Single Particle Motion

SULI Introductory Course 2025

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My path through plasma physics began with SULI!

1995 – 2013

Grew up in Oklahoma, Texas, Louisiana & Ohio

2013 – 2017

Undergrad at ‘Bama

Studied physics & math

2017 – 2018

Fulbright scholar at the Max Planck Institute for Plasma Physics

2018 – 2024

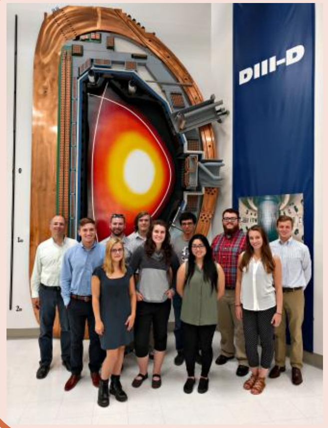
PhD student in Plasma Physics at Princeton University

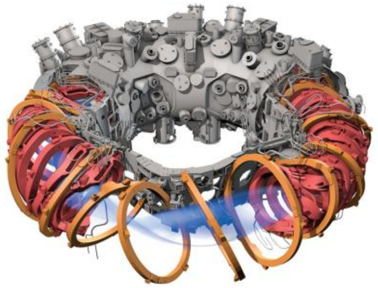
2024 – now

Scientist at CFS working on fast ions and burning plasmas in SPARC and ARC

Summer 2016

SULI student at DIII-D!



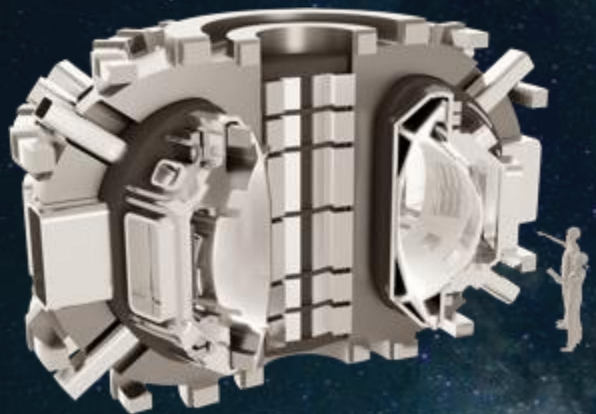


CFS will advance fusion energy with concrete milestones

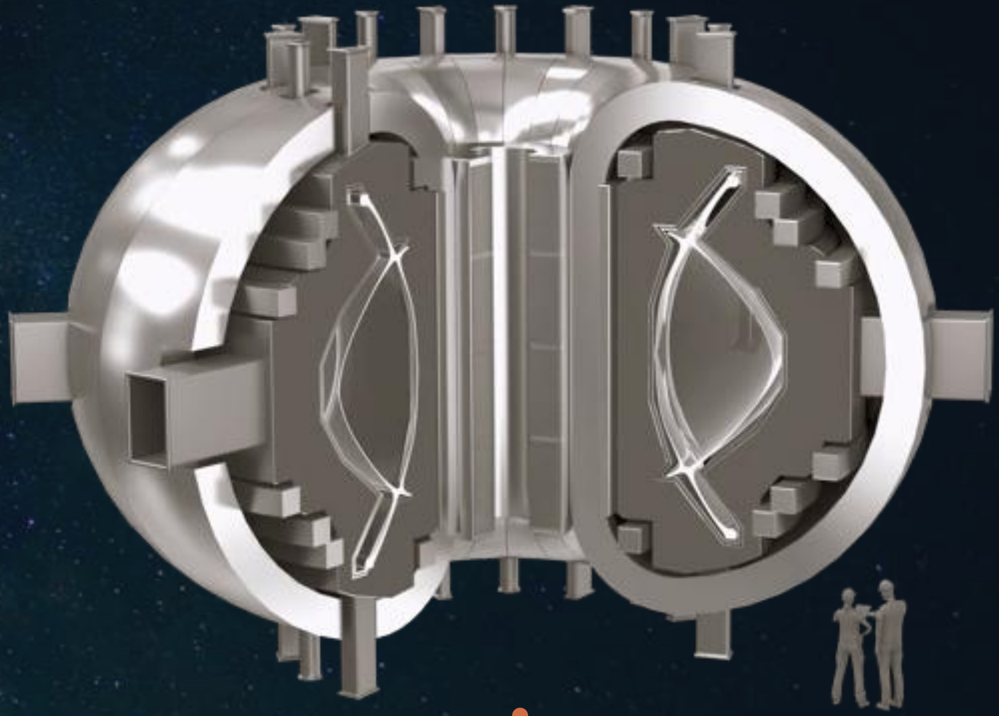
R&D — Commercial demo — Commercial powerplant

Grounded in established tokamak physics

Magnet tech
COMPLETED



SPARC
UNDER CONSTRUCTION

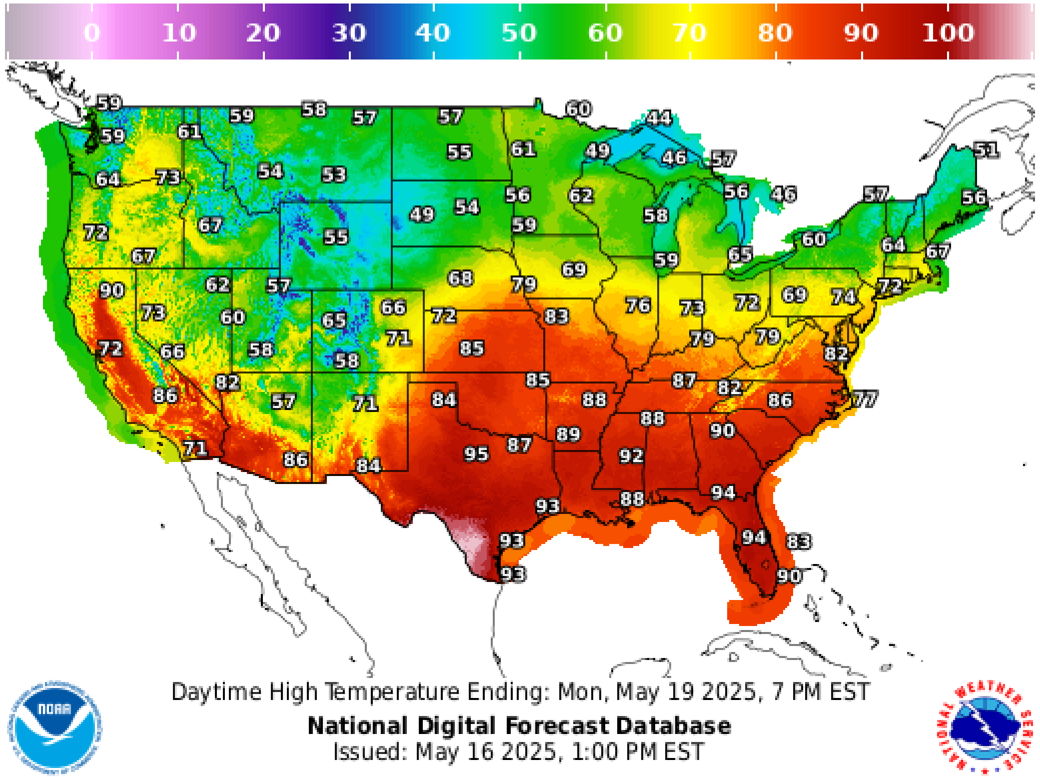


ARC
EARLY 2030s

What is a vector field?

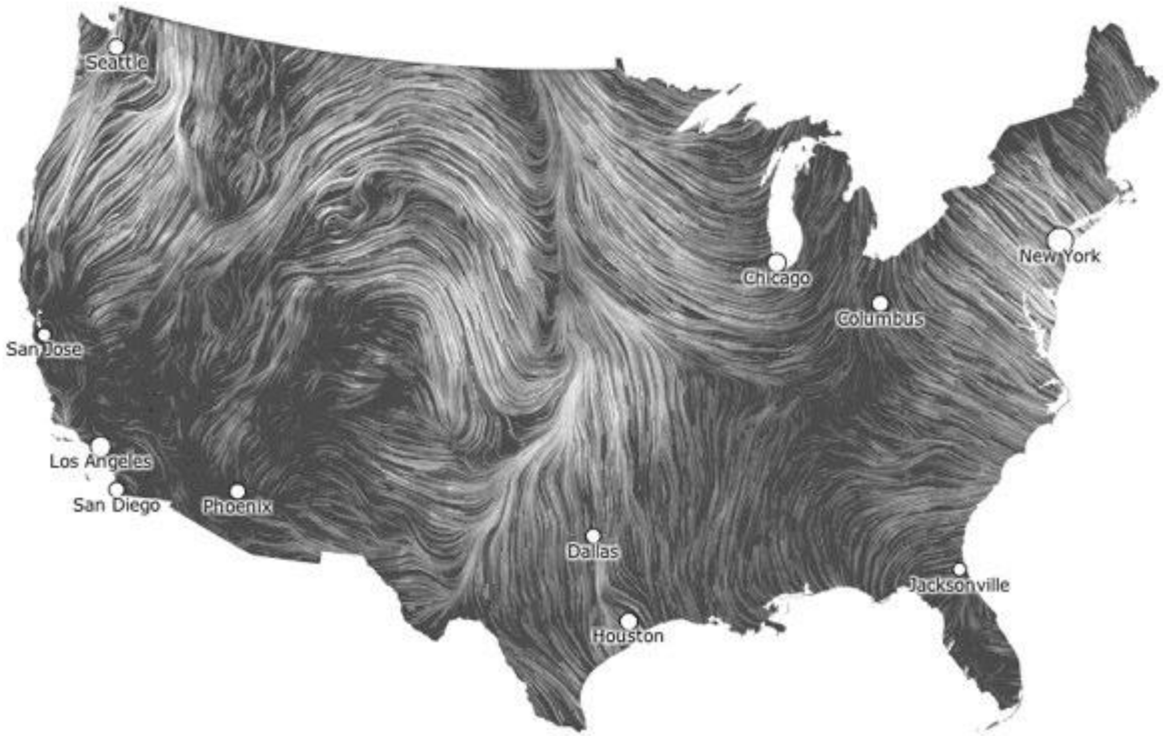
- Scalar field: has a **magnitude** at each point in space

