addressed.

Still challenging, both analytically and computationally.

Simplify further.

Plasma models



Fluid Theory: What's different?

- Kinetic theory:
  - Klimontovich: Examine evolution of individual particle trajectories in 6D phase space and evolution of electromagnetic fields.
  - Boltzmann: Examine evolution of smoothed particle densities in 6D phase space and evolution of electromagnetic fields.
- Fluid theory:
  - Examine evolution of fluid elements characterized by macroscopic properties such as density, temperature, pressure in 3D configuration space

Fluid Theory: Quantities characterizing fluids

In our fluid theories, we will focus on macroscopic quantities of fluid elements:

$$n_s(\vec{x}, t) = \int f_s(\vec{x}, \vec{v}, t) \mathrm{d}\vec{v}$$

$$\vec{V}_s(\vec{x}, t) = \frac{1}{n_s} \int \vec{v} f_s(\vec{x}, \vec{v}, t) \mathrm{d}\vec{v}$$

$$\overleftrightarrow{P} = m_s \int \vec{v}\vec{v} f(\vec{x}, \vec{v}, t) \mathrm{d}\vec{v}$$

Fluid Theory: Limitations

If fluid quantities are to adequately describe plasma properties, we must have nearby particles remain near to their neighbors.

Fluid Theory: Limitations

If fluid quantities are to adequately describe plasma properties, we must have nearby particles remain near to their neighbors.

- High collisionality

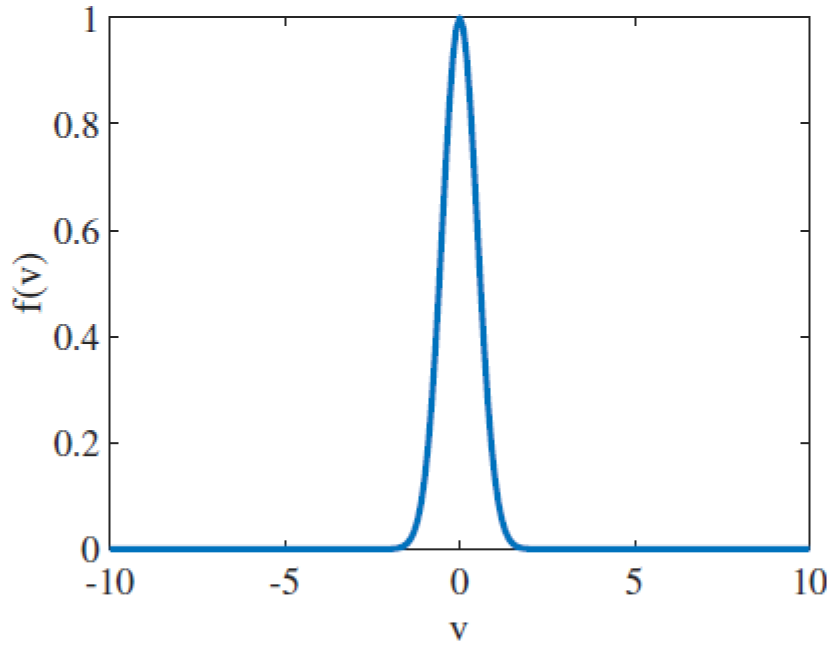
This is what allows neutral fluids to satisfy the above requirement

Fluid Theory: Limitations

If fluid quantities are to adequately describe plasma properties, we must have nearby particles remain near to their neighbors.

- High collisionality
- Low temperature

Little spread in particle velocities means nearby particles will remain nearby as they move.

