Introduction to waves in plasma

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Tokamak Energy

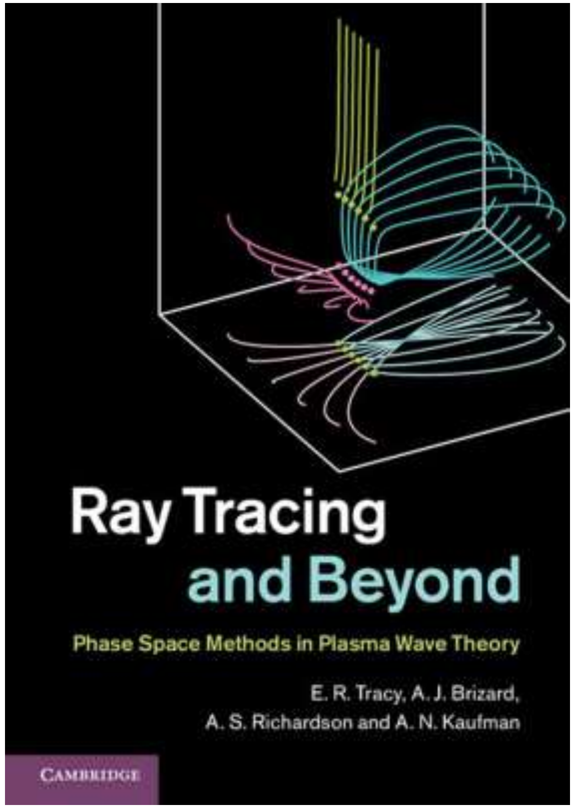
SULI lecture series, 2025

About me

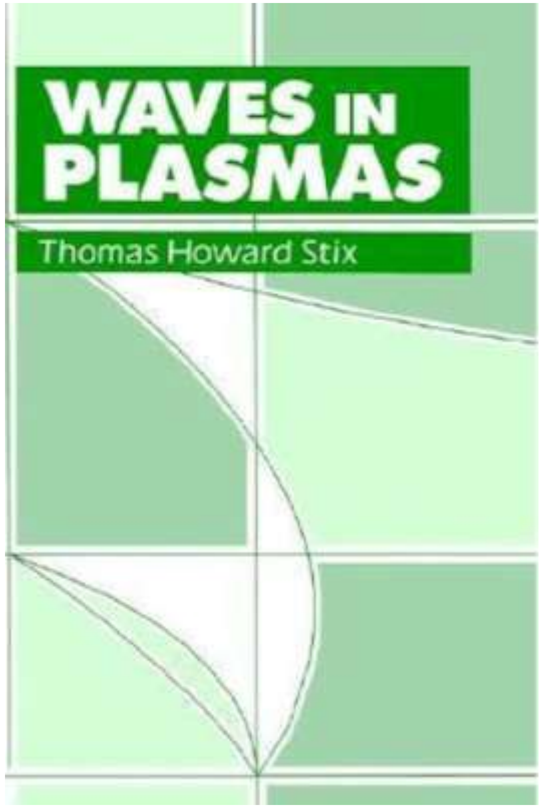
- **2011-2015**: MIT Undergrad
  - Started as Nuclear Engineer (22), added Physics (8) near end
  - Informal exposure to fusion via talking with PSFC faculty
- **Summer 2014**: NUF/SULI participant
  - Attended these lectures at PPPL, then sent to DIII-D (General Atomics)
  - Helped assess new upgrades to neutral beam source
- **2015 – 2022**: PPPL Graduate student
  - Started specializing in wave physics
  - Studied wave heating & current drive in NSTX-U, mode-conversion models, and more accurate models for wave propagation (Thesis!)
- **2022 – 2024**: U. Oxford postdoc
  - Studied electron transport caused by whistler waves in lab-astro context
  - But, talking w/ UK fusion microwave community made me get ‘back in the game’
- **Sept. 2024 – now**: Tokamak Energy staff researcher
  - Model wave heating & current drive in ST40 tokamak & fusion pilot plants

* Disclaimer: ‘Cartoon’ physics coming up *

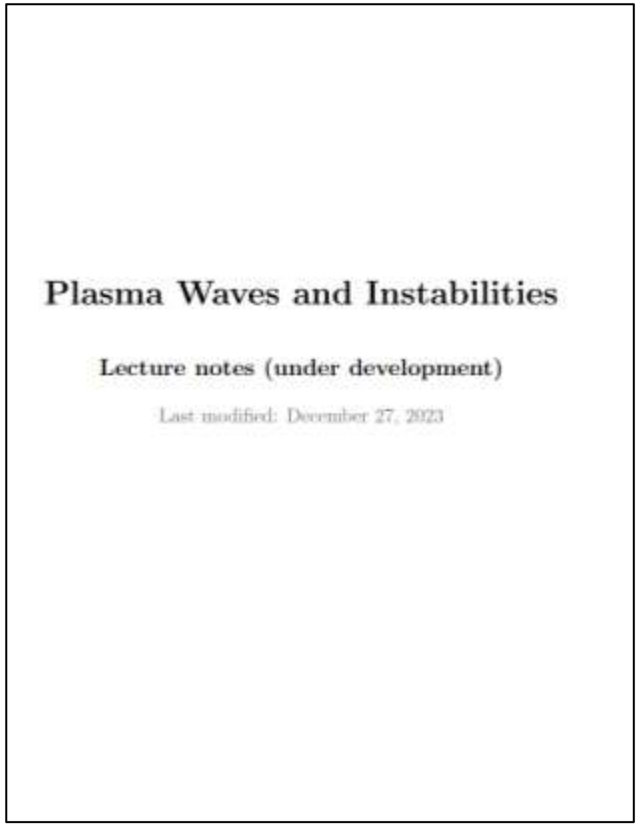
- Rigorous (grad-level) intros to plasma waves can be found in:



Ray tracing and beyond,
Tracy, Brizard, Richardson & Kaufman



Waves in Plasmas,
Stix

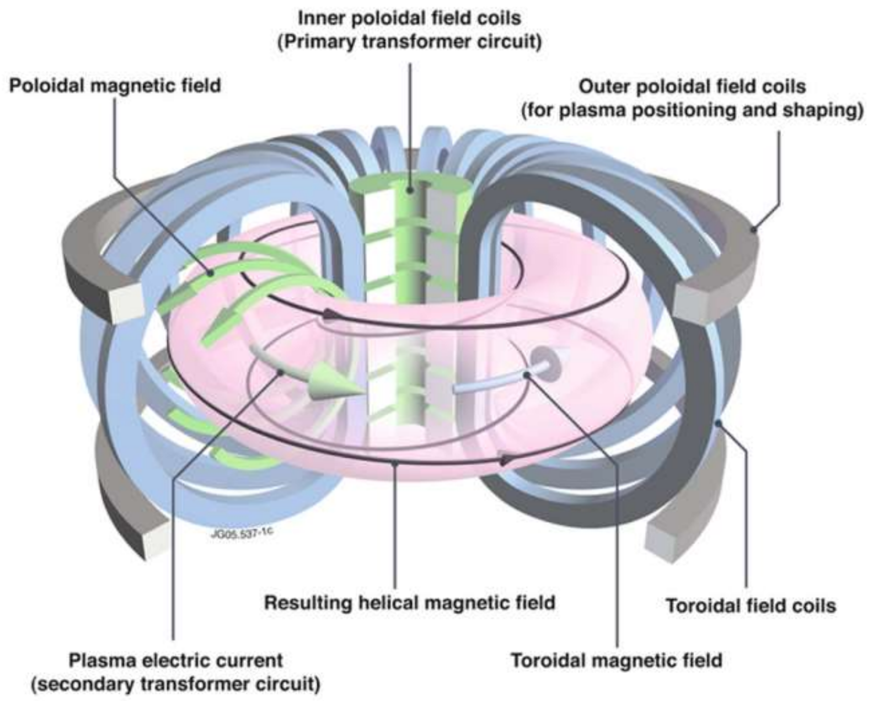

Plasma Waves and Instabilities

Lecture notes (under development)

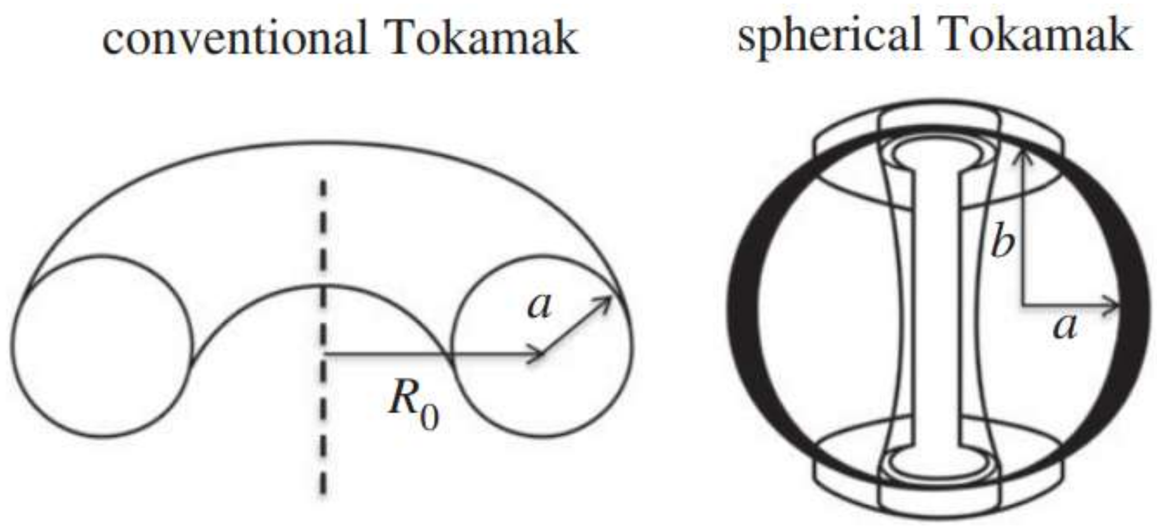
Last modified: December 27, 2023

Plasma wave lecture notes,
Dodin
https://www.princeton.edu/~idodin/ast553/main.pdf

Motivation – RF heating & current drive in STs

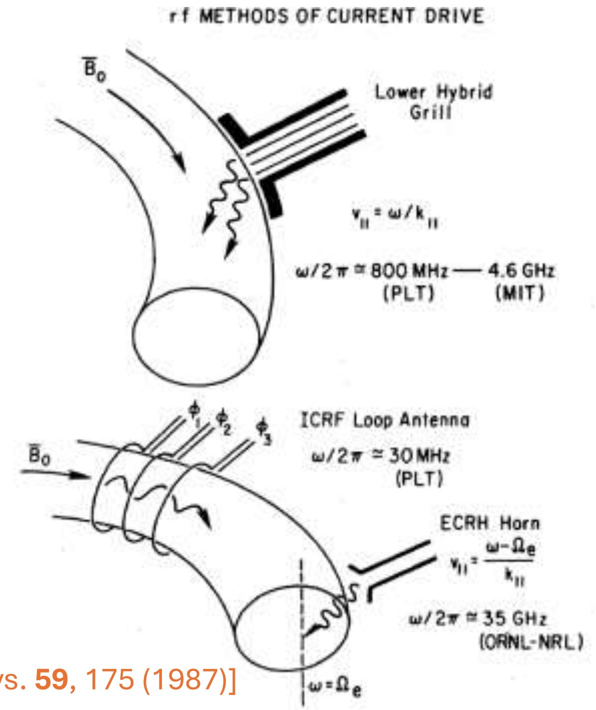


[Eurofusion (2011) & Poli PoP **25**, 055602 (2018)]