Example: Temperature



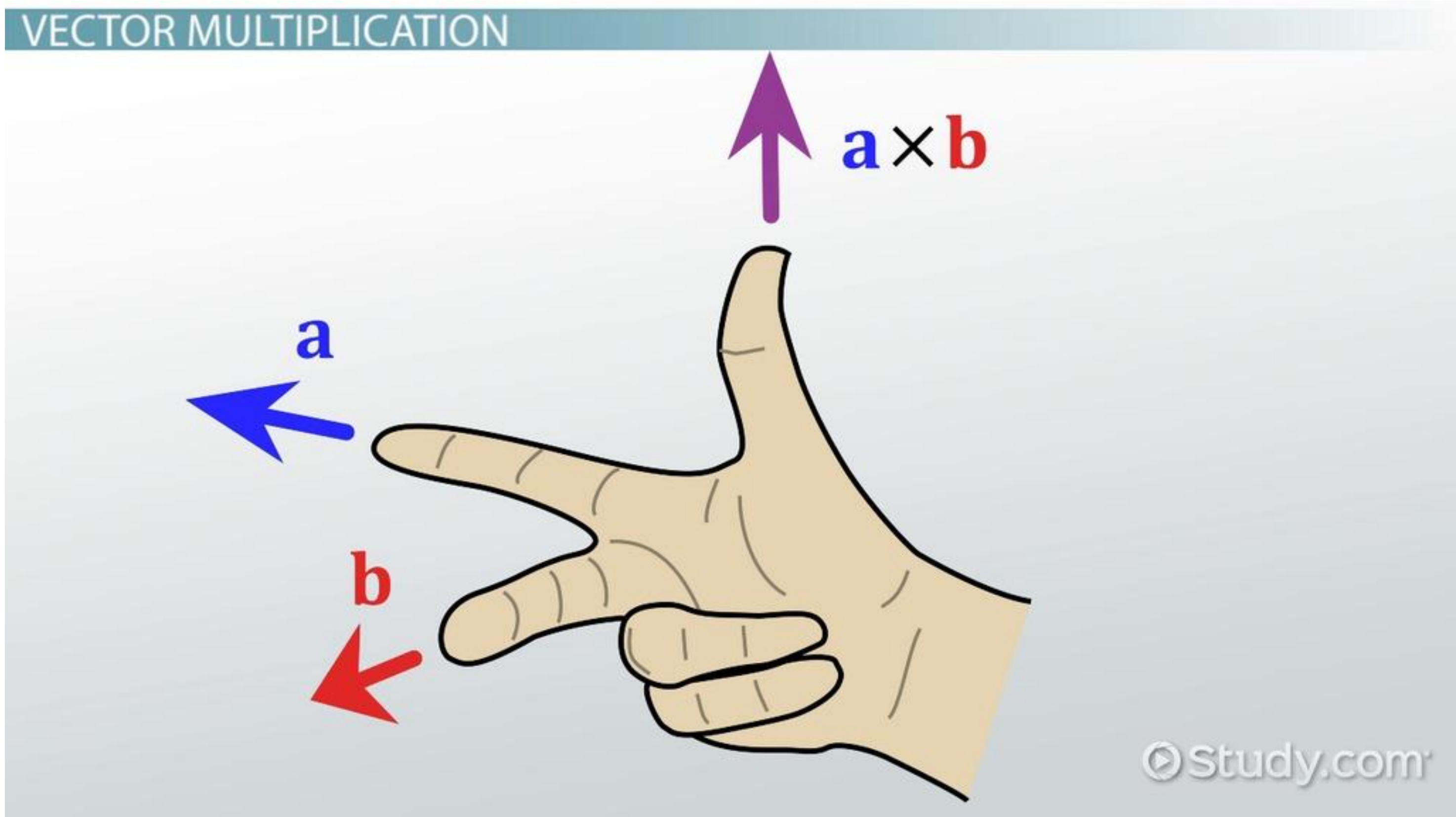
- Vector field: has a **magnitude** and **direction** at each point in space

Example: Wind Velocity



Right-hand rule

- When taking a *cross product*, always remember the right-hand rule!

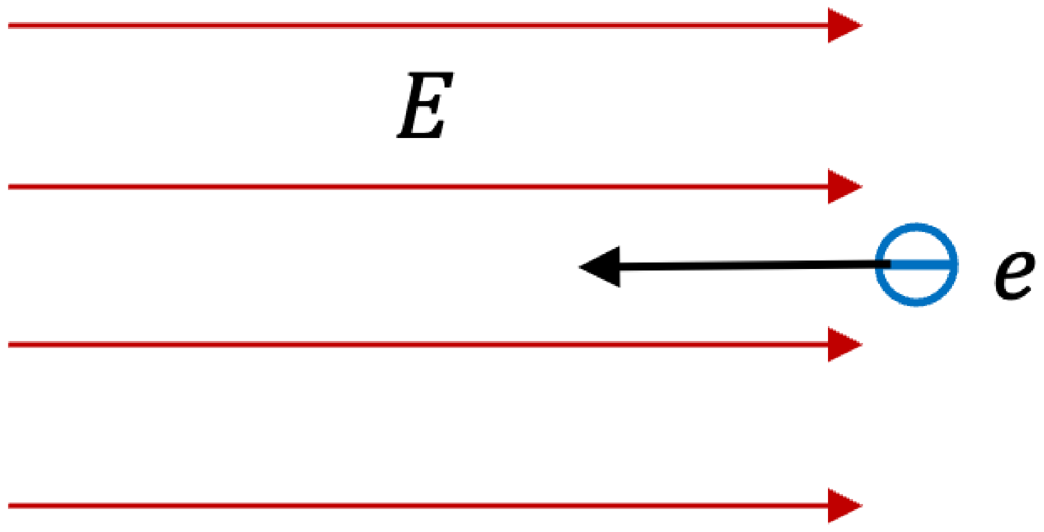


Basics of magnetic confinement: the Lorentz force

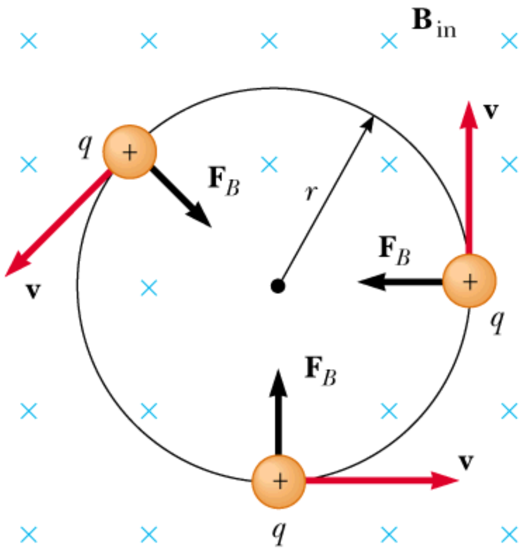
- Motion of charged particles in electromagnetic fields is defined by the **Lorentz force**:

$$\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$$

In an electric field: force is **parallel** to the field



In a magnetic field: force is **perpendicular** to the field (*and* the particle velocity!)



What do we know about the Lorentz force?