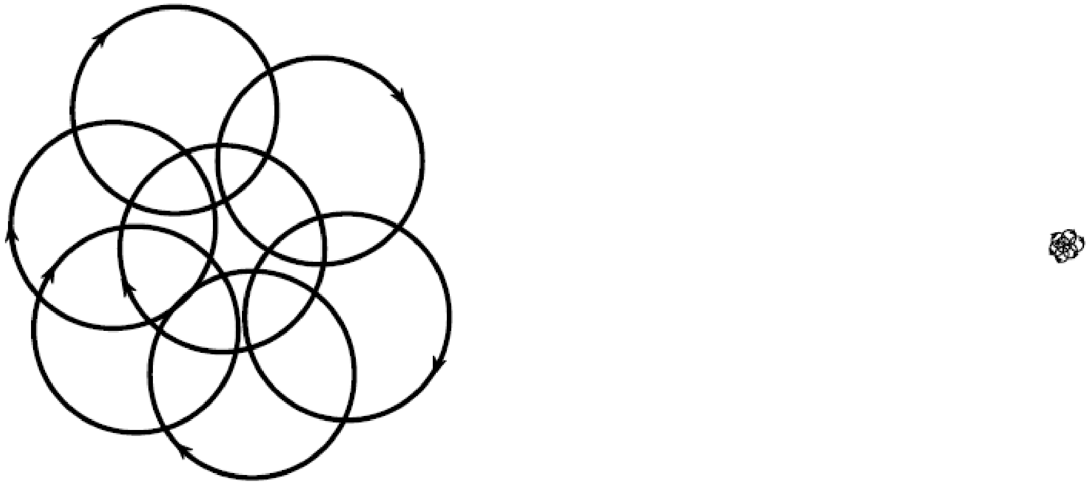


Fluid Theory: Limitations

If fluid quantities are to adequately describe plasma properties, we must have nearby particles remain near to their neighbors.

- High collisionality
- Low temperature
- Strong magnetic field

In fusion plasmas, fluid theories tend to work best when describing motion perpendicular to the magnetic field. Kinetic treatment may still be required in parallel direction.



Weak magnetic field Strong magnetic field

Fluid theory: Moment equations of Boltzmann

$$\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s = \left(\frac{\partial f_s}{\partial t}\right)_c$$

Use moments to find equations describing evolution of fluid quantities:

$$n_s(\vec{x},t) = \int f_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

$$\vec{V}_s(\vec{x},t) = \frac{1}{n_s}\int \vec{v} f_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

$$\overleftrightarrow{P} = m_s \int \vec{v}\vec{v} f(\vec{x},\vec{v},t)\mathrm{d}\vec{v}$$

Fluid theory: Moment equations of Boltzmann

$$\frac{\partial f_s(\vec{x},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla_x f_s + \frac{q_s}{m_s}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\nabla_v f_s = \left(\frac{\partial f_s}{\partial t}\right)_c$$

Use moments to find equations describing evolution of fluid quantities:

$$n_s(\vec{x},t) = \int f_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v} \qquad \text{0th moment:} \quad \int [\text{Boltzmann}]\mathrm{d}\vec{v}$$

$$\vec{V}_s(\vec{x},t) = \frac{1}{n_s}\int \vec{v} f_s(\vec{x},\vec{v},t)\mathrm{d}\vec{v} \qquad \text{1st moment:} \quad \int m_s\vec{v}\,[\text{Boltzmann}]\mathrm{d}\vec{v}$$

$$\overleftrightarrow{P} = m_s\int \vec{v}\vec{v} f(\vec{x},\vec{v},t)\mathrm{d}\vec{v} \qquad \text{2nd moment:} \quad \int \frac{1}{2}m_s\vec{v}\vec{v}\,[\text{Boltzmann}]\mathrm{d}\vec{v}$$

Fluid theory: Zeroth-order moment

$$\underbrace{\frac{\partial n_s(\vec{x},t)}{\partial t}}_{\text{Rate of density change}} + \underbrace{\nabla_x \cdot (n_s \vec{V}_s)}_{\text{Flux into and out of fluid element}} = \int \left(\frac{\partial f_s}{\partial t} \right)_c \mathrm{d}\vec{v}$$

- This is essentially a conservation law: Conservation of particles.

Fluid theory: Zeroth-order moment

$$\underbrace{\frac{\partial n_s(\vec{x},t)}{\partial t}}_{\text{Rate of density change}} + \underbrace{\nabla_x \cdot (n_s \vec{V}_s)}_{\text{Flux into and out of fluid element}} = \cancelto{0}{\int \left(\frac{\partial f_s}{\partial t}\right)_c} \mathrm{d}\vec{v}$$

- This is essentially a conservation law: Conservation of particles.
- Collisions tend to change individual particle velocities but do not greatly change a particle’s position.

