Introduction to Plasmas, Part I I

Single Particle Motion

Priyanjana Sinha, Princeton Plasma Physics Laboratory

Email: psinha@pppl.gov

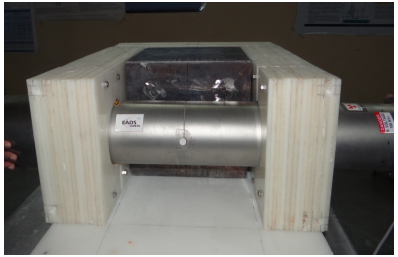
SULI Introductory Course on Plasma Physics
6th June 2023

My journey so far....

2009 -2012 | 2012 -2015 | 2015-2019 | (since Sep. 2019)

Undergrad: Hansraj College,
University of Delhi

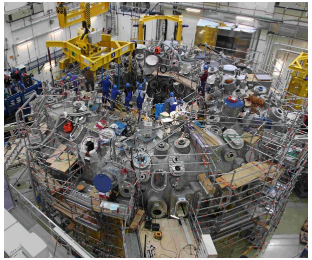
Masters in Nuclear Sci. & Tech
University of Delhi



Institute for Plasma Research, Gujarat
Master thesis project

IPP Max Planck Institute for Plasma Physics

Ph.D in plasma physics



Worked on the world's largest stellarator W7-x located in Germany

PPPL
PRINCETON
PLASMA PHYSICS
LABORATORY

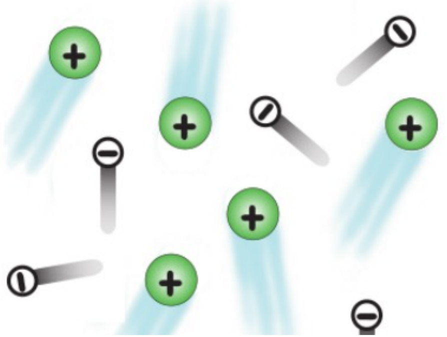
Associate Research
Physicist

Need for plasma confinement!

We require high temperatures for fusion roughly T ∽ 5 − 10 keV or 50 − 100 M Kelvin

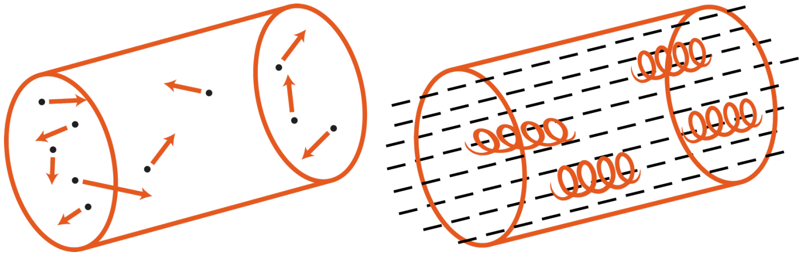
For ions: $k_B T = \frac{1}{2} \; v_{th}{}^2$

$\mathbf{F} = -\nabla p$



$v_{th} = \; 6\,X\,105\; m/s \longrightarrow$ very high velocities

Stop the motion from high p to low p region
(serious issue!) → confine the plasma



Popular solution: use magnetic field

→ Need to understand e⁻ and ion motion in inhomogeneous magnetic field

Outline

- Motion of particle in uniform magnetic field
- Magnetic mirrors
- Motion of particle in crossed fields
  - drift of charged particle in E ⊥ B
  - drift of charge particle due to ∇B
  - drift of charge particle due to curvature in B
- Tokamaks

Motion of particles in uniform magnetic field

Motion of particles in uniform magnetic field – Simple case

Force experienced by charged particle (q) with velocity $\boldsymbol{v}$ in presence of electric fields and magnetic field

Lorentz force: $\boxed{\mathbf{F} = q(\boldsymbol{E} + \boldsymbol{v}\, X\, \boldsymbol{B})}$

For electron: charge = - e mass = m

Assuming external magnetic field in z direction:

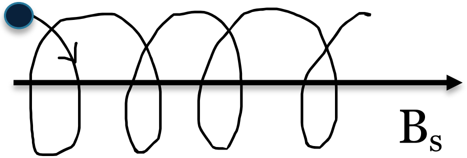
$$\mathbf{B_s} \,||\, \hat{z} \text{ and } E = 0$$

$$m\,\frac{d\boldsymbol{v}}{dt} = -\,e\,(\boldsymbol{v} \times \mathbf{Bs})$$

$$\frac{d\boldsymbol{v}}{dt} = -\,\omega_c\,(\boldsymbol{v} \times \hat{z})$$

Gyro-frequency or cyclotron frequency:

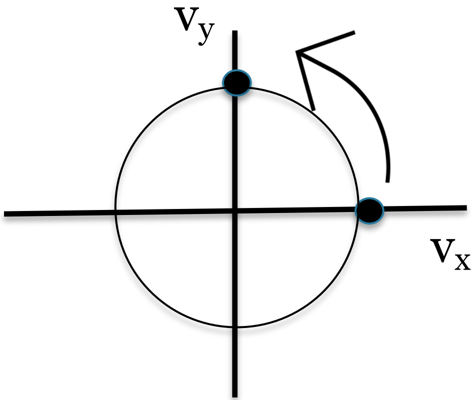
$$\boxed{\omega_c = \frac{eB_s}{m}}$$



Motion of particles in uniform magnetic field – Simple case

$$\left.\begin{array}{l} x \text{ direction : } \dfrac{dv_x}{dt} = - \omega_c v_y \\[2ex] y \text{ direction : } \dfrac{dv_y}{dt} = \omega_c v_x \end{array}\right\} \longrightarrow \text{coupled}$$

$$z \text{ direction : } \frac{dv_z}{dt} = 0$$



$$\frac{d^2v_x}{dt^2} = - \omega_c^2 v_x \qquad \frac{d^2v_y}{dt^2} = - \omega_c^2 v_y$$

$$v_x = v_\perp \cos(\omega_c t + \delta) \qquad v_y = + v_\perp \sin(\omega_c t + \delta)$$