- Simplest case:
  - Straight, uniform $\vec{B}$
  - $\vec{E} = 0$

$$\vec{F} \cdot \vec{B} = 0 \longrightarrow \frac{dv_{\parallel}}{dt} = 0$$

$$\frac{d\vec{v}_{\perp}}{dt} = ??$$

Effect on the perpendicular velocity

- Take the time derivative of the Lorentz force…

$$\frac{d^2\vec{v}}{dt^2} = \frac{d^2\vec{v}_\perp}{dt^2} = \frac{q}{m}\left(\frac{d\vec{v}}{dt} \times \vec{B}\right) = \left(\frac{q}{m}\right)^2 \left(\left(\vec{v} \times \vec{B}\right) \times \vec{B}\right)$$

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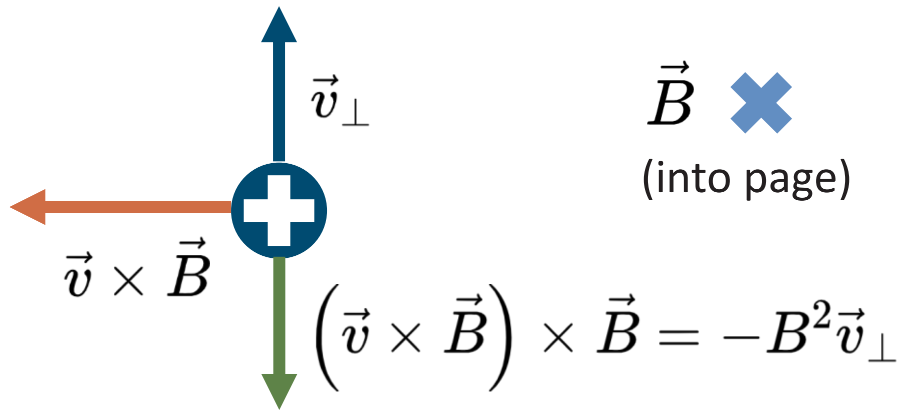
and use right-hand rule twice to find the direction…



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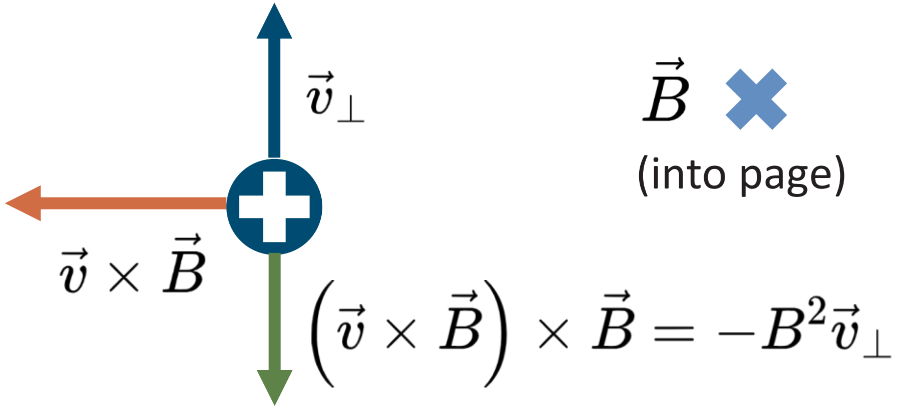


and we get: $$\frac{d^2\vec{v}_\perp}{dt^2} = -\left(\frac{qB}{m}\right)^2 \vec{v}_\perp$$

It's just a simple harmonic oscillator!

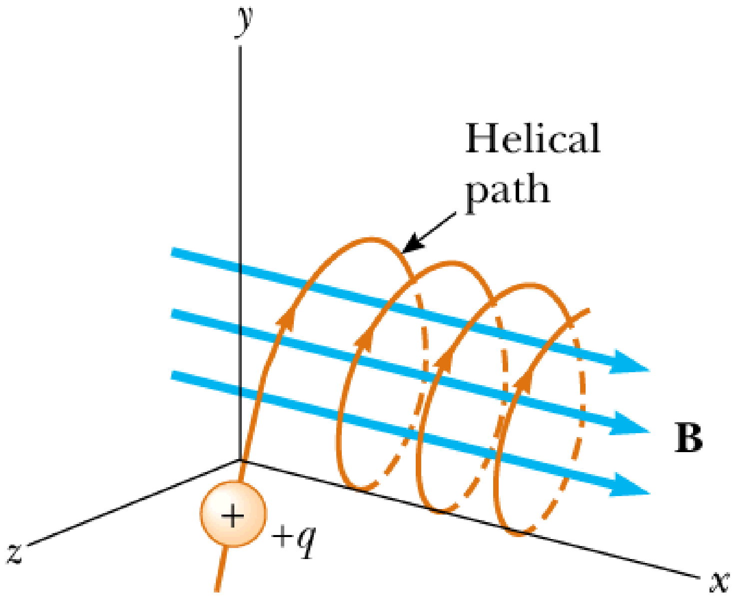
$$\frac{d^2\vec{v}_\perp}{dt^2} = -\omega_c^2\vec{v}_\perp \qquad \omega_c = \frac{qB}{m}$$

- So: the particle is rotating around the magnetic field with frequency ω_c!

$$\frac{d|v_\perp|}{dt} = 0, \ \frac{d\vec{v}_\perp}{dt} \neq 0$$

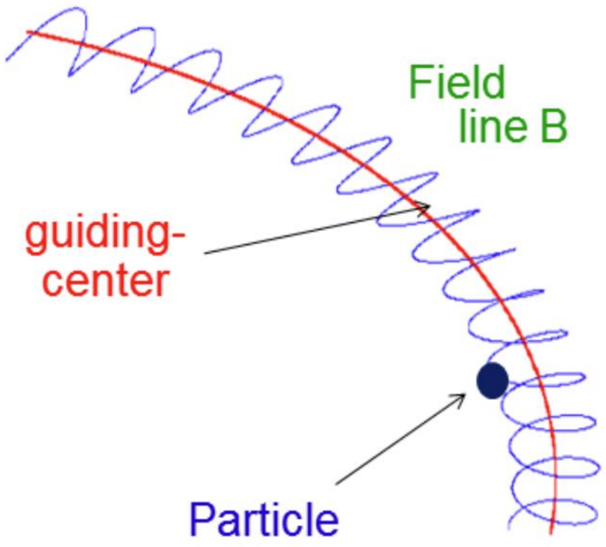
- Direction of rotation depends on charge q
- Radius of gyration:

$$\rho = \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}$$



Guiding center approximation

- Can separate two parts of the motion:
  - Straight motion along $\vec{B}$
  - Rotation around $\vec{B}$ (“gyroorbit”)
- Average over the orbit to get “guiding center motion”



Guiding center approximation: limitations

Q: What happens if $\vec{B}$ isn't straight or uniform?

Guiding center approximation: limitations

Q: What happens if $\vec{B}$ isn't straight or uniform?

A: We can *still* use the guiding center approximation if a few conditions are met:

$$\rho \ll B/\nabla B \qquad \omega_c \gg \frac{\partial B}{\partial t}\frac{1}{B}$$

Is this approximation valid in SPARC?

$B \approx 12\ \mathrm{T}$

$T \approx 20\ \mathrm{keV}$

- Assume $v_\perp = v_\parallel$

Hydrogen ion

$$f = \frac{\omega_c}{2\pi} = 182\ \mathrm{MHz}$$
$$\rho = 1.2\ \mathrm{mm}$$

Electron

$$f = 336\ \mathrm{GHz}$$
$$\rho = 28\ \mu\mathrm{m}$$

Particle drifts

- We never really have a perfectly straight, uniform magnetic field with zero other forces
- Add small perturbations, get *drifts*: $v_\perp = \omega_c \rho + v_D \ \ (v_D \ll \omega_c \rho)$

Particle drifts: arbitrary constant force

Q: What happens when you add an arbitrary, constant force to the picture?

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A: If $\vec{F} \parallel \vec{B}$, the equations of motion can be decoupled!

$$\frac{dv_{\|}}{dt} = \frac{F}{m}, \quad \frac{d\vec{v}_{\perp}}{dt} = \frac{q}{m}\left(\vec{v} \times \vec{B}\right)$$

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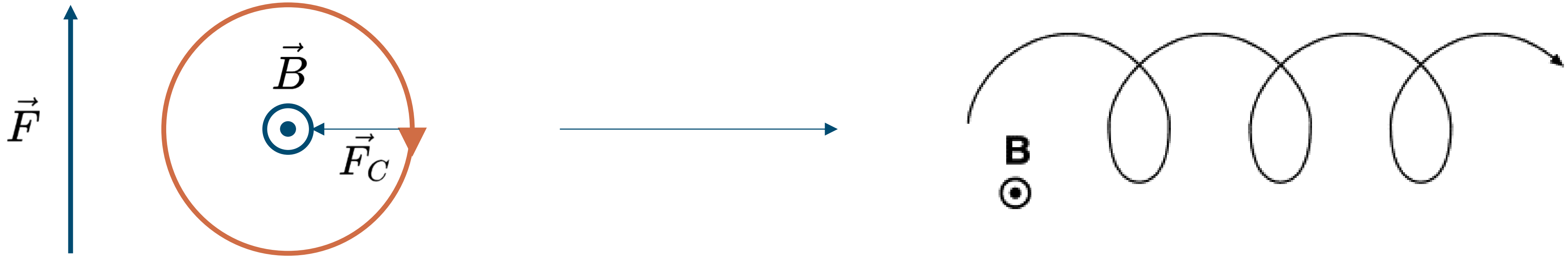
But what if $\vec{F} \perp \vec{B}$?