Fluid theory: Zeroth-order moment

$$\underbrace{\frac{\partial n_s(\vec{x},t)}{\partial t}}_{\text{Rate of density change}} + \underbrace{\nabla_x \cdot (n_s \vec{V}_s)}_{\text{Flux into and out of fluid element}} = \cancelto{0}{\int \left(\frac{\partial f_s}{\partial t}\right)_c \mathrm{d}\vec{v}}$$

- This is essentially a conservation law: Conservation of particles.
- Collisions tend to change individual particle velocities but do not greatly change a particle's position.
- Multiply by species mass or species charge to get statements for mass conservation and charge conservation.

Fluid theory: First-order moment

$$m_s n_s \frac{\partial \vec{V}_s}{\partial t} + m_s n_s \vec{V}_s \cdot \nabla_x \vec{V}_s = -\nabla P_s + q_s n_s(\vec{E} + \vec{V}_s \times \vec{B}) + \vec{K}_s(\vec{x}, t)$$

Fluid theory: First-order moment

$$\underbrace{m_s n_s \frac{\partial \vec{V}_s}{\partial t} + m_s n_s \vec{V}_s \cdot \nabla_x \vec{V}_s}_{\text{Convective derivative of momentum density}} = -\nabla P_s + q_s n_s (\vec{E} + \vec{V}_s \times \vec{B}) + \vec{K}_s(\vec{x}, t)$$

Momentum changes due to:

Fluid theory: First-order moment

$$\underbrace{m_s n_s \frac{\partial \vec{V}_s}{\partial t} + m_s n_s \vec{V}_s \cdot \nabla_x \vec{V}_s}_{\text{Convective derivative of momentum density}} = -\nabla P_s + \underbrace{q_s n_s(\vec{E} + \vec{V}_s \times \vec{B})}_{\text{Lorentz force term}} + \vec{K}_s(\vec{x}, t)$$

Momentum changes due to:

- Lorentz force acting on our fluid element

Fluid theory: First-order moment

$$\underbrace{m_s n_s \frac{\partial \vec{V}_s}{\partial t} + m_s n_s \vec{V}_s \cdot \nabla_x \vec{V}_s}_{\text{Convective derivative of momentum density}} = \underbrace{-\nabla P_s}_{\text{Pressure gradient term}} + \underbrace{q_s n_s (\vec{E} + \vec{V}_s \times \vec{B})}_{\text{Lorentz force term}} + \vec{K}_s(\vec{x}, t)$$

Momentum changes due to:

- a force acting on our fluid element
- imbalance in surrounding pressures (e.g. buoyancy)

Fluid theory: First-order moment

$$\underbrace{m_s n_s \frac{\partial \vec{V}_s}{\partial t} + m_s n_s \vec{V}_s \cdot \nabla_x \vec{V}_s}_{\text{Convective derivative of momentum density}} = \underbrace{-\nabla P_s}_{\text{Pressure gradient term}} + \underbrace{q_s n_s (\vec{E} + \vec{V}_s \times \vec{B})}_{\text{Lorentz force term}} + \underbrace{\vec{K}_s(\vec{x}, t)}_{\text{Collisional effects}}$$

Momentum changes due to:

- a force acting on our fluid element
- imbalance in surrounding pressures (e.g. buoyancy)
- particle collisions
  - Note: a species cannot change its own momentum through collisions. But, collisions with different species can change momentum. So,

$$\vec{K}_e(\vec{x}, t) = -\vec{K}_i(\vec{x}, t)$$

Fluid theory: Closure

Each moment equation depends on the next moment. Eventually, an assumption must be made to truncate the infinite series of moments.