[Costley Phil. Trans. R. Soc. A **377**, 20170439 (2019)]

- Spherical tokamaks have little room for solenoid
  - Even less space for reactors using HTS magnets because of neutron shielding
- So auxiliary heating & current drive needed
  - 'Non-solenoidal' operation has continued to be one of the most important topics in spherical tokamak research

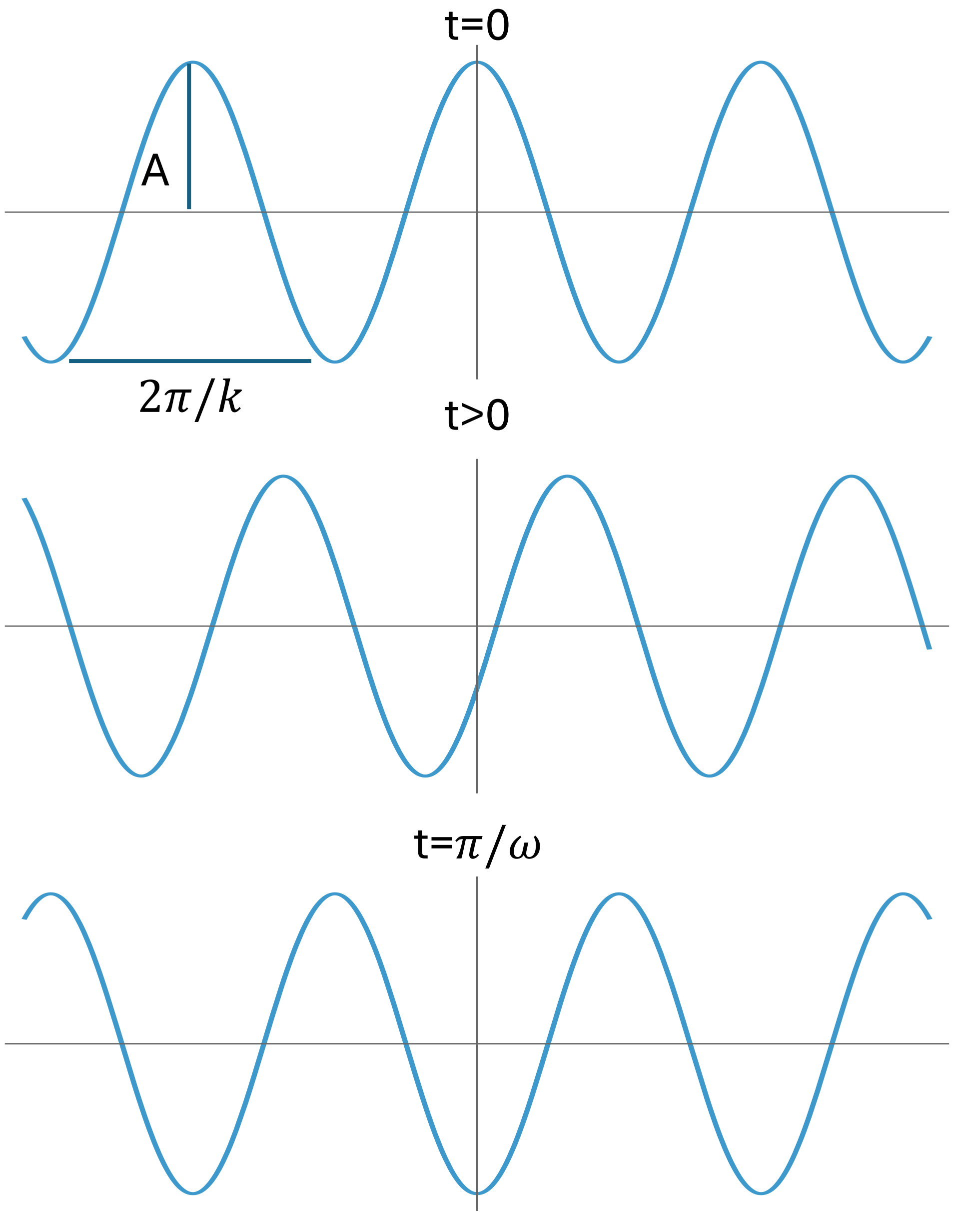


[Fisch Rev. Mod. Phys. **59**, 175 (1987)]

Wave propagation

What is a wave?

- Exact plane wave: $A\sin(kx - \omega t)$
  - A: amplitude, $\phi = kx - \omega t$: phase
    - $\frac{d\phi}{dx} = k, \qquad \frac{d\phi}{dt} = -\omega$
  - Phase velocity $v_p = \omega/k$
- 'Almost' plane wave: $A(x,t)\sin\phi(x,t)$
  - A and ϕ vary in space & time
    - $\frac{d\phi}{dx} \sim k, \qquad \frac{d\phi}{dt} \sim -\omega$
- **Dispersion relation** constrains ω, k, x, t
  - Ex: light wave has $\omega = ck$,
    - c the speed of light, no dependence on x, t
    - (But plasmas won't be so simple!)



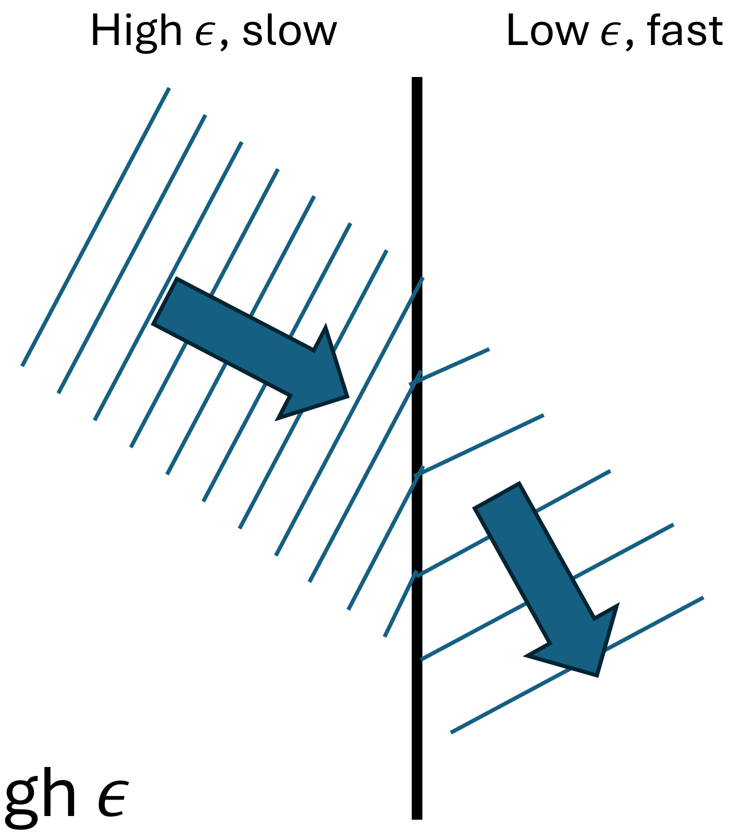
Refraction in dielectric media

- More generally for dielectric media,

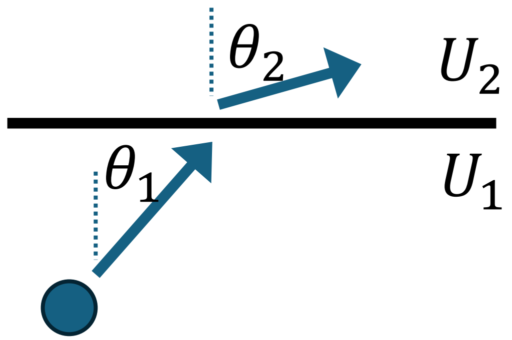
$$N^2 \equiv \left(\frac{c}{v_p}\right)^2 = \epsilon$$

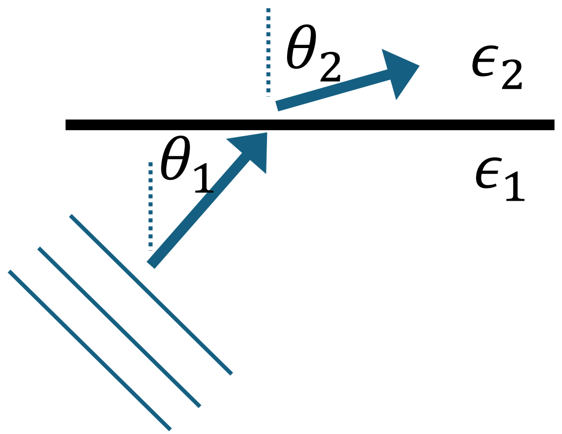
  - ϵ: relative permittivity, $\epsilon = 1$ for vacuum
  - Larger ϵ means smaller v_p, and vice versa

- At interface, waves bend away from low ϵ to high ϵ
  - $\epsilon = 0$: 'cutoff', everything reflected away
  - $\epsilon = \infty$: 'resonance', everything attracted into it
    - We will see examples of these in plasmas soon!



Refraction directly analogous to classical motion





Energy conservation: $|p_{1,2}|^2 = 2m(E - U_{1,2})$

Momentum conservation: $|p_1|\sin\theta_1 = |p_2|\sin\theta_2$

Dispersion relation: $|N_{1,2}|^2 = \epsilon_{1,2}$

Snell's law of refraction: $|N_1|\sin\theta_1 = |N_2|\sin\theta_2$

So $N \sim p, \quad \epsilon \sim -2U$

- We know continuum limit for classical motion in U(x) $\boxed{m\ddot{x} = -\nabla U}$
  - (m can be removed by reparameterizing $t = \sqrt{m}\,T$)
- Equation for continuum (gradient-index) optics $\boxed{x'' = +\frac{1}{2}\nabla\epsilon}$
  - Sometimes called 'F = ma' optics [Evans et al, Am. J. Phys. **54**, 876 (1986)]

Ray-tracing in gradient-index media

- Waves instead evolved using Hamilton's Eqs of motion (EOM)