where: $v_\perp^2 = v_x^2 + v_y^2$

assume $\delta = 0$

at $t = 0, \quad v_x = v_\perp, \quad v_y = 0$

at $t = \frac{\pi}{2}\frac{1}{\omega_c}, \quad v_x = 0, \quad v_y = v_\perp$

Motion of particles in uniform magnetic field – Simple case

For ions: charge = e mass = m_i

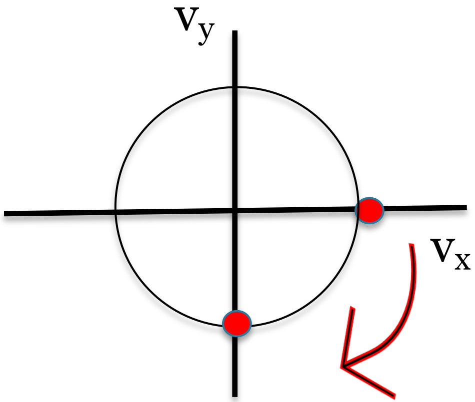
$$m_i \frac{dv}{dt} = e\,(\boldsymbol{v} \times \mathbf{Bs})$$

$$\frac{dv}{dt} = \omega_{ci}\,(\boldsymbol{v} \times \hat{z})$$

Gyro-frequency for ion:

$$\boxed{\omega_{ci} = \frac{eB_s}{m_i}}$$

→ much smaller than e$^-$



x direction : $\frac{dv_x}{dt} = \omega_{ci}\, v_y$

y direction : $\frac{dv_y}{dt} = -\,\omega_{ci}\, v_x$

z direction : $\frac{dv_z}{dt} = 0$

$$\left.\begin{matrix} \frac{dv_x}{dt} = \omega_{ci}\, v_y \\ \frac{dv_y}{dt} = -\,\omega_{ci}\, v_x \end{matrix}\right\} \longrightarrow \text{coupled}$$

$$v_x = v_\perp \cos(\omega_{ci} t + \delta)$$

$$v_y = -\,v_\perp \sin(\omega_{ci} t + \delta)$$

Taking into account the charge of particle

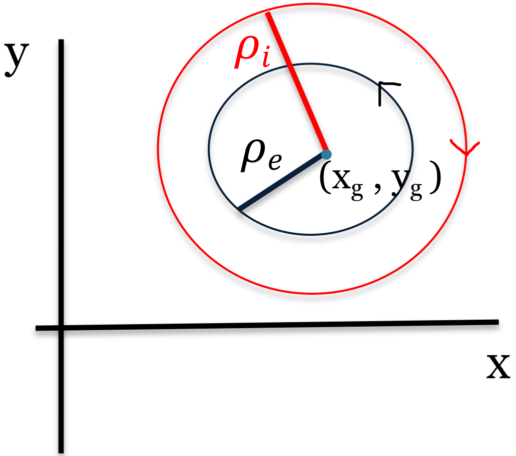
Motion of particles in uniform magnetic field – Simple case

$$\frac{dx}{dt} = v_x = v_\perp \cos\left(\omega_{c\ or\ ci}\, t + \delta\right)$$

$$\frac{dy}{dt} = v_y = \underset{-}{+}\, v_\perp \sin\left(\omega_{c\ or\ ci}\, t + \delta\right)$$

$$x = x_g + \frac{v_\perp}{\omega_c} \sin\left(\omega_{c\ or\ ci}\, t + \delta\right)$$

$$y = y_g \underset{+}{-} \frac{v_\perp}{\omega_c} \cos\left(\omega_{c\ or\ ci}\, t + \delta\right)$$



$$(x - x_g)^2 + (y - y_g)^2 = \left(\frac{v_\perp}{\omega_c}\right)^2$$

Equation of a circle with center (x_g, y_g)

Larmor radius for e$^-$ = $\rho_e = \dfrac{v_\perp}{\omega_c} \approx m_e^{1/2}\ T_e^{1/2}$

Larmor radius for ion = $\rho_i = \dfrac{v_\perp}{\omega_{ci}} \approx m_i^{1/2}\ T_i^{1/2}$

$$\boxed{as\ m_i > me\ \rightarrow \rho_i > \rho_e}$$

Motion of particles in uniform magnetic field – Simple case

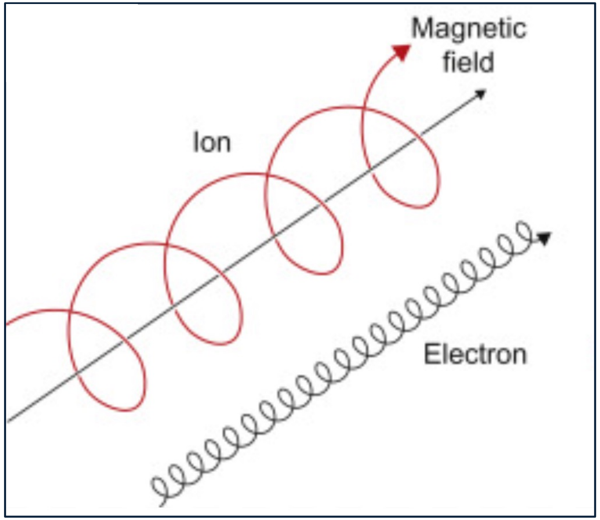
z direction : $\frac{dv_z}{dt} = 0$

$$v_z = vo_z$$

$$\frac{dz}{dt} = v_{oz}$$

$$z = zg + \ voz\ t$$

→ Charge particles move freely along z axis



$$\boldsymbol{r} = [\rho_{e\ or\ i}\sin\left(\omega_{c\ or\ ci}\,\mathrm{t} + \delta\right) + \mathrm{xg}\,]\,\hat{\mathrm{x}} \underset{+}{-} [\rho_{e\ or\ i}\cos\left(\omega_{c\ or\ ci}\,\mathrm{t} + \delta\right)] + \mathrm{y_g}\,]\,\hat{\mathrm{y}} + [z_g + \ voz\ t]\hat{z}$$