Today, we will assume an isotropic distribution function which leads to:

$$P_s(\vec{x},t) \equiv n_s(\vec{x},t)T_s(\vec{x},t)$$

Fluid theory: Two-fluid model

$$\frac{\partial n_s(\vec{x},t)}{\partial t} + \nabla_x \cdot (n_s \vec{V}_s) = 0$$

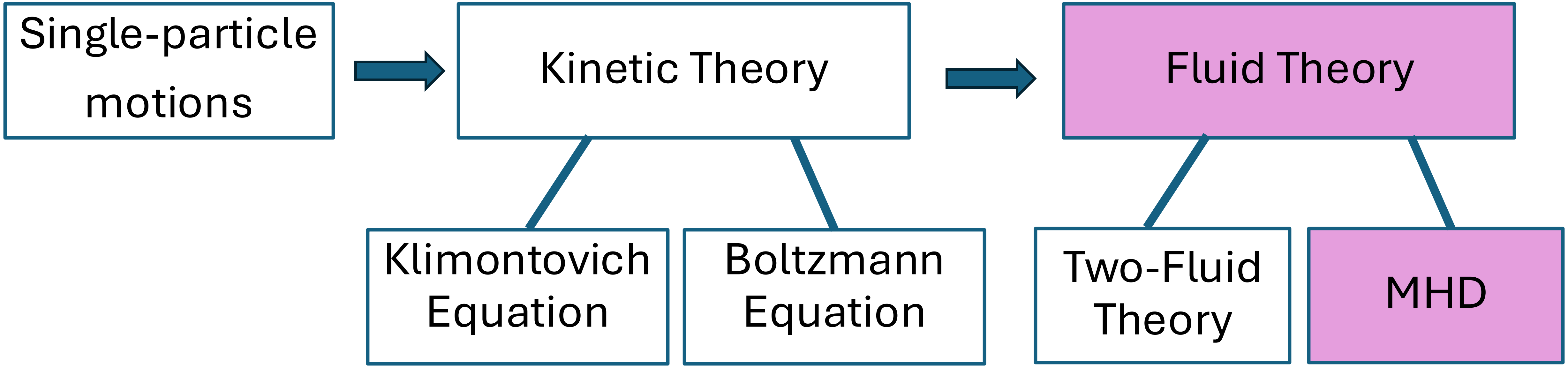
$$m_s n_s \frac{\partial \vec{V}_s}{\partial t} + m_s n_s \vec{V}_s \cdot \nabla_x \vec{V}_s = -\nabla P_s + q_s n_s (\vec{E} + \vec{V}_s \times \vec{B}) + \vec{K}_s(\vec{x},t)$$

+ Maxwell equations

This represents the equations governing interacting electron and ion fluids passing through each other.

This can still be complicated when dealing with realistic geometric effects. Let's simplify down to a one-fluid picture.

Plasma models



Fluid theory: Quantities describing single fluid

To simplify further, we'll treat the plasma as a single fluid and examine the evolution of joint quantities such as

$$\rho_M(\vec{x}) \equiv m_e n_e(\vec{x}) + m_i n_i(\vec{x}) \approx m_i n_i(\vec{x})$$

$$\rho_c(\vec{x}) \equiv q_e n_e(\vec{x}) + q_i n_i(\vec{x}) = e(n_i - n_e)$$

$$\vec{V} \equiv \frac{1}{\rho_M}(m_i n_i \vec{V}_i + m_e n_e \vec{V}_e)$$

$$\vec{J} \equiv q_i n_i \vec{V}_i + q_e n_e \vec{V}_e$$

$$P = P_e + P_i$$

Fluid theory: MHD equations

By combining electron and ion moment equations together and remembering definitions for one-fluid quantities, we find:

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot (\rho_M \vec{V}) = 0$$

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J} = 0$$

+ Maxwell's equations

$$\rho_M \frac{\partial \vec{V}}{\partial t} = -\nabla P + \rho_c \vec{E} + \vec{J} \times \vec{B}$$

Generalized Ohm's Law

$$\frac{m_e m_i}{\rho_M e^2} \frac{\partial \vec{J}}{\partial t} = \frac{m_i}{2\rho_M e} \nabla P + \vec{E} + \vec{V} \times \vec{B} - \frac{m_i}{\rho_M e} \vec{J} \times \vec{B} - \frac{1}{\sigma} \vec{J}$$