Newton's EOM

$$\boxed{m\ddot{x} = -\nabla U}$$

Hamiltonian function

$$\boxed{H(x,p) = \frac{p^2}{2m} + U(x) = E}$$

Hamilton's EOM

$$\boxed{\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}}$$

Newton's EOM

$$\boxed{x'' = +\frac{1}{2}\nabla\epsilon}$$

Dispersion function

$$\boxed{D(x,N) = \frac{1}{2}\left[N^2 - \epsilon(x)\right] = 0}$$

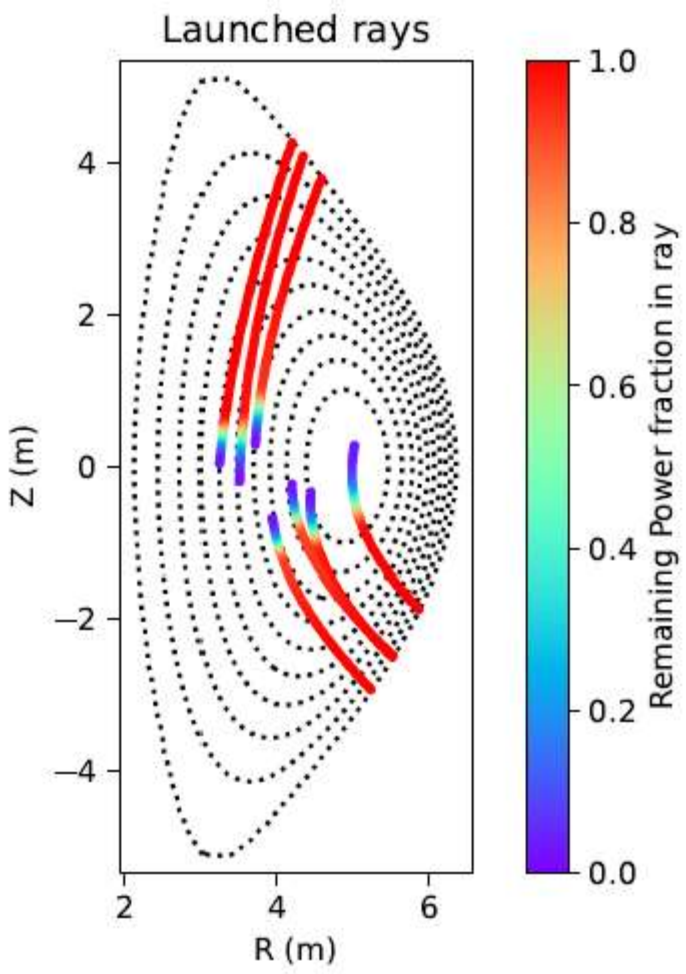
Hamilton's EOM

$$\boxed{N' = -\frac{\partial D}{\partial x}, \quad x' = \frac{\partial D}{\partial N}}$$

  - (Can multiply $H \rightarrow \alpha H$, $D \rightarrow \alpha D$ without changing physics, just changing time units)

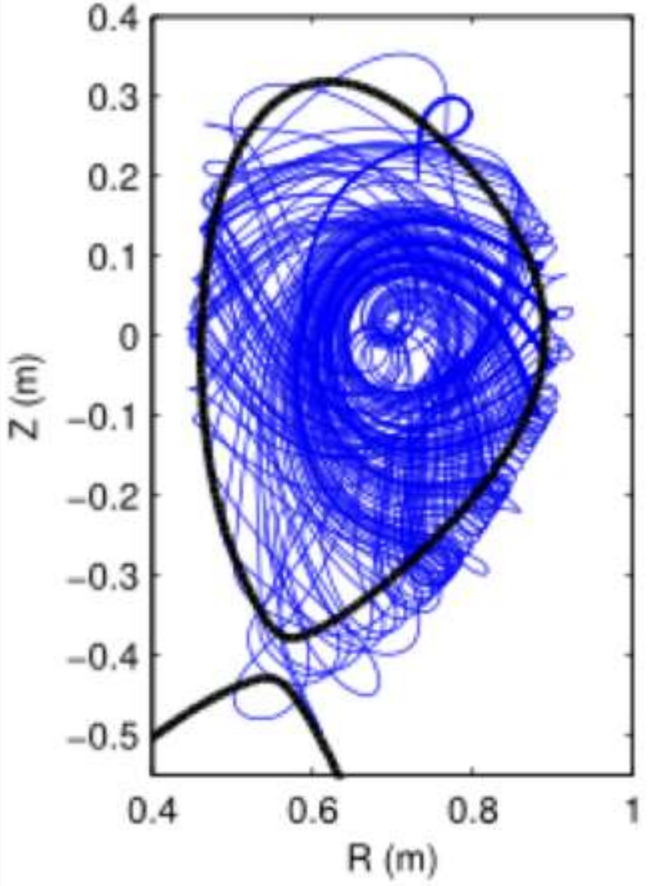
- Simulating waves in plasmas requires **good algorithms** for solving classical equations of motion!

Examples of ray-tracing in fusion



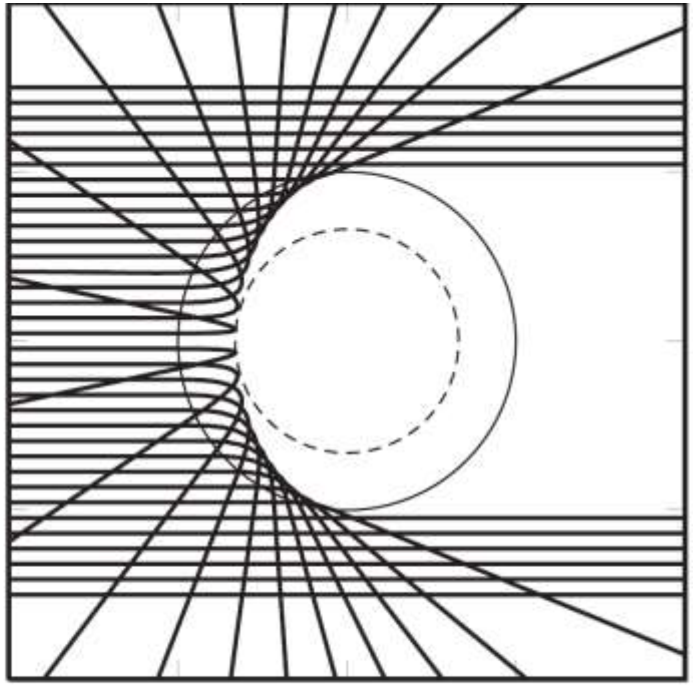
Electron-cyclotron waves in tokamak

[Lopez et al, PPCF **67**, 055012 (2025)]



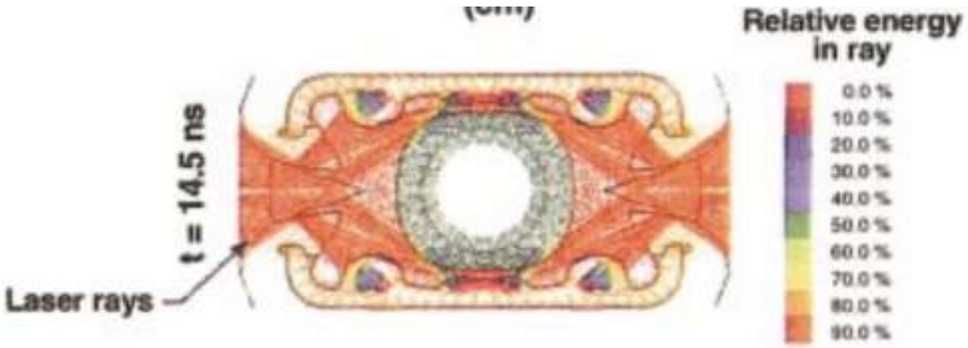
Lower-hybrid waves in tokamak

[F Poli, APS DPP meeting (2017)]



Laser beam reflecting off spherical plasma

[Tracy et al, Ray-tracing & Beyond p. 160]



Laser beam propagation in hohlraum

[Lindl et al, PoP **11**, 339 (2004)]

Plasma dispersion relation

General Plasma wave equation

- Time-dependent Maxwell equations:

$$\frac{d\vec{B}}{dt} = -c\nabla \times \vec{E}, \quad \frac{d\vec{E}}{dt} = c\nabla \times \vec{B} - 4\pi\vec{j}$$

- Curl of B eq, d/dt of E equation, then substitution gives:

$$\boxed{\frac{d^2\vec{E}}{dt^2} = c^2\nabla^2\vec{E} - c^2\nabla(\nabla \cdot \vec{E}) - 4\pi\frac{d\vec{j}}{dt}}$$

- Having response model for j as a function(al) of E then gives wave eq. for E

Plasma vs Vacuum

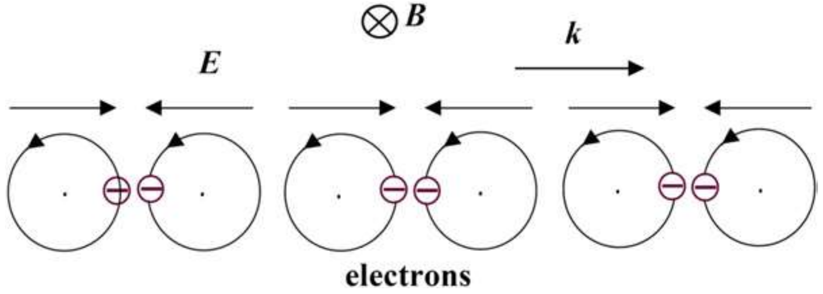
Plasma

$$\frac{d^2\vec{E}}{dt^2} = c^2\nabla^2\vec{E} - c^2\nabla(\nabla \cdot \vec{E}) - 4\pi\frac{d\vec{j}}{dt}$$

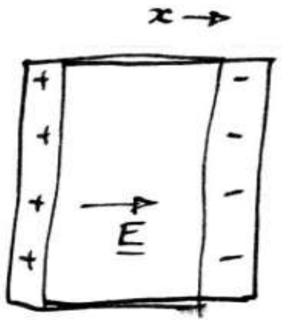
Vacuum

$$\frac{d^2\vec{E}}{dt^2} = c^2\nabla^2\vec{E}$$

- $\nabla \cdot \vec{E} \neq 0$ for plasma (charged plasma particles)
  - Plasma can support longitudinal (electrostatic) oscillations!



[Laqua, PPCF **49**, R1 (2007)]



[MIT OCW course 22.611j]

- $j \neq 0$ for plasma (charged plasma particles set in motion by E)
  1. **Anisotropic,**
     - (Different dispersion relation for different propagation directions)
  2. **Birefringent,**
     - (Different dispersion relation for different E-field polarizations)
  3. **Frequency-dependent** response!
     - Resonance responses of plasma (cyclotron motion, plasma oscillations, etc.)