Particle drifts: arbitrary constant force

- Zeroth order: helical orbit with frequency ω_c and radius ρ



- Bottom of orbit: $F_C = qv_\perp B + F$
- Top of orbit: $F_C = qv_\perp B - F$
- $\rho \propto F_C^{-1}$
- So: the orbit alternatively compresses and expands, leading to a drift!
  - Opposite directions for ions and electrons

Particle drifts: arbitrary constant force

- We now have a different differential equation: $m\frac{d\vec{v}}{dt} = q\left(\vec{v} \times \vec{B}\right) + \vec{F}$

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- Let $v = v_{\parallel} + v_{gyro} + v_{drift}$, assume that v_{drift} is small and constant

$$m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}_g}{dt} + m\frac{d\vec{v}_D}{dt} = q\left(\vec{v}_g \times \vec{B} + \vec{v}_D \times \vec{B}\right) + \vec{F}$$

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$$\vec{F} = -q(\vec{v}_D \times \vec{B})$$

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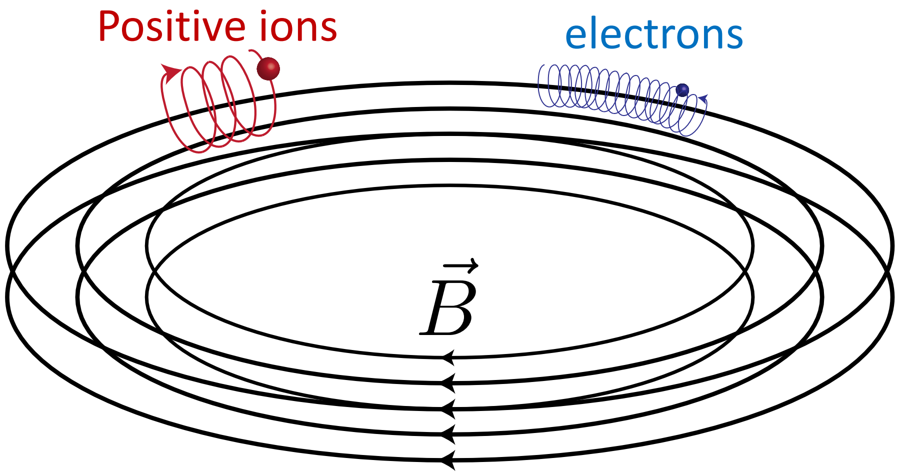
$$\boxed{\vec{v}_D = \frac{\vec{F} \times \vec{B}}{qB^2}}$$

What happens when the field isn't straight?

- You could confine a plasma in an infinitely long magnetic field, but we can't build that!
- To confine along $\vec{B}$, need a *toroidal* field
- Now we've got *two* orbital motions:
  - Particle around the guiding center
  - Guiding center around the center of the torus



“Force” associated with magnetic field curvature

- The guiding center frame rotates with the guiding center as it circles the torus
- In this rotating reference frame, the particle experiences a *centrifugal* force
  - This force isn’t “real”; in an inertial reference frame, it disappears

Curvature drift

- The centrifugal force is $\vec{F} = \frac{mv_{\parallel}^2}{R_C}\hat{r}$
- This gives us a drift velocity of:

$$\vec{v}_D = mv_{\parallel}^2 \frac{\vec{R}_C \times \vec{B}}{qB^2 R_C^2}$$

- Is $v_D \ll \omega_c \rho$?

Curvature drift in SPARC

- Assume $v_\perp = v_\parallel = \sqrt{E/m}$
- Hydrogen ion or electron in SPARC:

$$|v_D| = \frac{2E_\parallel}{qBR_C} = \frac{20\ keV}{(1.6 \times 10^{-19}\ \mathrm{C}) * (12\ \mathrm{T}) * (1.85\ \mathrm{m})} = 900\ \mathrm{m/s}$$

- $\omega_c \rho$ = $1.4 \times 10^6\ \mathrm{m/s}$ (hydrogen ion) or $5.9 \times 10^7\ \mathrm{m/s}$ (electron)

Curvature drift: consequences

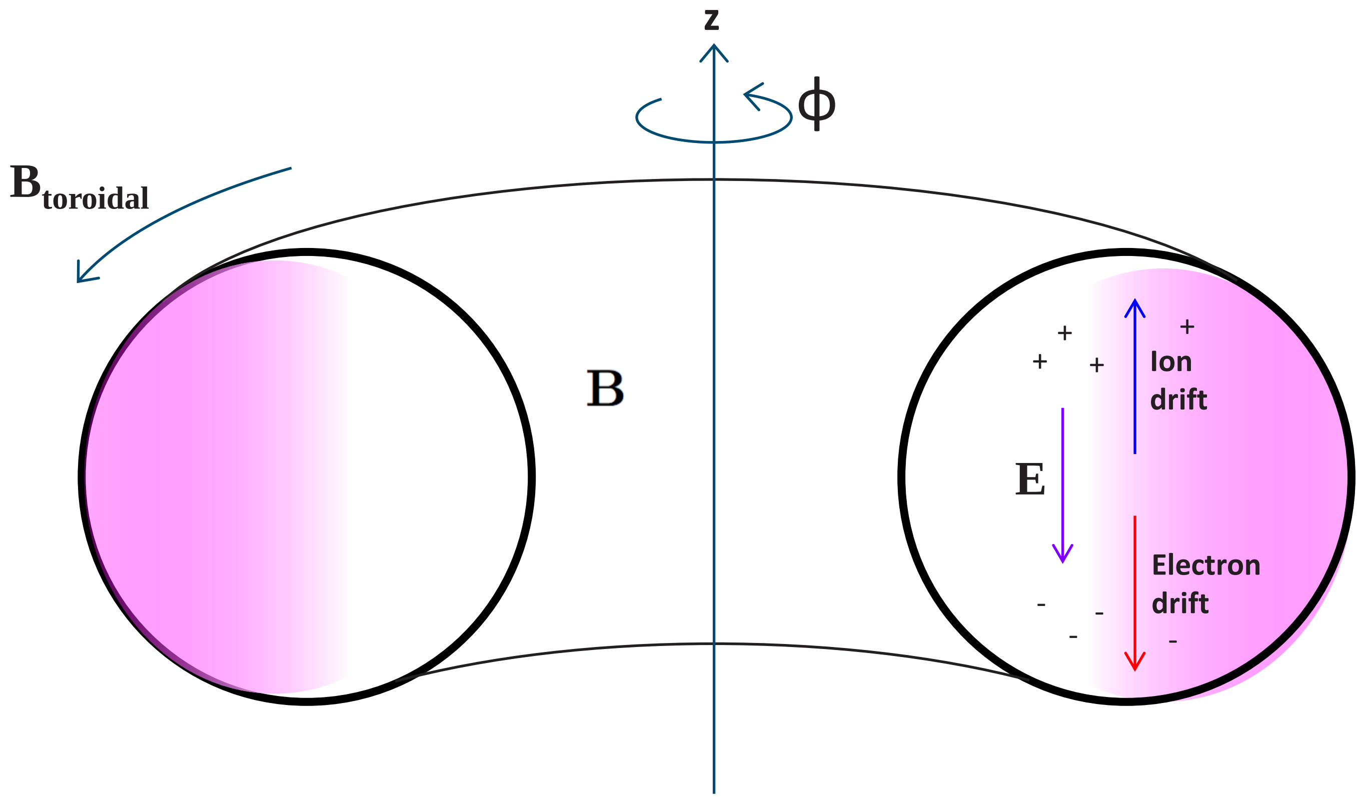
$$\vec{v}_D = m v_{\parallel}^2 \frac{\vec{R}_C \times \vec{B}}{q B^2 R_C^2}$$

We can see two important features of the curvature drift:

1. It's perpendicular to both $\vec{B}$ and $\vec{\mathrm{R}}$, meaning that, in a toroidal field, it points in the *vertical* direction.
2. It depends on charge q, meaning it points in *different* directions for electrons and ions.

Curvature drift: consequences

Positive and negative charges separate, creating a vertical electric field!