Fluid theory: MHD equations

By combining electron and ion moment equations together and remembering definitions for one-fluid quantities, we find:

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot (\rho_M \vec{V}) = 0$$

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J} = 0$$

$$\rho_M \frac{\partial \vec{V}}{\partial t} = -\nabla P + \rho_c \vec{E} + \vec{J} \times \vec{B}$$

$$\frac{m_e m_i}{\rho_M e^2} \frac{\partial \vec{J}}{\partial t} = \frac{m_i}{2\rho_M e} \nabla P + \vec{E} + \vec{V} \times \vec{B} - \frac{m_i}{\rho_M e} \vec{J} \times \vec{B} - \frac{1}{\sigma} \vec{J}$$

14 equations, 14 unknowns

+ Maxwell's equations

Generalized Ohm's Law

Fluid theory: Ideal MHD equations

If we assume high conductivity and low frequency, we can simplify a little bit more. We call this Ideal MHD.

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot (\rho_M \vec{V}) = 0$$

$$\rho_M \frac{\partial \vec{V}}{\partial t} = -\nabla P + \vec{J} \times \vec{B}$$

$$\nabla \times (\vec{V} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Fluid theory: Ideal MHD equations

If we assume high conductivity and low frequency, we can simplify a little bit more. We call this Ideal MHD.

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot (\rho_M \vec{V}) = 0$$

$$\rho_M \frac{\partial \vec{V}}{\partial t} = -\nabla P + \vec{J} \times \vec{B}$$

$$\nabla \times (\vec{V} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

10 equations,
10 unknowns

Fluid theory: Ideal MHD successes

- Macroscopic equilibrium and stability of plasmas
  - Design of tokamaks, stellarators, mirrors, RFPs.
    - Drifts not present in single-particle picture
      - e.g. The diamagnetic drift perpendicular to pressure gradients
    - Investigations of instabilities in fusion plasmas
      - Kink modes, ballooning modes, Tearing instabilities, edge localized modes
- MHD waves
  - Alfvén, fast magnetosonic, and slow magnetosonic waves
- Magnetic convection and dissipation
  - Solar dynamo, Parker spiral, magnetic reconnection
- Plasma turbulent cascades

Key takeaways

- Each method can provide valuable insight, allowing us to build intuition and to interpret results.
- Insights can be amplified by using multiple models.

	Magnetohydrodynamics	Two Fluids	Gyrokinetics	Kinetics	Everything
Description	The plasma is one continuous fluid - ions have all the mass, but electron carry all the current.	Break the ions & electrons into two continuous, mingling fluids.	Only track superparticles' straight motion - and ignore the corkscrewing.	Assign particles a speed and location based on a distribution. Track super particles through space.	Track every particle, at all times.
Strengths	Easily solved.	Simple bulk effects like drift waves & reconnection can be understood.	Captures most of kinetic model, but much easier to solve - can model an entire Tokamak.	Many things captured, can get powerful results like the linear velocity-space instabilities.	Most accurate model possible.
Weakness	Most things not captured: most plasma waves, leakage, kinetic instabilities, structures etc.	Many things not captured: plasma instabilities, large effects & non-equilibrium effects. Assumes bell curves.	Non-physical behavior over long times: resonances & adiabatic invariants can be lost.	Tough to solve: hard to apply to full-size reactors. Loses some effects: like plasma microdensity and collective thomson scattering.	Typically impossible to solve.
Mathematics	Navier-stokes, Lorentz force, Maxwells' equations.	Navier-stokes, Lorentz force, Maxwells' equations.	Vlasov-Maxwell Expansion Equation	Vlasov-Maxwell Equation	Klimontovich Model

Plasma as a fluid (Chalkboard)

Plasma as a gas (Computer Required)

Simplicity

Detial

Taken from J. Williams presentation, Plasma/Fusion summer school, 2022

Any questions?