Some estimates for ITER like machine……

For ITER like machines: $T_i = T_e = 10$ keV and B = 5T

For ions : $v_{Ti} = \sqrt{\frac{k_B\, Ti}{m_i}}$ ≈ 700 km/s $r_L = \frac{m v_{Ti}}{qB} \approx$ 3 mm

For electrons : rL ≈ 0.05 mm (60 times smaller)

Gyro-frequency of electrons : $\omega_c = \frac{eB_s}{m}$ → frequency of wave to heat plasma for ions is much lower -- 1/1836 times

$$\omega_c = \frac{1.6\ x\ 10\ ^{-19}\ x\ 5}{9.1\ x\ 10\ ^{-}31}$$

$$\omega_c = 8.5\ x\ 10\ 11\ rad/s$$

$\omega_c = 135\ GHz$ → to heat the electrons in plasma the wave should be of this frequency

Magnetic Mirrors

Magnetic mirrors - role of B_r field in confinement

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\frac{1}{r}\frac{\partial(rBr)}{\partial r} + \frac{\partial B_z(z)}{\partial z} = 0$$

→ B depends on on r and z, no azimuthal dependence

$$r\,Br = -\frac{\partial B_z(z)}{\partial z}\frac{r^2}{2} + c_1$$

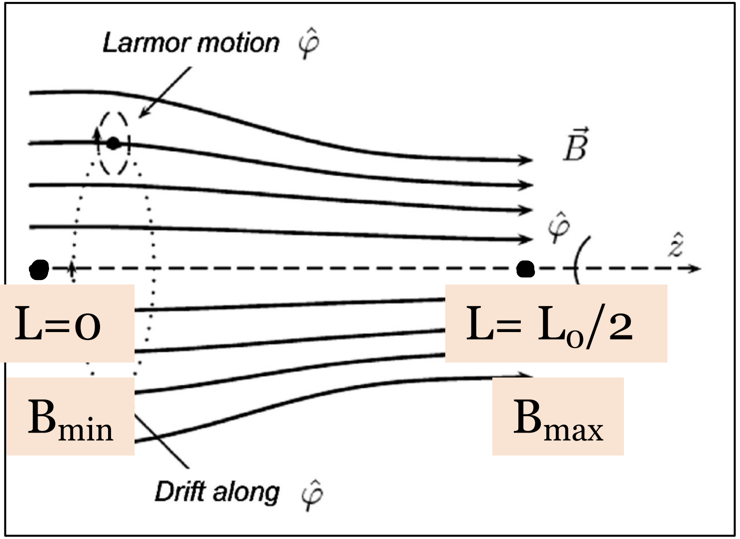
$$Br = -\frac{\partial B_z(z)}{\partial z}\frac{r}{2}$$

$$\boldsymbol{B} = \boldsymbol{B}_r + \boldsymbol{B}_z$$

$$m\frac{dv_z}{dt} = -q\,(\mathbf{v} \text{ x } \mathbf{B})_z$$

$$m\frac{dv_z}{dt} = -q\,(v_\varphi B_r)$$

→ exerts a force in - z direction



- Magnetic mirror was one of the earliest major approaches to fusion power
- Here, B_z is inhomogeneous and varying with z

Magnetic mirrors - role of B_r field in confinement

$$I = \frac{q}{T} = \frac{q\,\omega_c}{2\pi}$$

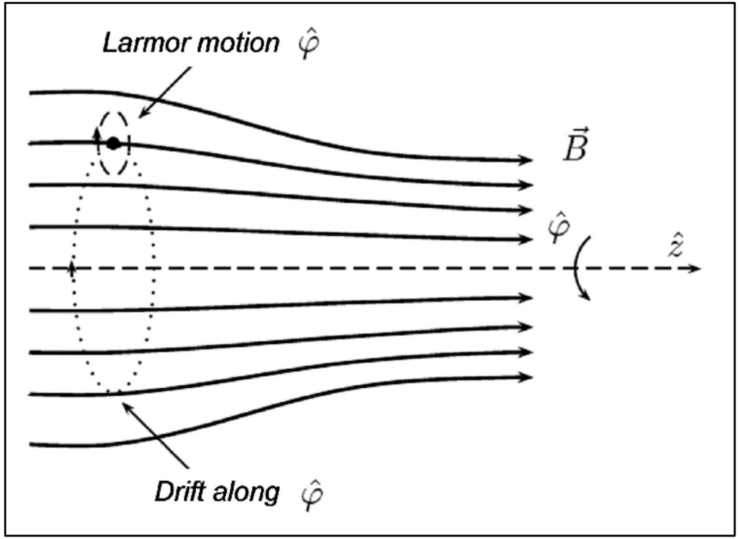
$$\text{Magnetic moment} = \mu = IA = \frac{q\,\omega_c}{2\pi}\,\pi\left(\frac{v_\perp}{\omega_c}\right)^2 = \frac{qv_\perp^{\ 2}}{2\omega_c} = \frac{mv_\perp^{\ 2}}{2B_{\ z}}$$

$$m\,\frac{dv_z}{dt} = \text{ - q}\,(\mathrm{v}_\varphi\ \mathrm{B_r})$$

$$Br = \text{ - }\frac{\partial B_z\,(z)}{\partial z}\,\frac{r}{2}$$

$$m\,\frac{dv_z}{dt} = \text{ - }\frac{qv_\perp^{\ 2}}{2\omega_c}\,\frac{\partial Bz}{\partial z} = \text{ - }\frac{mv_\perp^{\ 2}}{2B}\,\frac{\partial Bz}{\partial z}$$