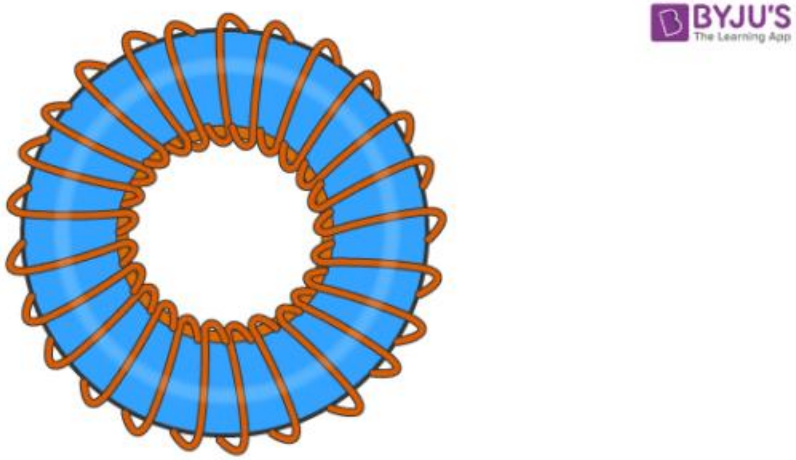


Field gradient

- How do we *make* a toroidal magnetic field?
- Example shown below: loops of current around plasma
- They're closer together on the inside than the outside:

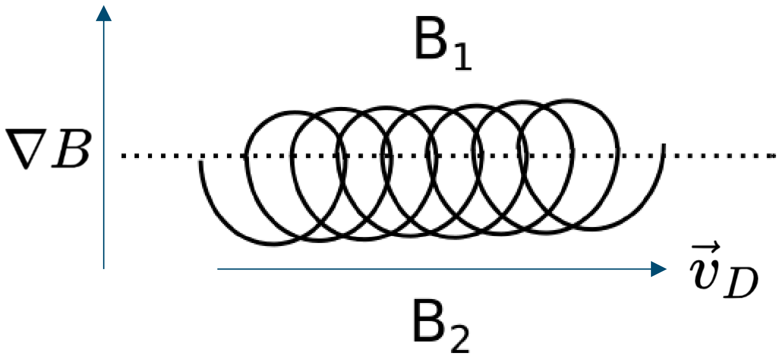
$$B_{in} > B_{out}$$

- $\vec{B}$ is neither straight nor uniform!



Gradient drift

- A difference in B creates a drift too!
- Why? $\rho \propto B^{-1}$
- Imagine we have two magnetic fields, $B_1 > B_2$



- Drift direction: $+\vec{B} \times \nabla B$ for ions, $-\vec{B} \times \nabla B$ for electrons

Gradient drift in a toroidal field

- $B_{in} > B_{out}$: ∇B is in the inward ($-\hat{r}$) direction
- $\vec{v}_D$ is in the $\vec{B} \times \nabla B$ direction, which is the same as $\hat{r} \times \vec{B}$
- ∇B and curvature drift are *additive*
- The ions drift in the $\hat{r} \times \vec{B}$ direction, the electrons in $-\hat{r} \times \vec{B}$
- Therefore, the electric field induced points in $-\hat{r} \times \vec{B}$

Vertical electric field

- Now we have an electric force on the particles, which leads to more drifts!

$$\vec{F} = q\vec{E} \qquad \vec{v}_D = \frac{q\vec{E} \times \vec{B}}{qB^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

- Same direction for ions and electrons!
- $\vec{E}$ points in the direction $-\hat{r} \times \vec{B}$
- $\vec{v}_D$ points in the direction $-(\hat{r} \times \vec{B}) \times \vec{B} = \hat{r}$
- Ions and electrons want to move *outwards*!!

How can we save our toroidal plasma??

- The honey wand on the right is our toroidal plasma
- Instead of a ∇B and curvature drift, we have gravity pulling the honey (plasma) down

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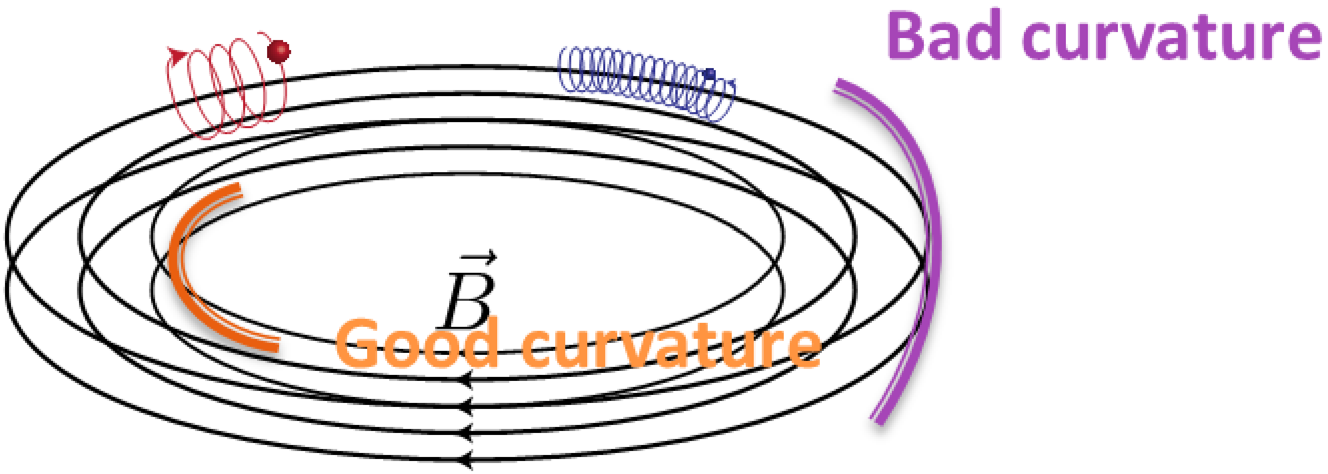
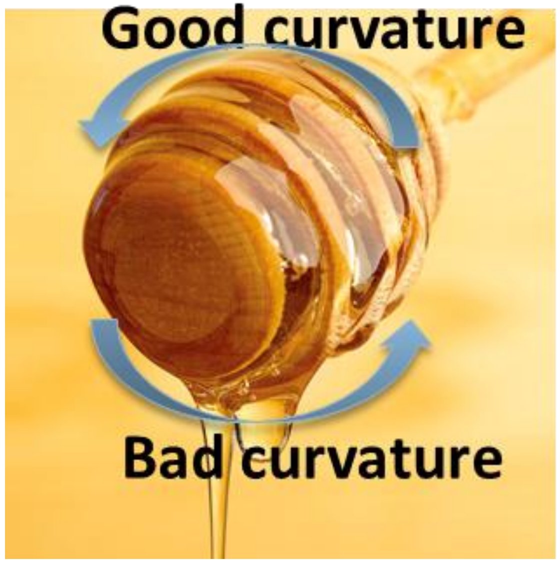
Q: How do you keep the honey on the wand?

A: You rotate it!

Plasmas have “good” and “bad” curvature

- On the inside, the drift is “good”: it pulls the plasma in towards its center
- On the outside, it’s “bad”, pulling the plasma outward
- If we twist the magnetic field, the particles will pass through regions of good and bad curvature, and the drifts will average out!



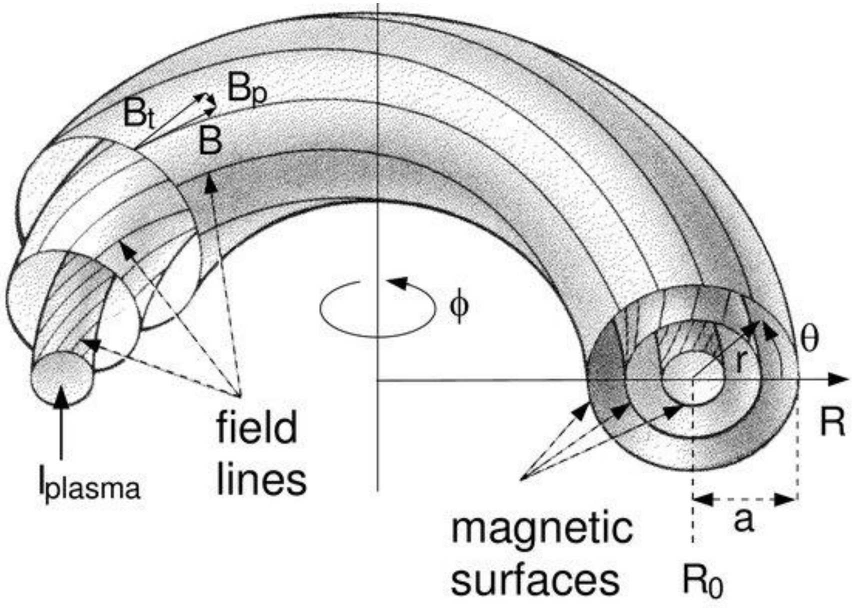


Adding a twist to the field

- $\vec{B} = B_{toroidal} + B_{poloidal}$
- We can describe a field line by “safety factor”

$$q = \frac{toroidal\ transits}{poloidal\ transits}$$

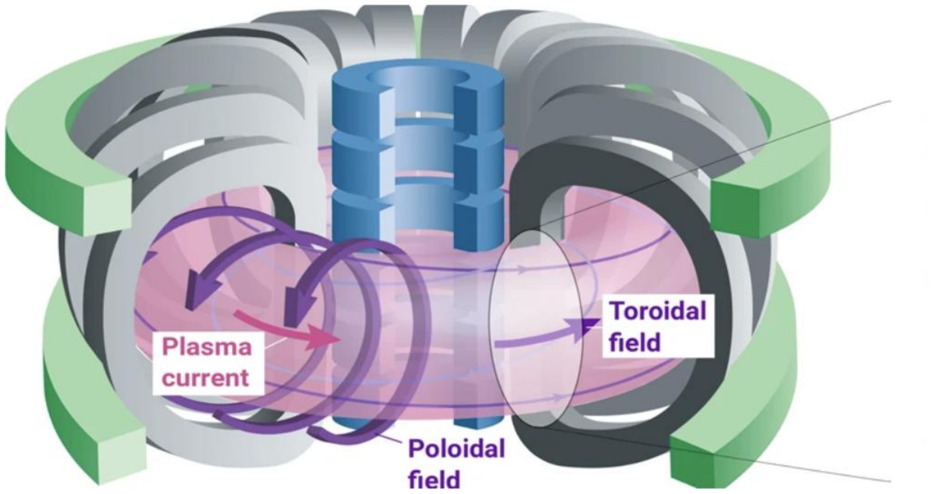
- If q is *irrational*, then a field line traces out a surface
- These are known as “flux surfaces” because q is determined by enclosed magnetic flux Ψ