Magnetic fields make anisotropy, birefringence

- Consider **harmonic modes** & **Ohm's law relation** for j & E

$$\vec{j}, \vec{E} \sim e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

(linear response)

$$\vec{j} = \bar{\bar{\sigma}} \cdot \vec{E}$$

(local response)

- Matrix relation ($\bar{\bar{\sigma}}$) between j & E = rich plasma wave dynamics!

$$\vec{j} = \left[\sigma \bar{\bar{I}} + \Delta\sigma \hat{b}\hat{b} + \sigma_\times \hat{b}\times\right] \vec{E}$$

  - **Uniform dielectric medium** (beginning slides)
  - **Simple birefringence** (crystal with single special axis)
  - **Magnetic fields add extra 'twist' to forces** (ExB drifts, diamagnetic flows)

- Similar decomposition for plasma transport too! ($\vec{q} = -\bar{\bar{\kappa}} \cdot \nabla T$)

Cold plasma dispersion relation

- Putting all that together gives wave equation

$$\boxed{\bar{\bar{D}} \cdot E = 0}$$

$$\bar{\bar{D}} = N^2 \bar{\bar{I}} - \vec{N}\vec{N} - \bar{\bar{\epsilon}}, \qquad \bar{\bar{\epsilon}} = \bar{\bar{I}} + \frac{4\pi i}{\omega}\bar{\bar{\sigma}}, \qquad \vec{N} = \frac{\vec{k}c}{\omega}$$

- For eigenmode to exist (with given N), E must be in **nullspace** of D
  - Requires $\det(D) = 0$

- This gives dispersion relation

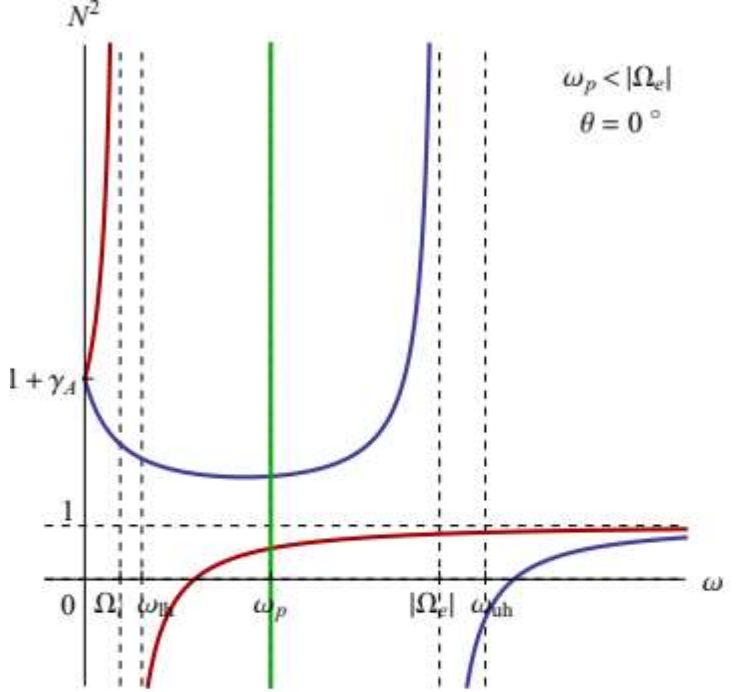
$$\boxed{N^2 = \frac{RL\sin^2\theta + PS(1+\cos^2\theta) \pm \sqrt{(RL-PS)^2\sin^4\theta + 4P^2D^2\cos^2\theta}}{2(S\sin^2\theta + P\cos^2\theta)}}$$

$$R, L = 1 - \sum_{species} \frac{\omega_{ps}^2}{\omega(\omega \pm \Omega_s)}, \qquad P = 1 - \sum_{species} \frac{\omega_{ps}^2}{\omega^2}, \qquad S = \frac{R+L}{2}, \qquad D = \frac{R-L}{2}$$

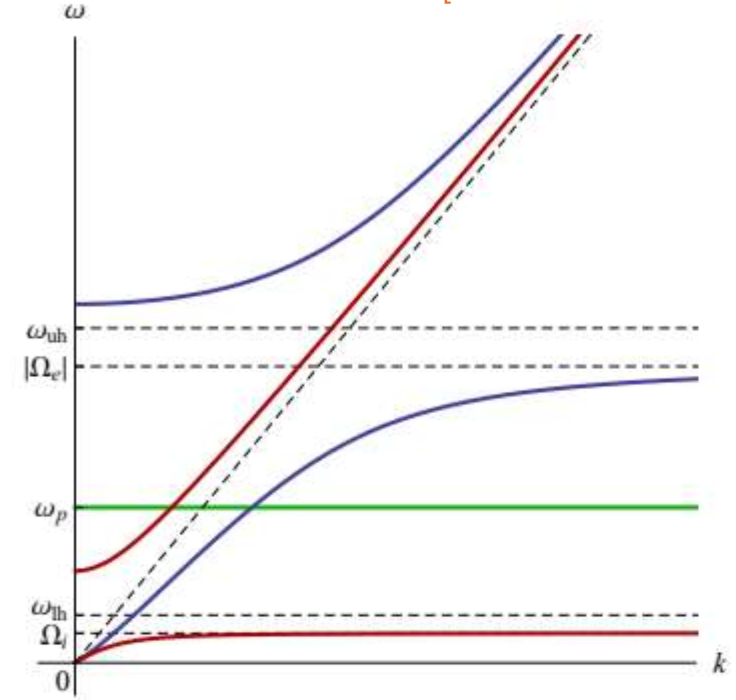
  - Notably depends on angle θ between $\vec{N}$ & $\hat{b}$, and frequency ω

Parallel propagation: $\theta = 0°$

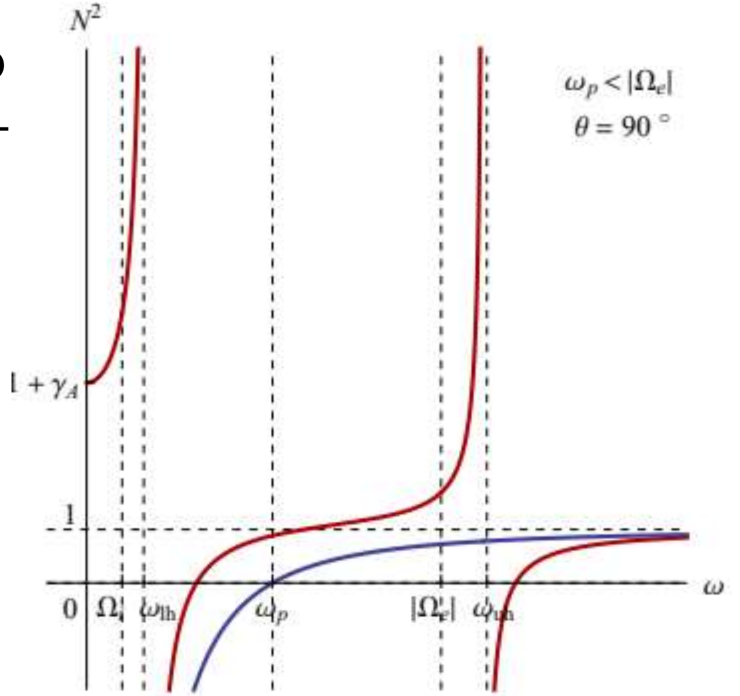
- **Green**: Electrostatic Plasma oscillation
  - Only exists where $\omega = \omega_p$
- **Red**: Left-hand polarized wave (L-mode)
  - Cutoff at 'left-hand' cutoff,
  - Resonance at $\omega = \Omega_i$
- **Blue**: Right-hand polarized wave (R-mode)
  - Cutoff at 'right-hand' cutoff
  - Resonance at $\omega = |\Omega_e|$
- For $\omega \to \infty$, L & R mode both become vacuum light wave with $N^2 = 1$



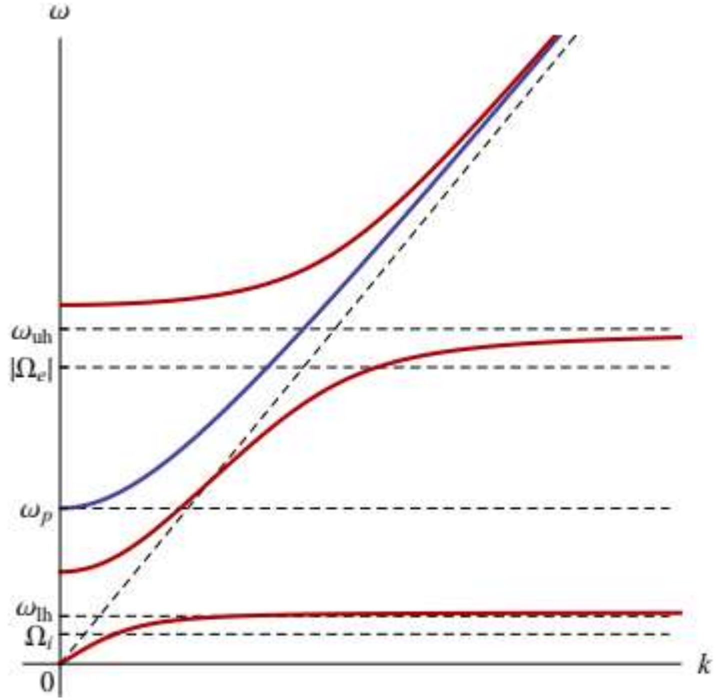
[I. Dodin lecture notes]



Perpendicular propagation : $\theta = 90°$



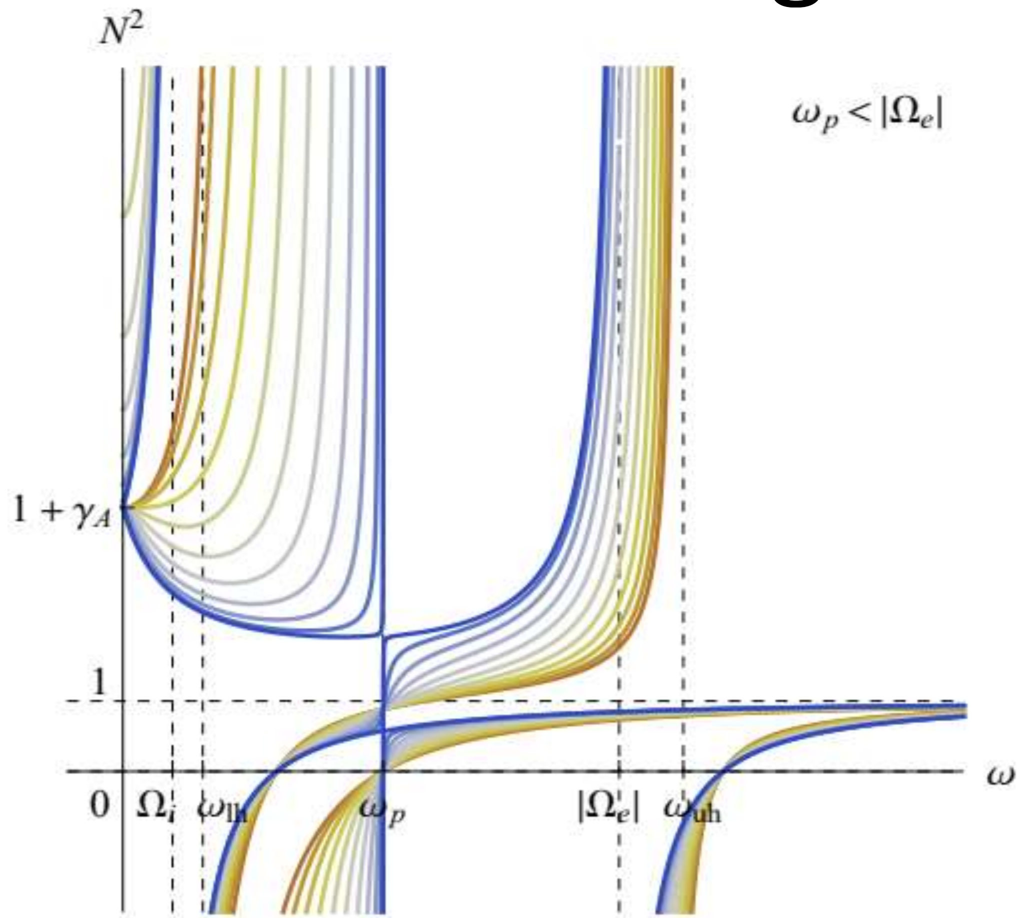
[I. Dodin lecture notes]