$$m\,\frac{dv_z}{dt} = \text{ - }\mu\,\frac{\partial Bz}{\partial z}$$



$$\rho = \frac{v_\perp}{\omega_c}$$

Magnetic mirrors – energy conservation

$$E = \frac{1}{2} m\, v_{\perp}{}^{2} + \frac{1}{2} m\, vz^{2}$$

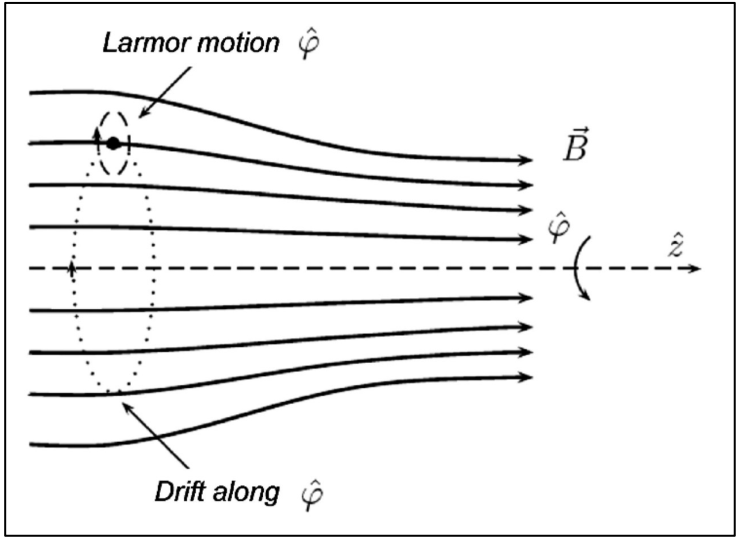
$$m \frac{dv}{dt} = q\,(\mathbf{v} \times \mathbf{B})$$

$$m\, \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} = 0$$

$\frac{1}{2} m\, v^{2} =$ constant of motion

Energy is constant, so $\frac{mv_{\perp}{}^{2}}{2} + \frac{mv_{z}{}^{2}}{2B}$ is constant

If $v_{\perp}$ increases, then v_z decreases



$B = B_r + B_z$

$B_r \rightarrow$ Very small

$B \approx B_z$

Magnetic mirrors – constant magnetic moment

$$\frac{d(\mu B)}{dt} = - \frac{1}{2}\frac{d(mvz^2)}{dt}$$

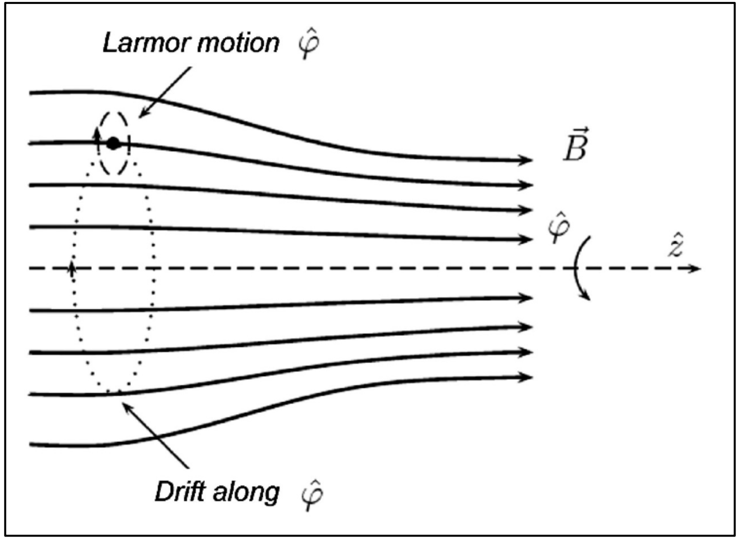
$$\frac{d(\mu B)}{dt} = -mv_z \frac{dv_z}{dt}$$

$$\frac{d(\mu B)}{dt} = mv_z\,(-\mu\,\frac{dB}{dZ})$$

$$\frac{d(\mu B)}{dt} = \mu\,\frac{dB}{dt}$$

$$\mu\,\frac{dB}{dt} + \mathrm{B}\frac{d\mu}{dt} = \mu\,\frac{dB}{dt}$$

$\boxed{\mathrm{B}\frac{d\mu}{dt} = 0}$ → μ is constant = $\frac{mv_\perp{}^2}{2B}$



$$\boxed{m\,\frac{dv_z}{dt} = -\,\mu\frac{\partial Bz}{\partial z}}$$

B increases, $v_\perp$ increases → v_z decreases (from energy conservation)

Magnetic mirrors – critical angle for confinement

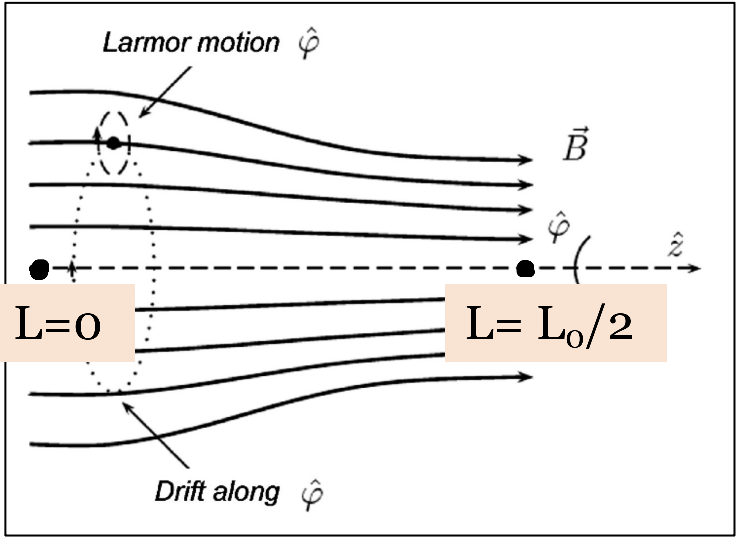
when $v_\perp$ = max = v_0, v_z = 0 → reverse its direction

→ particles bounce back and forth

→ trapped e/ions → confined

v_z not always zero at the throat → passing particles

What is the minimum value of θ that that will cause reflection?