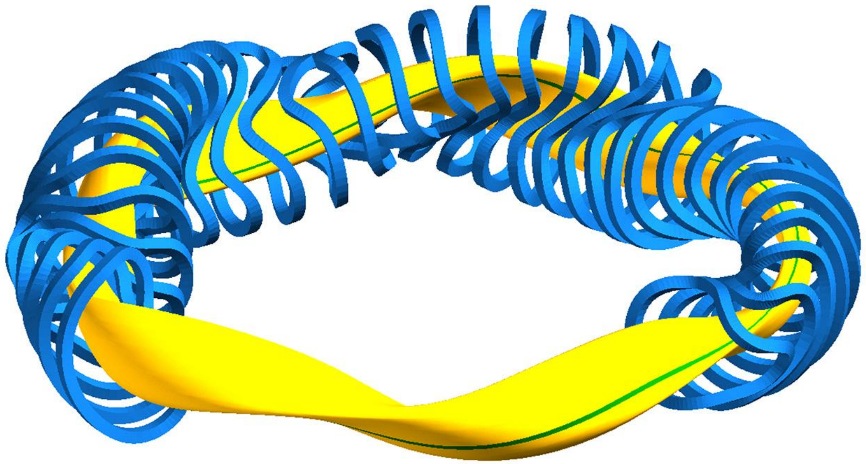


How do we twist this field?

Two popular ways of creating a helical magnetic field:

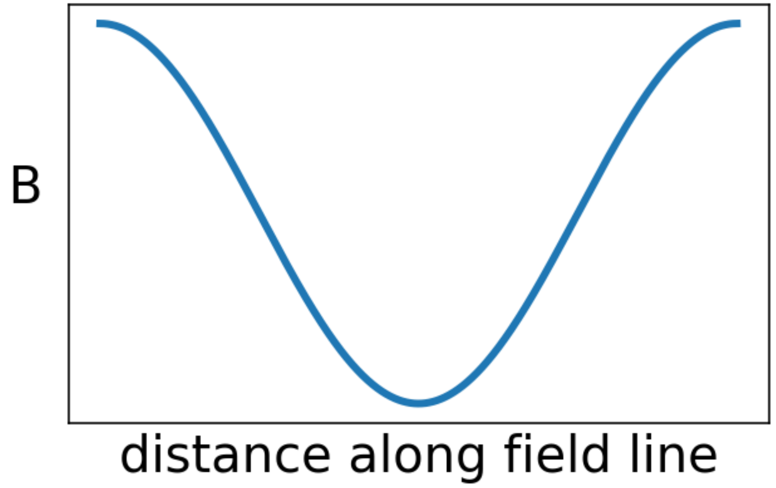
1. **Tokamak:** Symmetric, flat magnetic field coils create B_{tor}; a large current in the plasma creates B_{pol}
2. **Stellarator:** Both B_{tor} and B_{pol} are created by twisted magnetic field coils





A new problem?

- Now that the field lines are twisted, B isn't constant along each line!
- Magnetic field strength along a field line in an ideal tokamak looks like:



- We know what happens when we have $\nabla B \perp \vec{B}$, but what happens when $\nabla B \parallel \vec{B}$?

Adiabatic invariants

- From classical mechanics, we know that when a particle performs periodic motion in a coordinate q, it has an *action* defined as

$$\oint p_q dq$$

  where p_q is the generalized momentum associated with q
- For *small* perturbations of the periodic motion, the action is conserved
- Here, $q = \theta, p_\theta = m v_\theta r$

The first adiabatic invariant

$$\oint p_\theta d\theta = \int_0^{2\pi} m v_\theta \rho d\theta = 2\pi m v_\perp \rho =$$

$$2\pi m v_\perp \frac{m v_\perp}{qB} = \frac{4\pi m}{q} \frac{\frac{1}{2} m v_\perp^2}{B} = const.$$

- The first adiabatic invariant is thus $\frac{E_\perp}{B}$
  - This is conserved *if* the guiding center approximation holds!
- What does this mean?

Combine the adiabatic invariant with conservation of energy

- As we move along the field line, B increases, and so does $E_\perp$
- *But* the total energy of the particle is conserved!

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  A: The parallel energy!

Combine the adiabatic invariant with conservation of energy

- As we move along the field line, B increases, and so does $E_\perp$
- *But* the total energy of the particle is conserved!

  Q: Where is the energy coming from?

  A: The parallel energy!
- If B increases enough, *all* the parallel energy will be converted into perpendicular energy

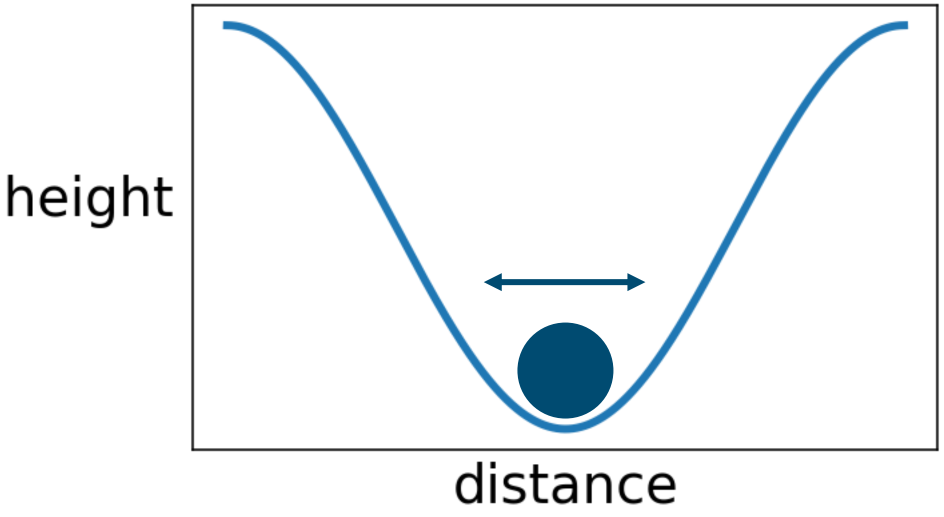
$$\boldsymbol{E} = \boldsymbol{E}_{||,0} + \boldsymbol{E}_{\perp,0} = \boldsymbol{E}_{\perp,f}$$

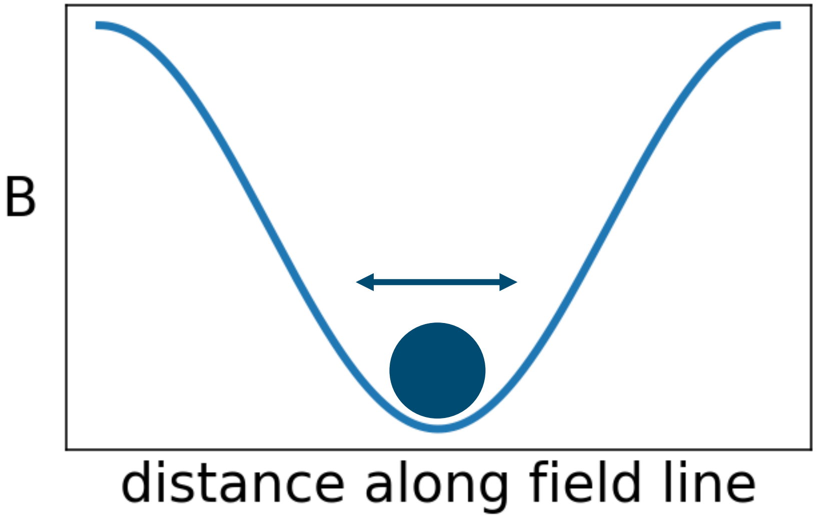
- What happens then?

Analogy: consider a ball trapped in a well

- As the ball rolls up the well, E_{kin} gets converted to U_{grav}
- If the well is too deep, it will stop and roll back down!