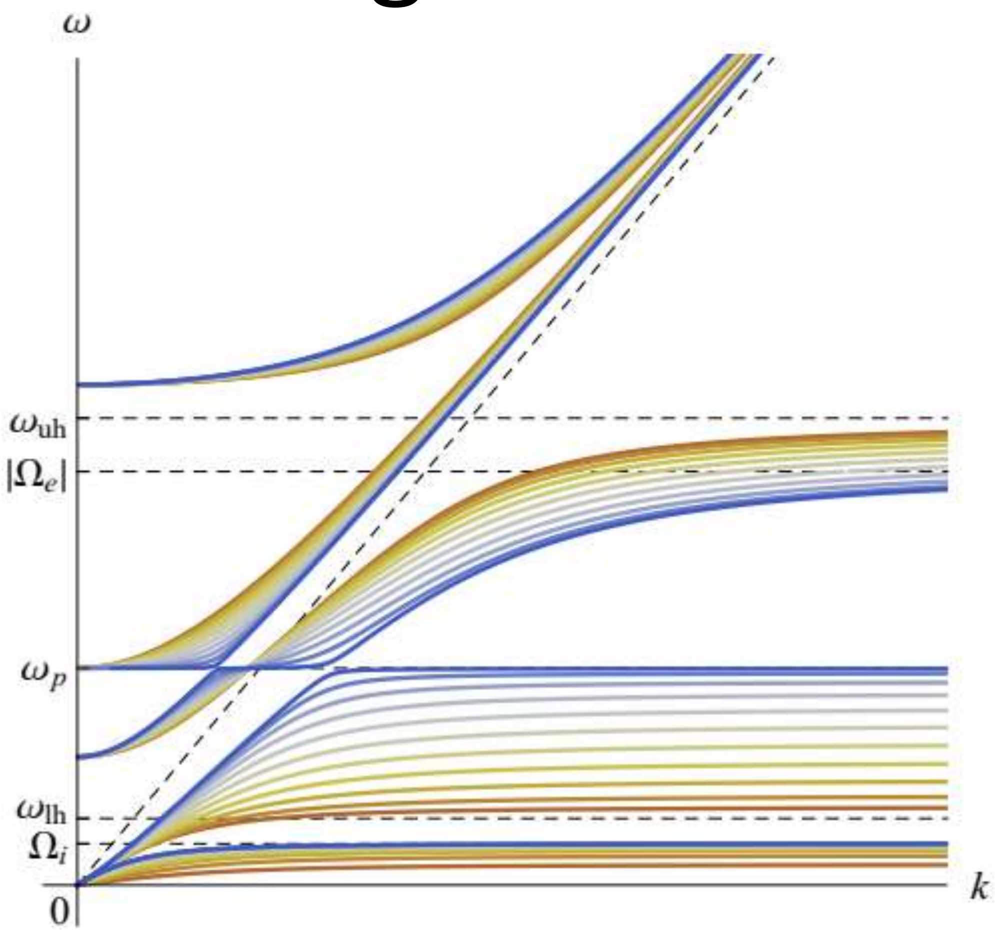


- **Red**: Extraordinary mode (X-mode)
  - E polarized perpendicular to B
  - Cutoff at 'left-hand' & 'right-hand' cutoff
  - Resonance at 'hybrid' resonances
- **Blue**: Ordinary mode (O-mode)
  - E polarized along B
  - Cutoff at $\omega = \omega_p$
  - No resonances
  - (Effectively unmagnetized)
- For $\omega \rightarrow \infty$, O & X mode both become vacuum light wave with $N^2 = 1$

Intermediate angles – mode mixing

[I. Dodin lecture notes]

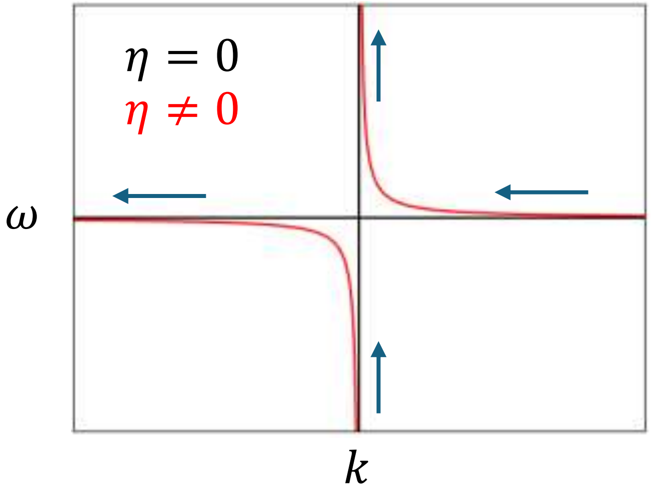


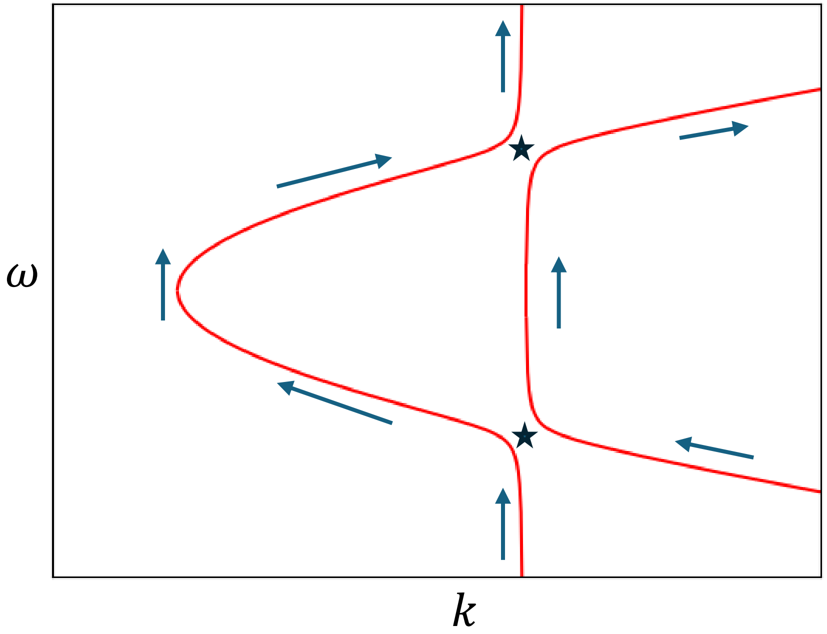


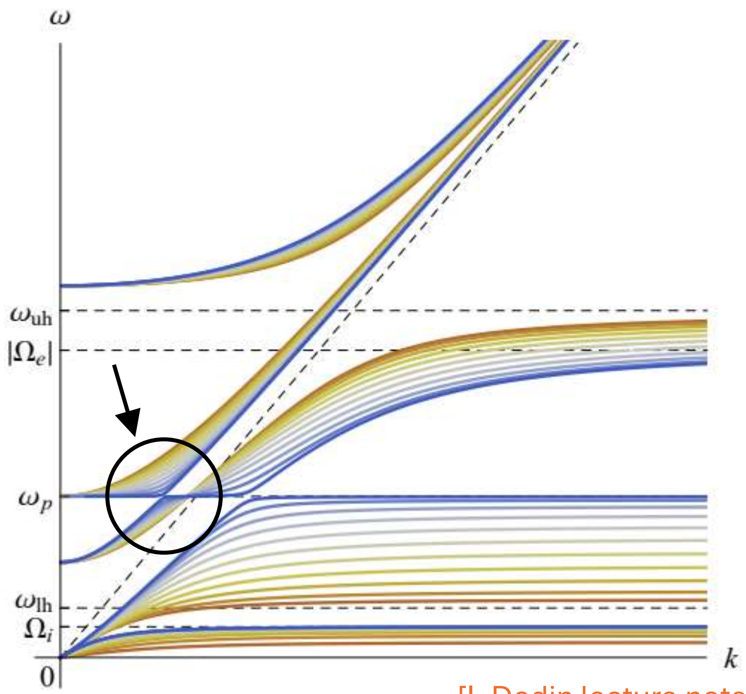
- Eigenmodes transition from L,R to O,X as ϴ changes from 0 to 90
  - (**blue**: $\theta = 0°$, transitioning to **red**: $\theta = 90°$)
- **But transition not smooth – crossings at $\boldsymbol{\theta = 0°}$ break apart for $\boldsymbol{\theta \neq 0°}$**