$$\mu = \frac{mv_0^2}{2B_{max}} = \frac{mv_{0\perp}^2}{2B_{min}}$$

$$v_0 = \sqrt{v_{0\perp}^2 + v_{0z}^2}$$

$$\frac{v_{0\perp}^2}{v_0^2} = \frac{B_{min}}{B_{max}} = \frac{1}{m_R}$$

$$\mathrm{Sin}\ \theta_0 = \left(\frac{1}{m_R}\right)$$

$\theta = \theta_c$ = critical angle

at L = 0
$v_\perp = v_{0\perp}$
$v_z = v_{0z}$

at L = L₀/2
$v_\perp = v_0$
$vz = 0$

Motion of particle in crossed field $\mathbf{E} \perp \mathbf{B}$

Motion of particle in crossed field E ⊥ B

Understand the dynamics of charged particles in inhomogeneous field

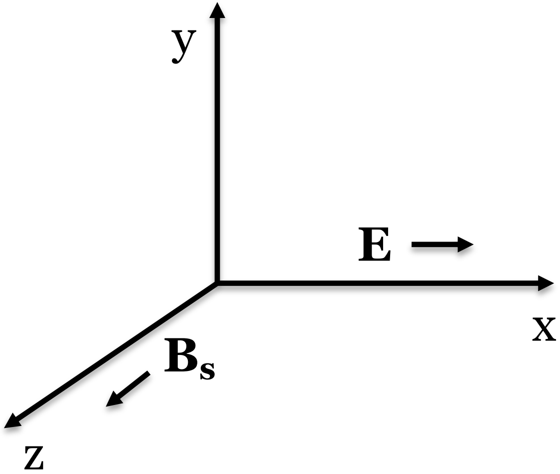
External fields: $\mathbf{B_s} \,||\, \hat{z}$ and $\mathbf{E} \,||\, \hat{x}$

For electron: $m\frac{d\boldsymbol{v}}{dt} = -\,e\mathbf{E} - e\,(\boldsymbol{v} \times \mathbf{B_s})$

X- component: $m\frac{dv_x}{dt} = -\,eE_x - e\,(v_y B_s)$

Y- component: $m\frac{dv_y}{dt} = -\,e\,v_x B_s$

$$v_y{}' = vy + \frac{E_x}{B_s} \rightarrow v_y = v_y{}' - \frac{E_x}{B_s}$$



Motion of particle in crossed field E ⊥ B

X- component: $$m\frac{dv_x}{dt} = -\,e\,(v_y{}'\,\text{x}\,B_s)$$

Y- component: $$m\frac{dv_y{}'}{dt} = e\,v_x\,B_s$$

$$v_x = v_\perp \cos(\omega_c t + \delta)$$

$$v_y{}' = v_\perp \sin(\omega_c t + \delta)$$

$$v_y = v_\perp \sin(\omega_c t + \delta) - \boxed{\frac{E_x}{B_s}} \longrightarrow \text{Additional force}$$

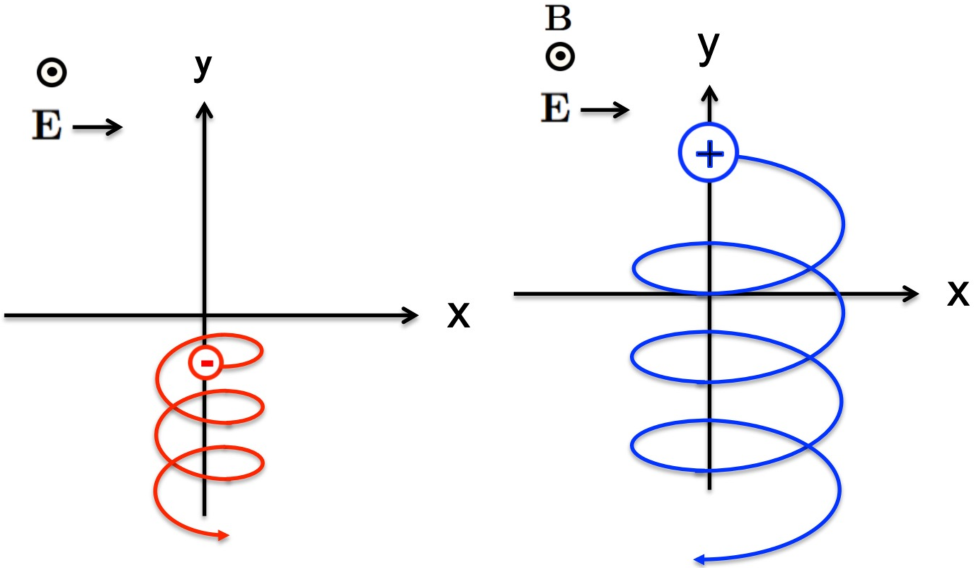
$$v_D = -\frac{E_x}{B_s} = -\left(\frac{\boldsymbol{E}_x\,\text{X}\,\boldsymbol{B}_s}{B_s{}^2}\right) \longrightarrow \text{DC drift of the particle which is independent of charge and mass}$$

Motion of particle in crossed field E ⊥ B

$$v_D = -\frac{E_x}{B_s} = \left(\frac{\boldsymbol{E_x} \,\mathrm{X}\, \boldsymbol{B_s}}{B_s{}^2}\right)$$

Takes particle away from lines of force/magnetic field region → serious problem!