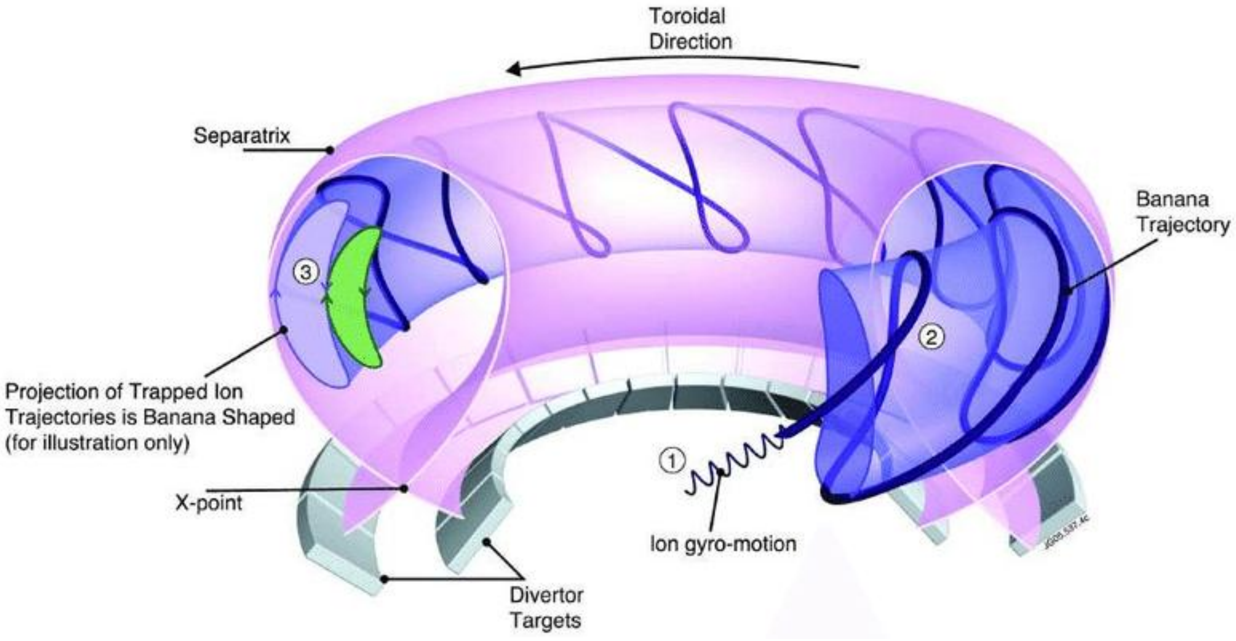
- As the particle follows the field line, $E_{\parallel}$ gets converted to $E_{\perp}$
- If the “magnetic well” is too deep, it will stop and reverse direction!





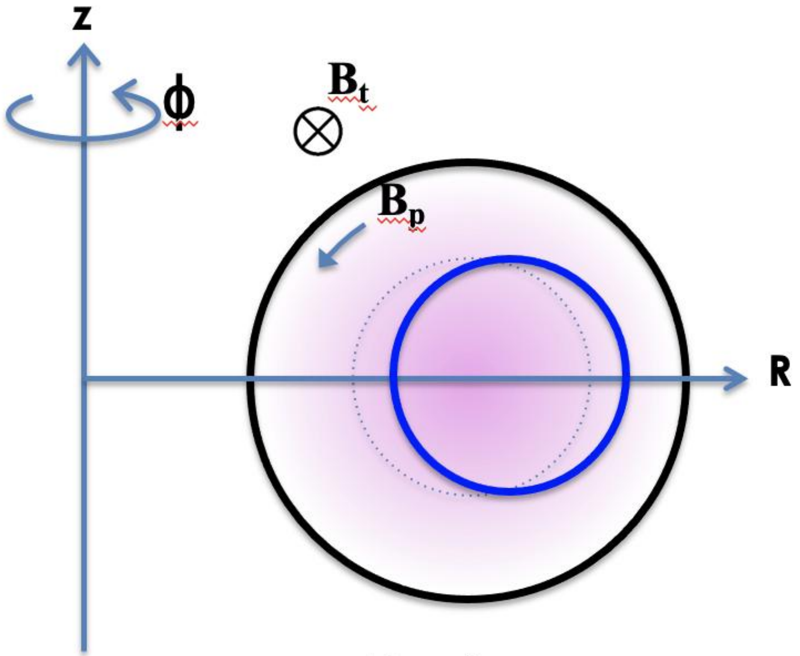
Mirrors and bananas

- This effect is called a “magnetic mirror,” because the particles are reflected
- In tokamaks, this leads to what we call “banana orbits” because they look like bananas in a *poloidal cross section*

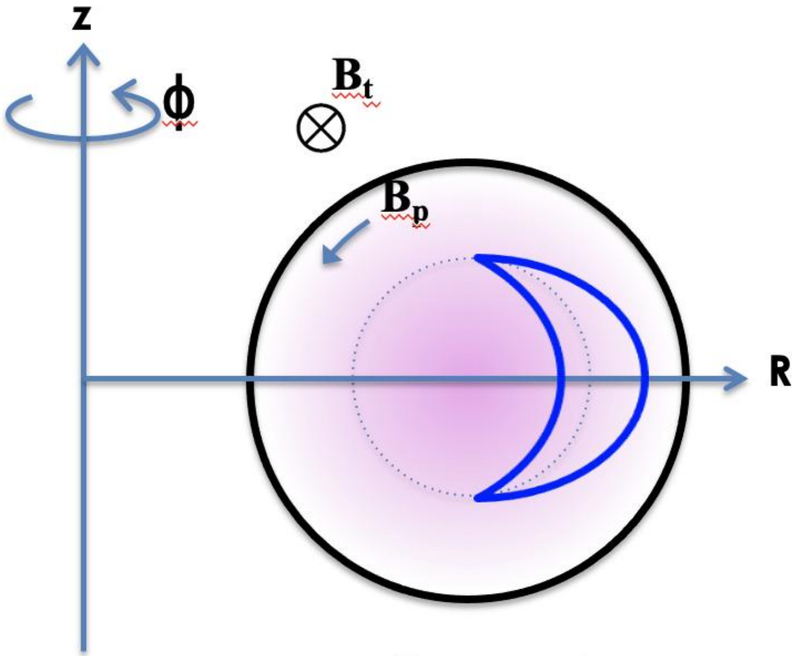


Single particle orbits in ideal tokamaks

- In ideal tokamaks, there are two types of collisionless particle orbits: “passing” and “trapped” (aka banana)
- Both drift away from flux surfaces, but thanks to axisymmetry, they always end up back where they started