Mode-conversion

$$D = \begin{pmatrix} \omega - \omega_c & \eta \\ \eta & k - k_c \end{pmatrix}$$



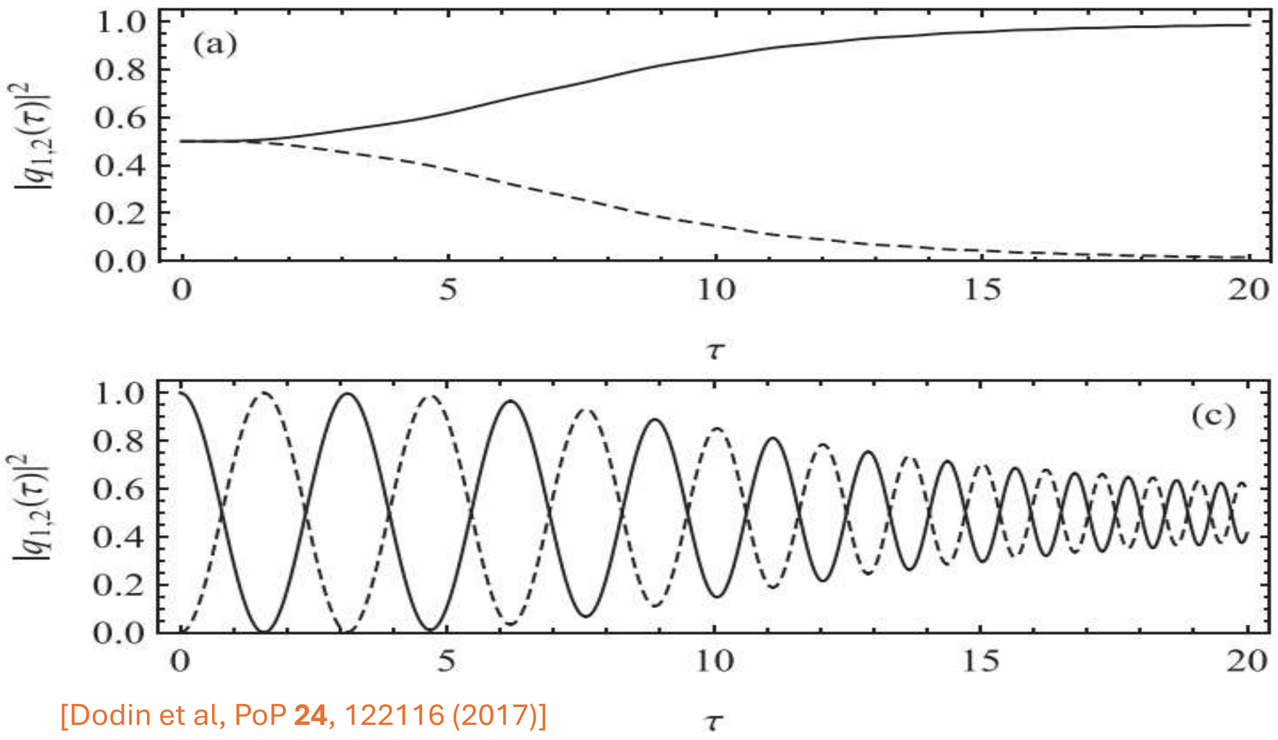




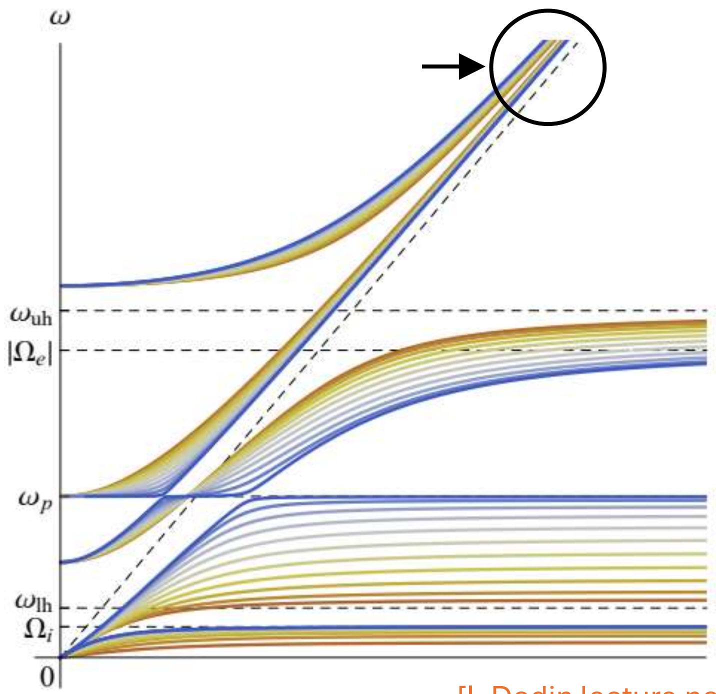
[I. Dodin lecture notes]

- Dispersion curves only cross for **exactly zero coupling** between modes
  - Nothing is ever exactly zero in reality, so generally curves cannot cross
- If gap is small enough, power can be transferred across $\sim \exp(-2\pi\eta^2)$
  - This is mode conversion!
    - (But what you call the 'converted' wave is matter of convention!)
- Rich **global** mode-conversion problems exist in plasmas
  - Ex: cutoff-resonance-cutoff triplet has wave interference in conversion efficiency

Mode-conversion



[Dodin et al, PoP **24**, 122116 (2017)]



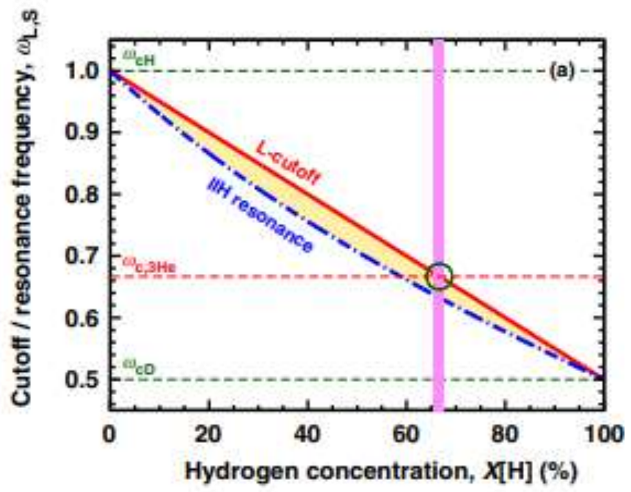
[I. Dodin lecture notes]

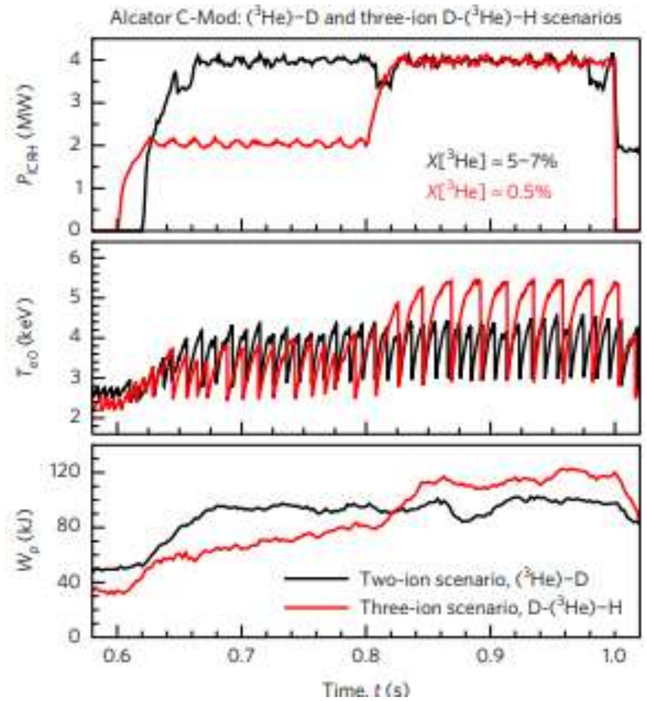
- Mode-conversion can also occur when dispersion curves are close, but not necessarily trying to cross each other
  - Can happen in low-density plasma (plasma edge!) where O & X are nearly identical (would be identical in vacuum)
- Launch **circularly polarized** waves to get **linear polarized O/X** in plasma

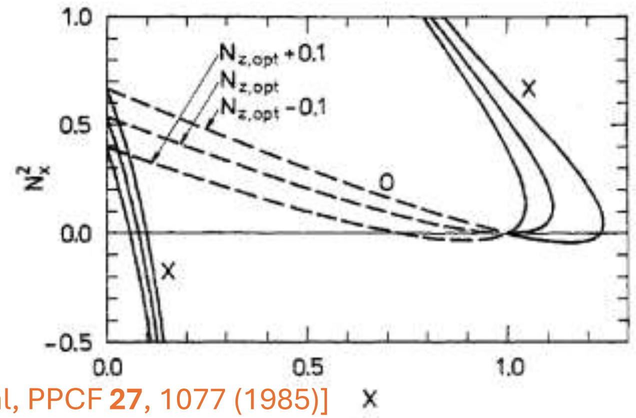
Getting waves to behave

- Must know **reflections**, **resonances**, & **mode-conversion** points to get waves where you want
  - ‘Standard schemes’ to do this are well-established
- New schemes continuing to be developed/refined:
  - Adjust fuel composition to put wave resonance near cyclotron resonance (3-ion heating)
    - Can deposit significant amounts of energy into fast ions
  - Inject waves at special angle to overlap two cutoffs (O-X-B conversion)
    - Allows EC waves to penetrate very dense plasma

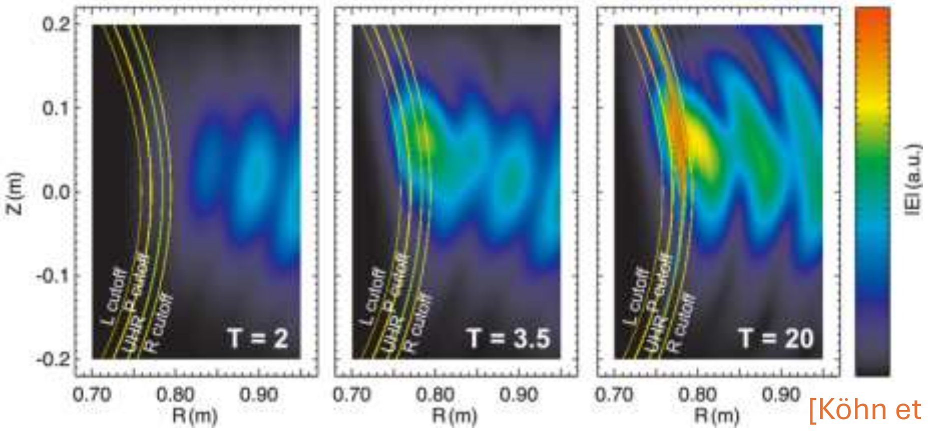
[Kazakov et al Nat. Phys. **13**, 973 (2017); Nucl. Fusion **55**, 032001 (2015)]







[Hansen et al, PPCF **27**, 1077 (1985)]



[Köhn et al, PoP **18**, 082501 (2011)]

Wave heating & current drive

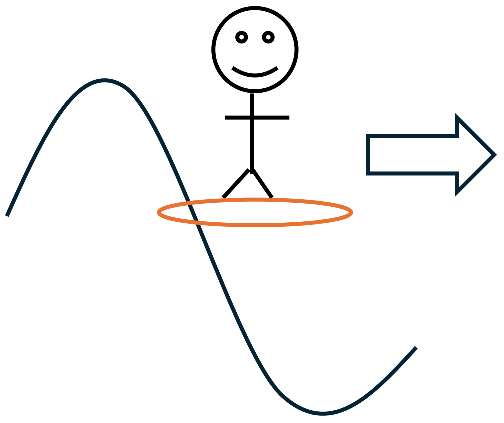
Heating plasmas with waves

- Waves can transfer energy to particles if resonance condition met:

$$\boxed{\omega - k_{\parallel} V_{\parallel} = n\Omega}$$

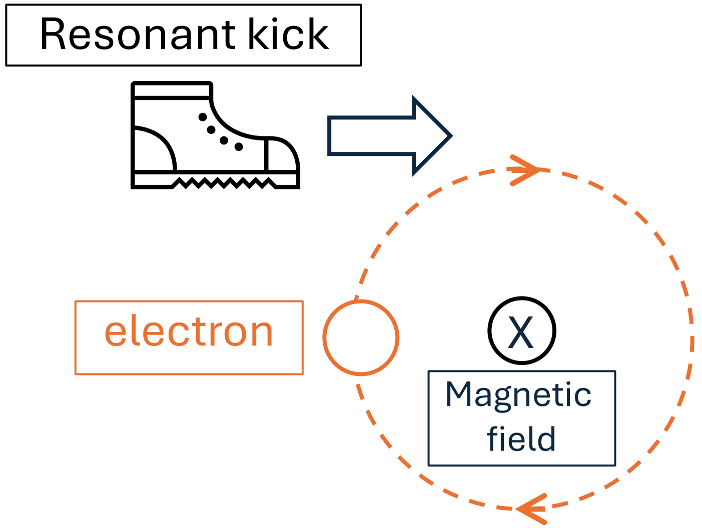
  - Doppler-shifted frequency must be a harmonic of cyclotron motion

Landau damping (n=0)



Accelerates particles parallel to B

Cyclotron damping (n≠0)