Both e⁻ and ions moves together in – y direction



Courtesy: Dr. Cami Collins

Generalized form of drift caused by external force F $\perp B_s$

$$m\,\frac{dv}{dt} = -\,\mathrm{e}\,(\mathbf{v}\ \mathrm{x}\ \mathbf{B_s}) + \mathbf{F}$$

$$\boldsymbol{F} = \frac{\boldsymbol{B_s}\mathrm{x}\,(\boldsymbol{F}\ \mathrm{x}\ \boldsymbol{B_s})}{B_s^{\ 2}} = \frac{F(\boldsymbol{B_s}\cdot\boldsymbol{B_s})}{B_s^{\ 2}} - \cancel{\frac{B_s\,(\boldsymbol{F}\cdot\boldsymbol{B_s})}{B_s^{\ 2}}}$$

$$\boldsymbol{F} = -\,\frac{(\boldsymbol{F}\ \mathrm{x}\ \boldsymbol{B_s})\ \mathrm{x}\ \boldsymbol{B_s}}{B_s^{\ 2}}$$

$$\boxed{m\,\frac{dv}{dt} = -\mathrm{e}\left[\mathbf{v} + \frac{\boldsymbol{F}\ \mathrm{x}\ \boldsymbol{B_s}}{B_s^{\ 2}}\right]\mathrm{x}\ \boldsymbol{B_s}}$$

assume: $\boldsymbol{v}' = \boldsymbol{v} + \dfrac{\boldsymbol{F}\ \mathrm{x}\ \boldsymbol{B_s}}{B_s^{\ 2}}$

$$m\,\frac{dv'}{dt} = -\,\mathrm{e}\,(\mathbf{v}'\ \mathrm{x}\ \mathbf{B_s})$$

$$\boxed{\boldsymbol{v}_{drift} = \frac{\boldsymbol{F}\ \mathrm{x}\ \boldsymbol{B_s}}{qB_s^{\ 2}}}$$

general expression

Other forces that can cause drift

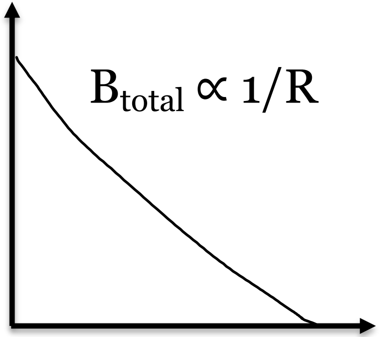
$$\boldsymbol{v}_{\mathrm{drift}} = \frac{\boldsymbol{F} \times \boldsymbol{B}_s}{qB_s^{\,2}}$$

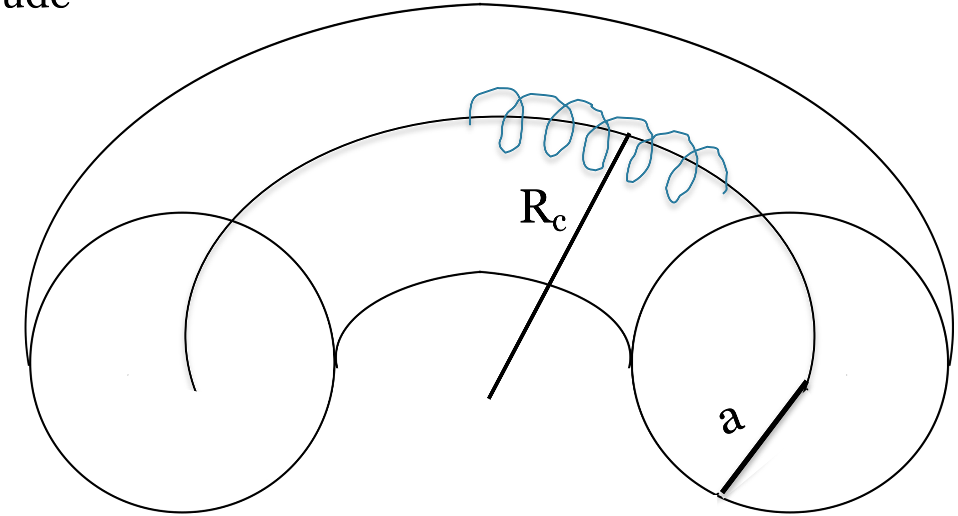
Gravitational force = F = mg

$m_i >> m_e \rightarrow \boldsymbol{v}_{di} \gg \boldsymbol{v}_{de}$

For electric force, this drift is similar in magnitude

Two other types of drifts that is present in tokamak: curvature drift and ∇ B





Drift of charge particle due to ∇ B

Drift of charge particle due to ∇ B

assume $B_s = B_s(x)\,\hat{z}$ → x dependence

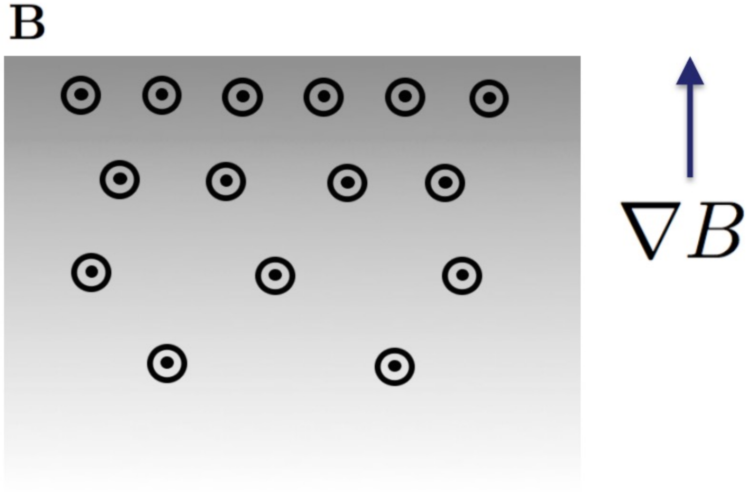
For electron: $m\,\frac{dv}{dt} = -\,e\,v \times B_s(x)\,\hat{z}$

$$B_s(x) = B_s(x_g + (x - x_g))$$

$$= B_s(x_g) + \frac{\partial Bs}{\partial x}\Big|_{\text{at } x = x_g}(x - xg)$$

$$= B_s(x_g) + \frac{\partial Bs}{\partial x}\,\frac{v_\perp}{\omega_c}\,\sin(\omega_c\,t)$$

$$m\,\frac{dv}{dt} = -\,e\,v \times \hat{z}\left(B_s(xg) + \frac{\partial Bs}{\partial x}\,\frac{v}{w_c}\,\sin(\omega_c\,t)\right)$$



Drift of charge particle due to Grad B drift

X direction:

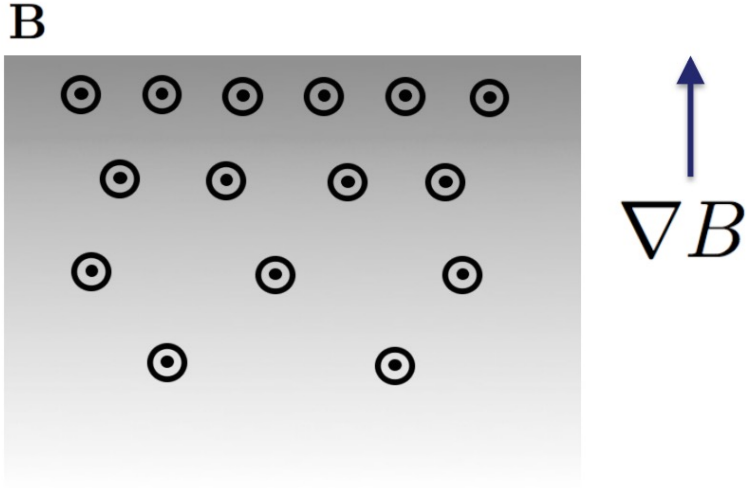
$$m\frac{dv_x}{dt} = -\,e\,v_y\left(B_s\,(xg) + \frac{\partial Bs}{\partial x}\,\frac{v_\perp}{w_c}\,\sin(\omega_c t)\right)$$

$$m\frac{dv_x}{dt} = -\,e\,v_y\,B_s\,(xg) - e\,v_\perp\,\sin(\omega_c t)\,\frac{\partial Bs}{\partial x}\,\frac{v_\perp}{w_c}\,\sin(\omega_c t$$

$$m\frac{dv_x}{dt} = -\,e\,v_y\,B_s\,(xg) - \boxed{\frac{e\,v_\perp^{\,2}}{2\omega_c}\,\frac{\partial Bs}{\partial x}}$$

$$m\frac{dv_x}{dt} = -\,e\,v_y\,B_s\,(xg) + F_x$$

Additional force



Y direction:

$$m\frac{dv_x}{dt} = -\,e\,v_x\left(B_s\,(xg) + \frac{\partial Bs}{\partial x}\,\frac{v_\perp}{\omega_c}\,\sin(\omega_c t)\right)$$

$$m\frac{dv_x}{dt} = -\,e\,v_x\,B_s\,(xg) - e\,v_\perp\,\cos(\omega_c t)\,\cancel{\frac{\partial Bs}{\partial x}\,\frac{v_\perp}{\omega_c}}\,\sin(\omega_c t)$$

$$v_x = v_\perp \cos(\omega_c t)$$

$$v_y = v_\perp \sin(\omega_c t)$$

Drift of charge particle due to Grad B drift

$$\mathbf{F} = \frac{qv_{\perp}^{2}}{2\omega_{c}}\nabla B_{s} = \frac{mv_{\perp}^{2}}{2B_{s}}\nabla Bs$$

$$\boldsymbol{v}_{\mathrm{D}} = \frac{\boldsymbol{F} \text{ x } \boldsymbol{B_s}}{qB_{s}^{2}}$$

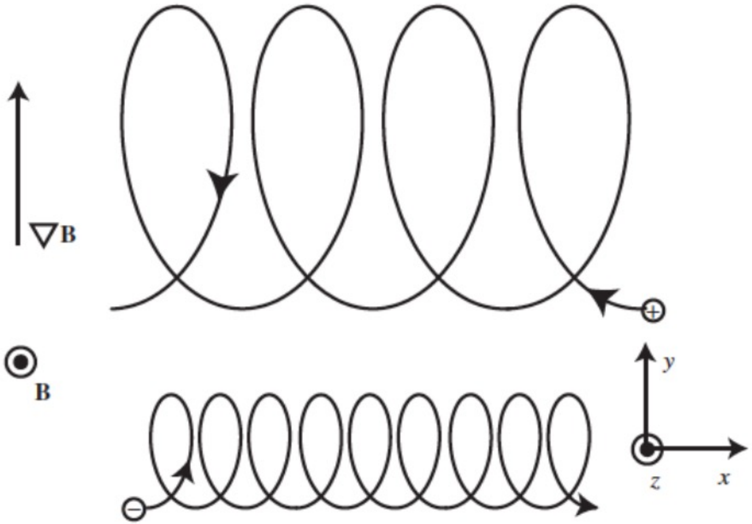
For ion : $\boldsymbol{v}_{\nabla \mathrm{B}} = \frac{mv_{\mathrm{i}\perp}^{2}}{2eB_{s}^{2}} \nabla Bs \text{ x } \mathbf{B_s}$

For e$^-$: $\boldsymbol{v}_{\nabla \mathrm{B}} = -\frac{m_{\mathrm{i}}v_{i\perp}^{2}}{2eB_{s}^{2}} \nabla Bs \text{ x } \mathbf{B_s}$

$m_i v_{\perp}^{2} \sim \mathrm{T_i}$ $\qquad$ $mv_{\perp}^{2} \sim \mathrm{T_e}$

If $\mathrm{T_e} = \mathrm{T_i} \rightarrow \boldsymbol{v}_{\nabla \mathrm{B}}$ are equal