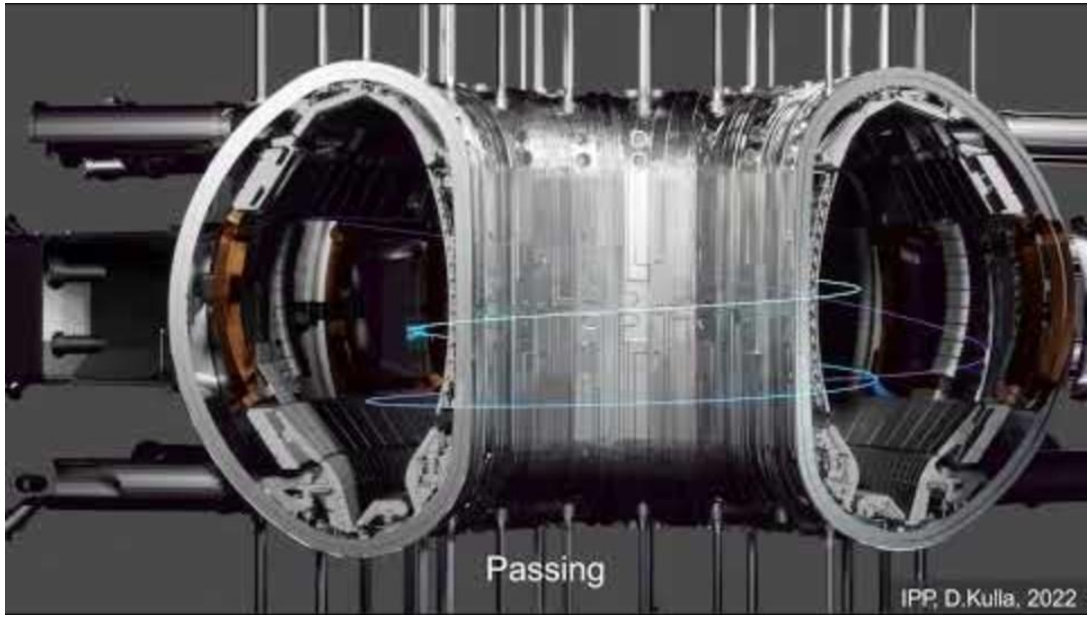


Passing



Trapped

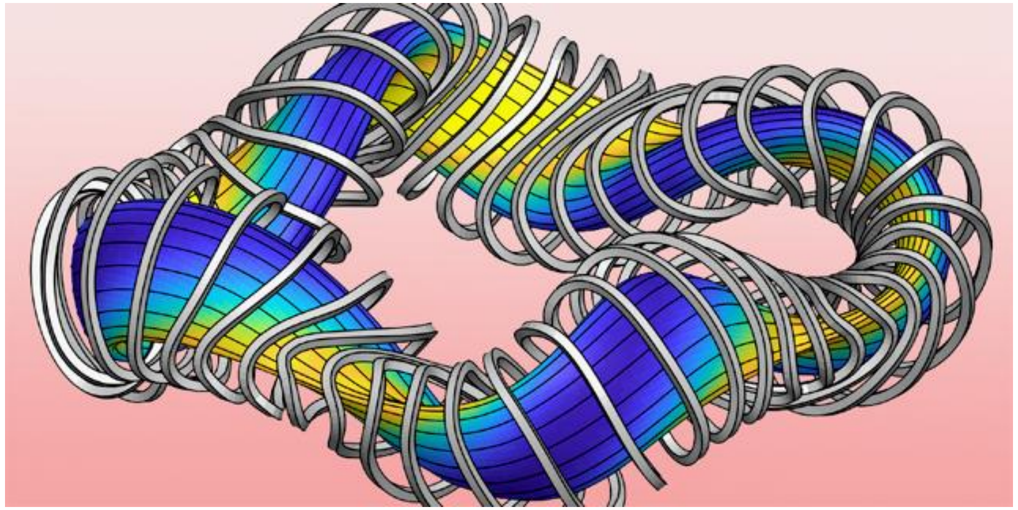
What does this actually look like?



Video credit: David Kulla, Max Planck Institute for Plasma Physics

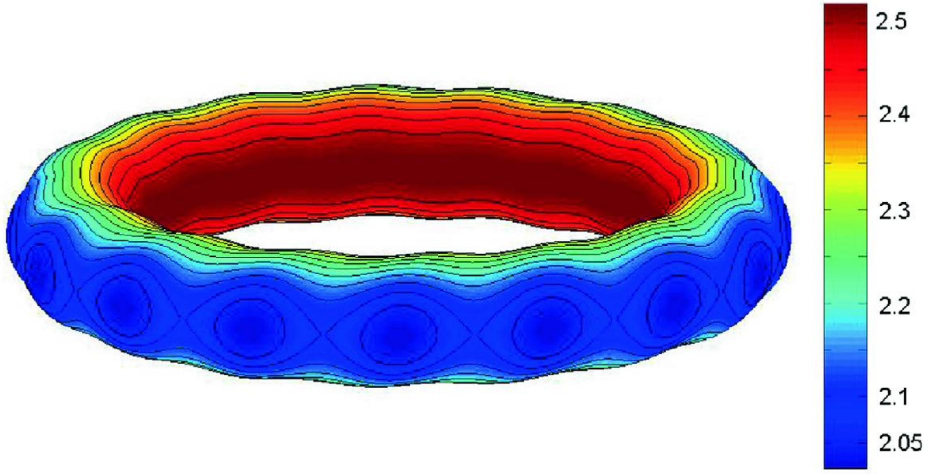
Single particle orbits in stellarators

- Stellarators are more complicated: lots of different orbit types, and the drift orbits don't necessarily close on themselves
- What does this mean?
- Stellarators don't *inherently* confine collisionless particle orbits
- Need *optimization* to fix this!



Real tokamaks

- Real life tokamaks *aren't* perfectly axisymmetric!
- Why not?
- Finite number of coils leads to "ripples" in B… and coils can be misaligned!
- This means that tokamaks can lose particles if:
  - Their banana orbits resonate badly with field ripple
  - Their drift orbits are big enough to take them out of the plasma



When does this matter?

- We've been assuming that we can treat the particles in the plasma as single particles interacting with $\vec{B}$
- In reality, the particles also interact with *each other*!
- <u>We still need a twisted field</u>
- But for most particles:
  - drifts away from flux surfaces are small
  - trapped particles are quickly de-trapped by collisions
- The exception: *fast ions*

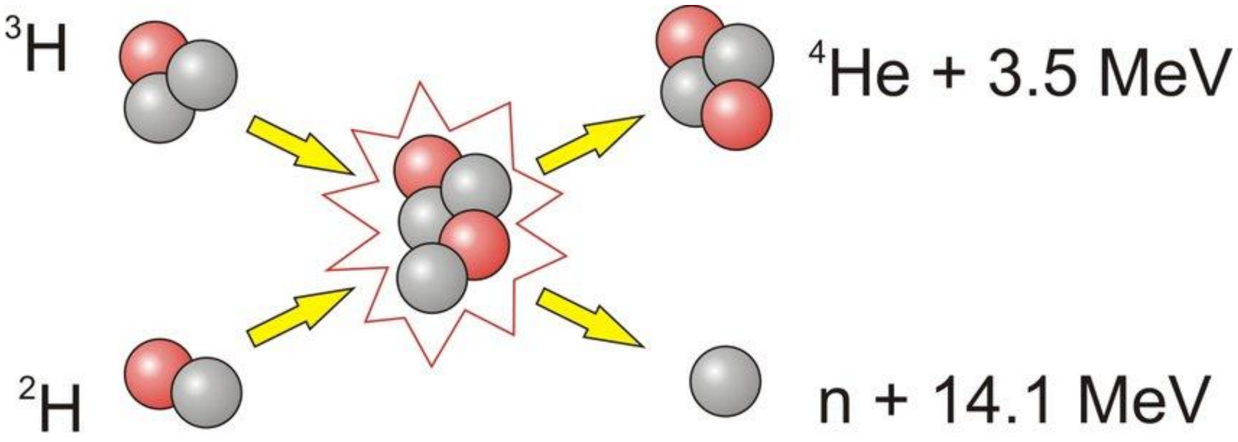
An introduction to fast ion physics

- Hotter/faster particles collide less often
- Curvature and ∇B drifts are proportional to kinetic energy
- So if you have a small population of very hot particles in a plasma, they will:

1. Travel far away from their original flux surfaces
2. Stay in one orbit type (trapped/passing) for a long time
3. Primarily interact with the magnetic field, *not* the other particles

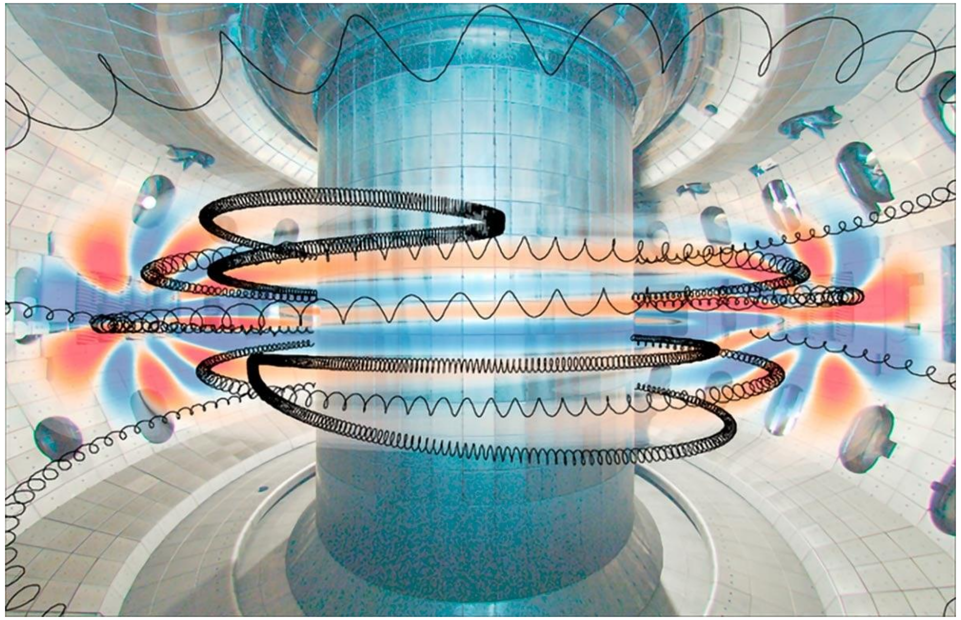
Where do we get fast ions?

- DT fusion produces 3.5 MeV alpha particles (helium ions), which are hundreds of times more energetic than the main plasma
- Heating systems like *neutral beam injection* (NBI) and *ion cyclotron resonance heating* (ICRH) can create ions with 100's to 1000's of keV



The importance of fast ion confinement

- Confining fast ions (whether from fusion, NBI, or ICRH) is how a fusion device heats itself!
- If you lose them, they hit the wall and start melting things…
- Can use orbit following codes to simulate what fast ions do in your plasma



Conclusions

- In a straight, uniform magnetic field, charged particles follow helical paths around the field lines; this is the basic premise of magnetic confinement
- Stronger magnetic field means smaller orbits, better confinement
- When you bend the magnetic field or add other forces/fields, particle orbits can *drift* across field lines
- To compensate, you need a *toroidal* and a *poloidal* field
- But twisting the field causes particles, especially energetic, collisionless ones (“fast ions”), to get “trapped”
- Confinement of fast ions is an important area of research, especially in stellarators, but also in tokamaks