Accelerates particles perpendicular to B

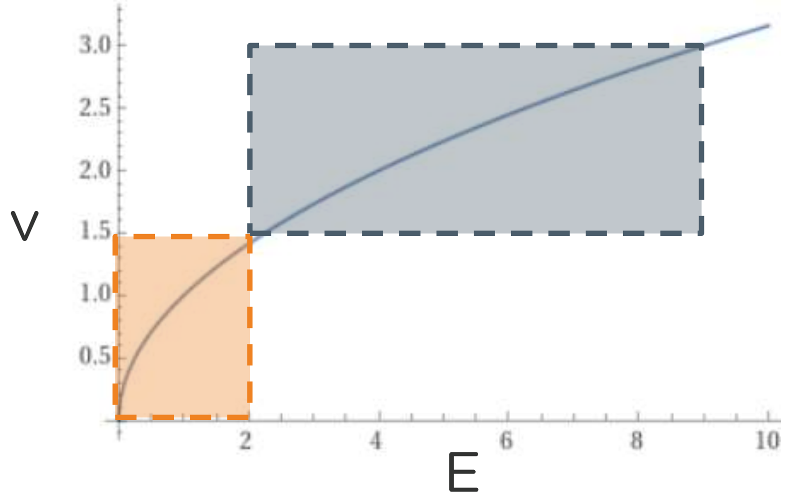
‘Easy’ to heat plasma, but driving current requires heating in special way

Easier to push slow electrons, efficient to push fast ones

- Kinetic energy: E = ½ m V^2

$$\boxed{\Delta E = m\, V\, \Delta V}$$

  - Thus, 'Easier' to push slow things



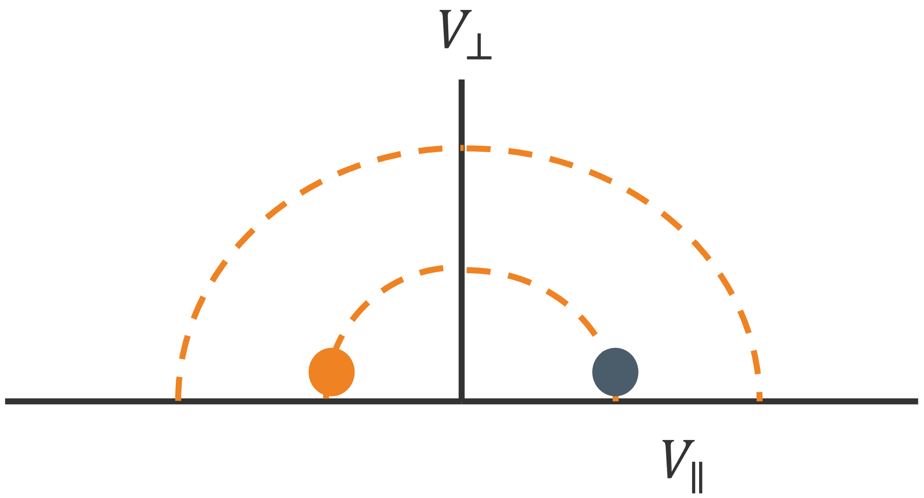
- But faster plasma particles collide less, need less frequent pushing

$$\boxed{\Delta P = \frac{\Delta E}{\Delta t} \sim \nu \Delta E \propto \frac{\Delta V}{V^2}, \qquad \frac{\Delta j}{\Delta P} \propto V^2}$$

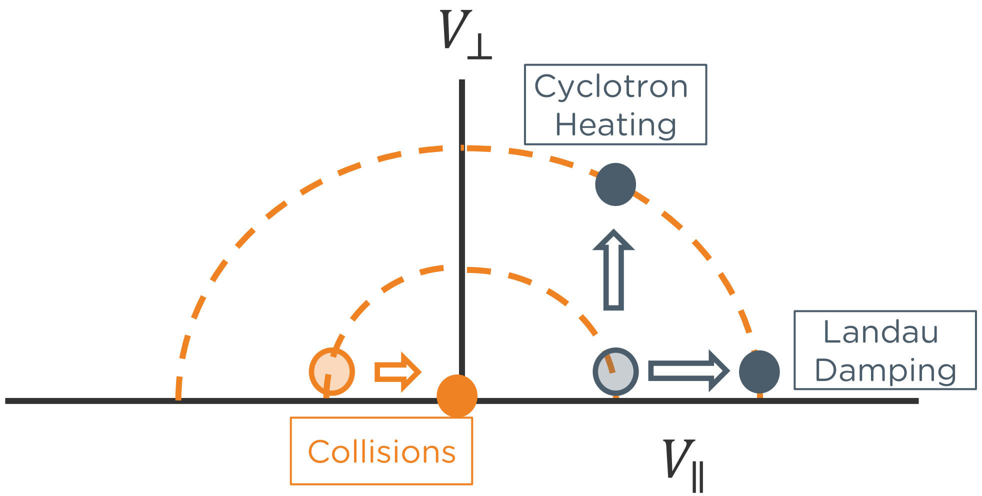
  - Thus, 'Efficient' to push fast things
  - (Recall $\nu \sim 1/V^3$ for plasma collisions)

Landau & Cyclotron damping can drive current

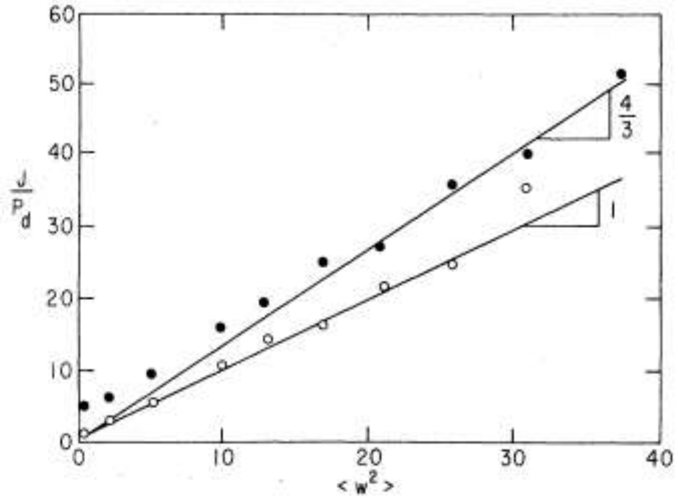
Average $V_{\|}$=0



Average $V_{\|}$ >0



- Because hotter particles collide less, **asymmetric** heating drives current
  - **Landau damping** (parallel pushes) directly drives current
  - **Cyclotron heating** (perpendicular pushes) drives current after collision time



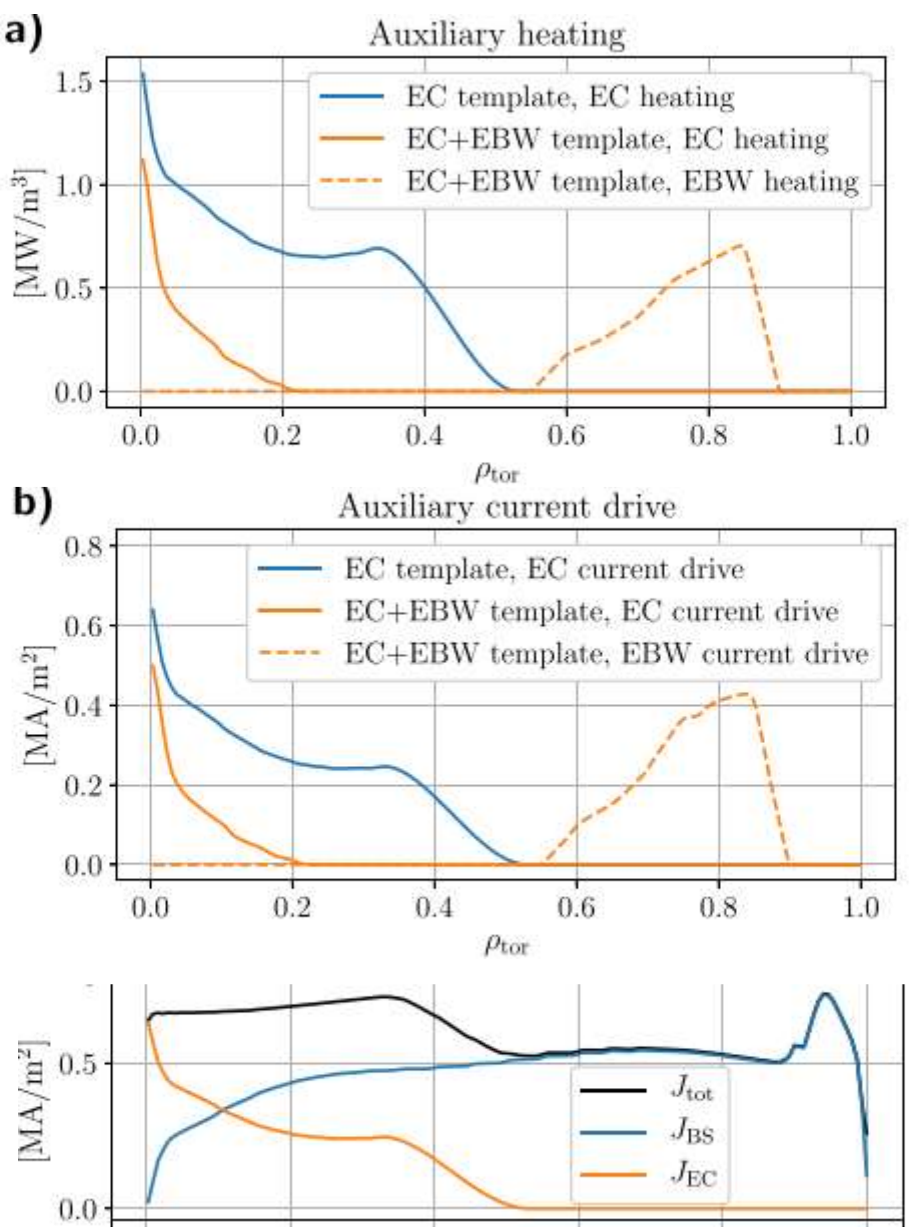
Landau (black) 4/3x better than Cyclotron (white)

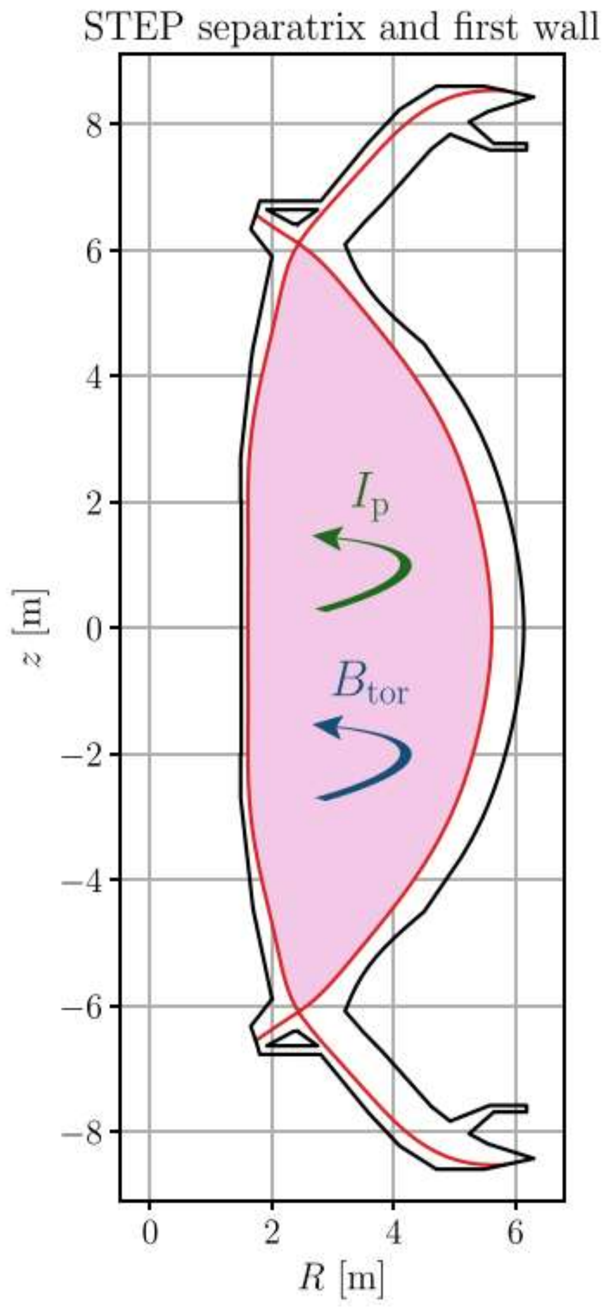
[Fisch, RMP **59**, 175 (1987)]

Ex: STEP

		EC [86]	EC+EBW [87]
P_{fus}	[GW]	1.65	1.77
Q_{sci}	[ ]	10.94	13.60
P_{EC}	[MW]	150.00	9.00
P_{EBW}	[MW]	—	120.00
$\langle n_e \rangle$	$[10^{19}\ \mathrm{m}^{-3}]$	15.40	17.35
$\langle n_e \rangle_{\mathrm{line}}/n_{\mathrm{GW}}$	[%]	100.13	100.02
$\langle T_e \rangle$	[keV]	9.81	9.05
$\langle T_i \rangle / \langle T_e \rangle$	[ ]	1.07	1.04
I_p	[MA]	20.10	22.88
I_{BS}/I_p	[%]	87.88	81.74
I_{EC}	[MA]	2.17	0.15
I_{EBW}	[MA]	—	3.95
$I_{\mathrm{EC}}/P_{\mathrm{EC}}$	$[\mathrm{kA\ MW}^{-1}]$	14.48	16.14
$I_{\mathrm{EBW}}/P_{\mathrm{EBW}}$	$[\mathrm{kA\ MW}^{-1}]$	—	32.89
β_N	[ ]	4.40	3.93
B_{axis}	[T]	2.59	2.57
l_i	[ ]	0.50	0.39
$l_i(3)$	[ ]	0.28	0.22
q_{min}	[ ]	2.18	3.14
q_{95}	[ ]	9.35	8.00
P_{rad}	[MW]	336.02	336.72

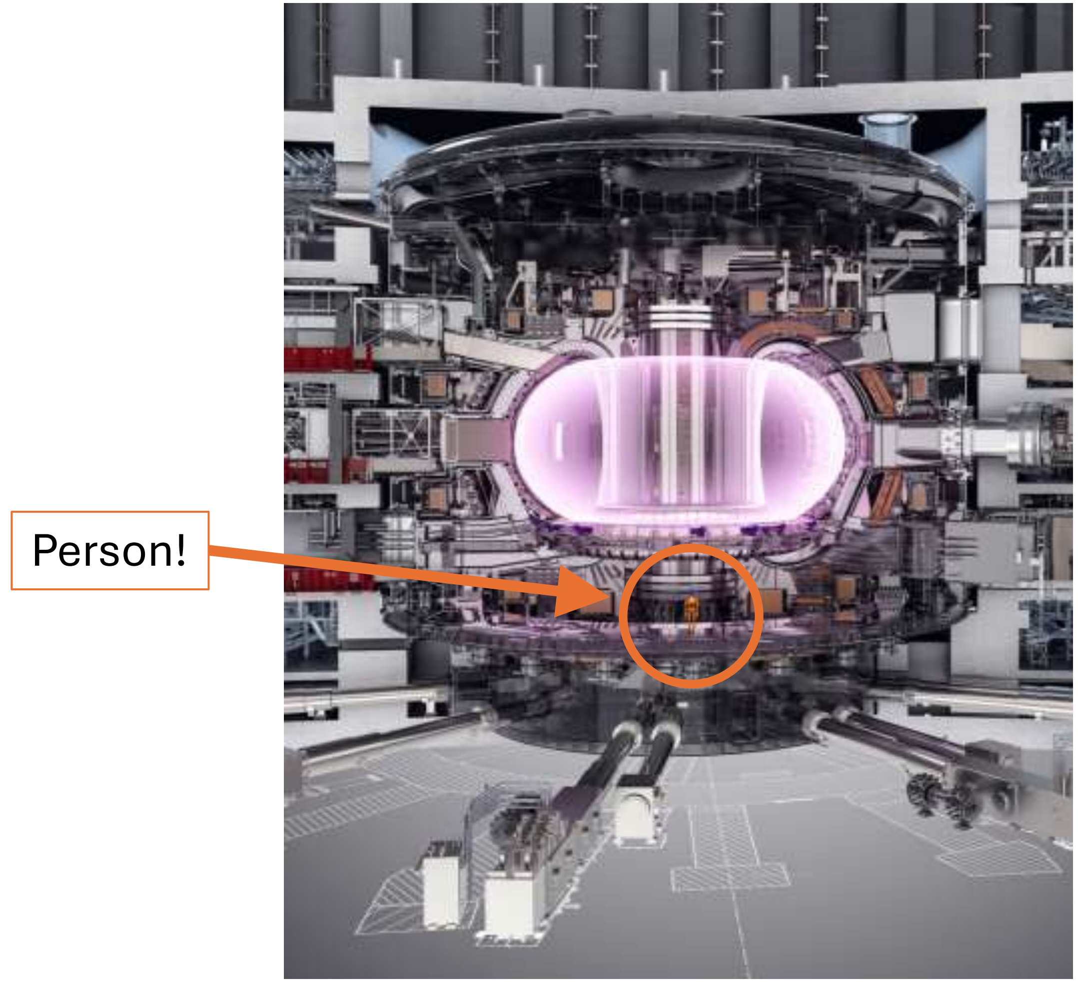
[Tholerus et al, NF **64**, 106030 (2024)]

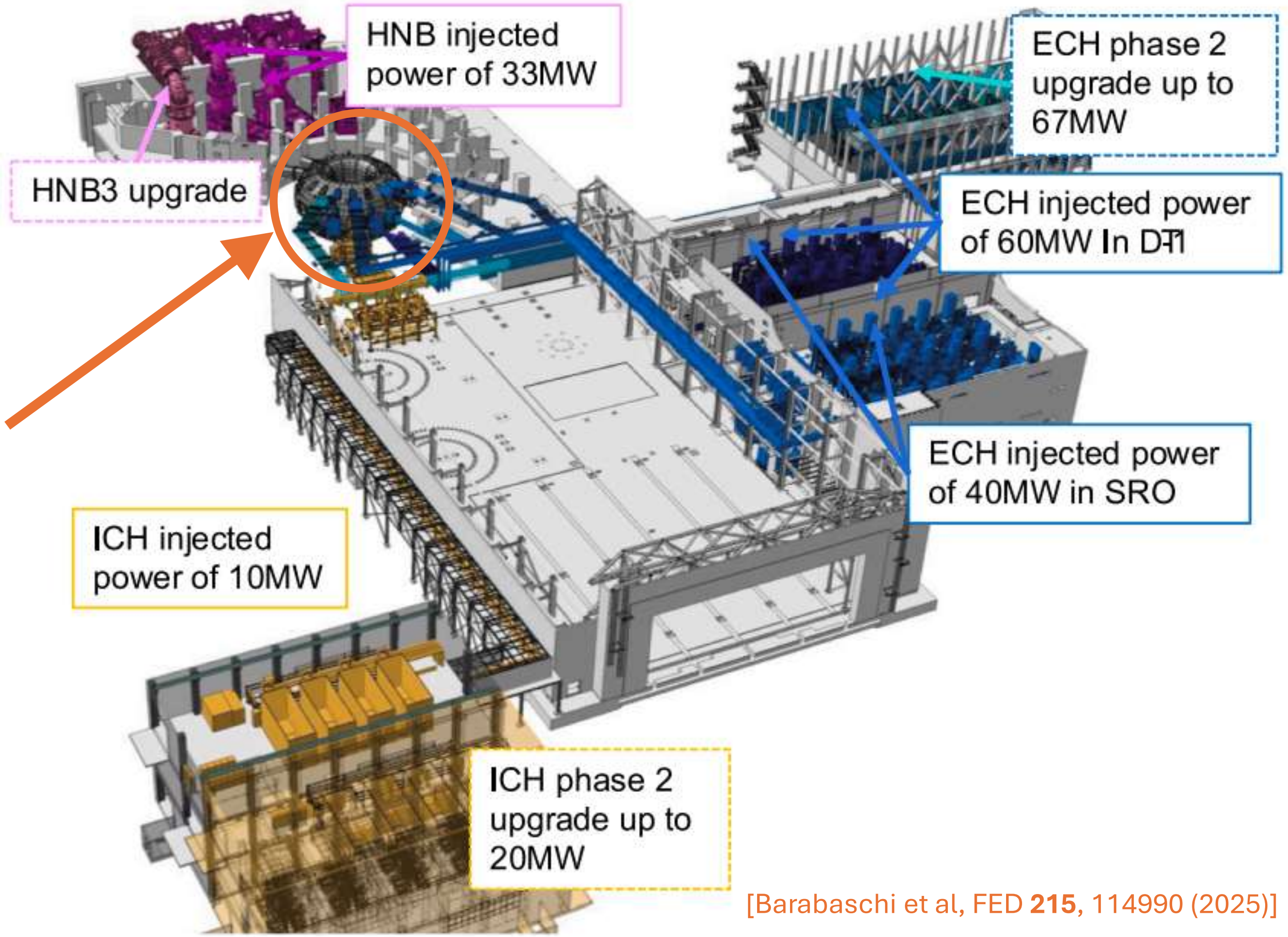




Ex: ITER

[ITER website]





[Barabaschi et al, FED **215**, 114990 (2025)]

- ‘Plasma’ is only small fraction of total heating system! Also physics of:
  - **Sources** (ex: gyrotron efficiency),
  - **Delivery system** (ex: efficient waveguide bends)
  - **Launching system** (ex: IC antenna designs)

Summary

- Externally launched plasma waves will be an important actuator in future fusion devices to provide heating and current drive
- Plasma is a rich medium that can support a variety of waves with complicated dispersion relations
  - It can take some creativity to get waves to go where you want them to
    - Dispersion relation depends on **propagation angle**, **wave frequency**, and **plasma** (density, magnetic field, temperature, fuel composition, etc.)
- Mode conversion can transfer energy from one type of wave to another at special locations
  - May be good ('pass the baton' to wave with better accessibility) or may be bad
- Waves can transfer energy to particles when resonance condition is met
  - If this transfer is given to particles moving preferentially in one direction versus the opposite direction, toroidal current can be driven as well