Space charge separation is created, leads to **E XB** drift



Larmor radius: $\rho = \frac{v_{\perp}}{\omega_c} = \frac{m v_{\perp}}{eB_s}$

The gyro-radius will be smaller where $\mathrm{B_s}$ is larger and vice-versa

Drift of charge particle due to curvature in B

Drift of charge particle due to curvature in B

Assume $B_s \parallel \hat{\phi}$

Radial motion

$$m\,(r\ \ - r\dot{\phi}^2) = e\,(\mathbf{v}\ \mathrm{x}\ \mathbf{B})_r$$

$$m\,(r\ \ - r\dot{\phi}^2) = e\ \ v_z\, B_s$$

$$m\,\frac{dv_r}{dt} = e\ \ v_z\, B_s + mr\dot{\phi}^2$$

$$m\,\frac{dv_r}{dt} = e\ \ v_z\, B_s + \boxed{mr\dot{\phi}^2} \longrightarrow$$

<u>Additional force</u>

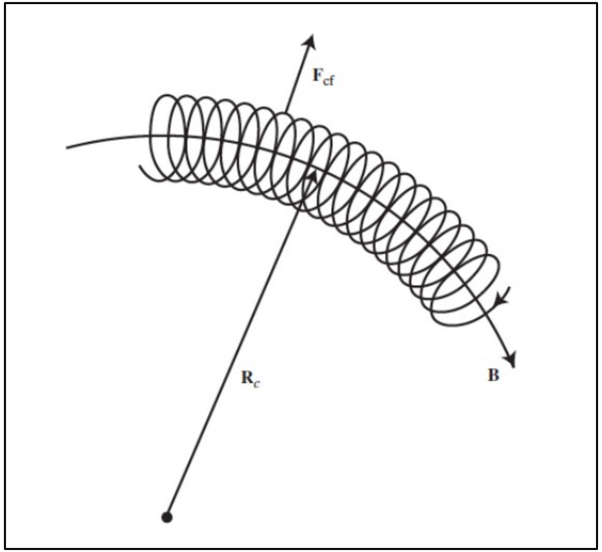
$$F = mr\dot{\phi}^2\ \hat{R}_c$$

$$F = mR_c\,\dot{\phi}^2\ \hat{R}_c$$

$$F = \frac{m\,v_{\parallel}^{\ 2}}{R_c}\ \hat{R}_c$$

Z motion

$$m\,\frac{dv_z}{dt} = -\ e\ \ v_r\, B_s$$



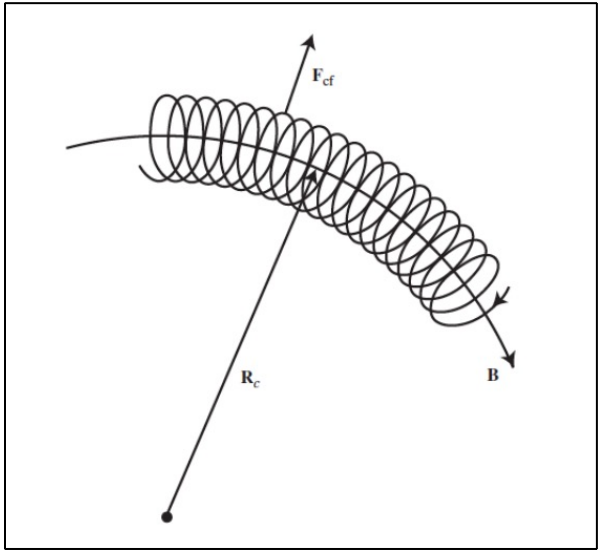
$$v_\phi = r\,\dot{\phi} = v_{\parallel}$$

Drift of charge particle due to curvature in B

$$\boldsymbol{v}_{curv} = \frac{\boldsymbol{F} \times \boldsymbol{B_s}}{qB_s{}^2}$$

For e⁻ : $\boldsymbol{v}_{curv} = -\frac{m\,v_{\parallel}{}^2}{eR_c}\frac{\hat{R}_c \times \mathrm{B_S}}{B_s{}^2}$

For ion : $\mathrm{v}_{curv} = \frac{m\,vi_{\parallel}{}^2}{eR_c}\frac{\hat{R}_c \times \mathrm{B_S}}{B_s{}^2}$



$$\mathrm{v}_{curv} \sim \frac{T}{R_c\,Bs}$$

If $T_e = T_i$, v_{curv} similar in magnitude for e⁻ and ions

Production of magnetic field is expensive

Total drift on charge particle due to curvature in B and grad B

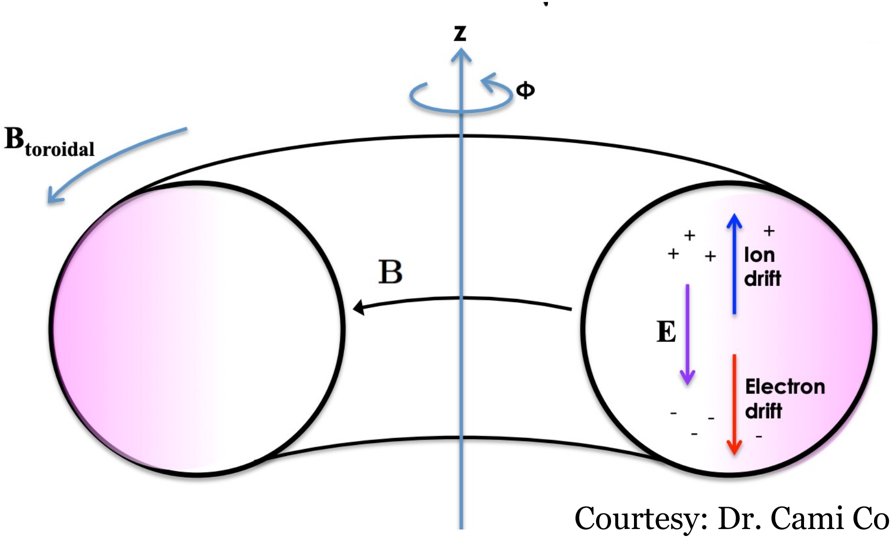
Gradient drift + curvature drift→ in same direction for each species, so they add up

$$\boldsymbol{v}_{curv} + \boldsymbol{v}_{\nabla \mathrm{B}} = \frac{m}{qR_c} \frac{\boldsymbol{R_c} \, \mathrm{x} \, \mathbf{B_S}}{B_s{}^2} \left(v^{\,2}_{\parallel} + \frac{1}{2} v^{\,2}_{\perp} \right)$$

Charge separation causes **E** to be generated

$$v_D = \left(\frac{\boldsymbol{E} \, \mathrm{x} \, \boldsymbol{B_s}}{B_s{}^{\,2}} \right)$$

Suppress the drift → tokamak configuration



Courtesy: Dr. Cami Collins

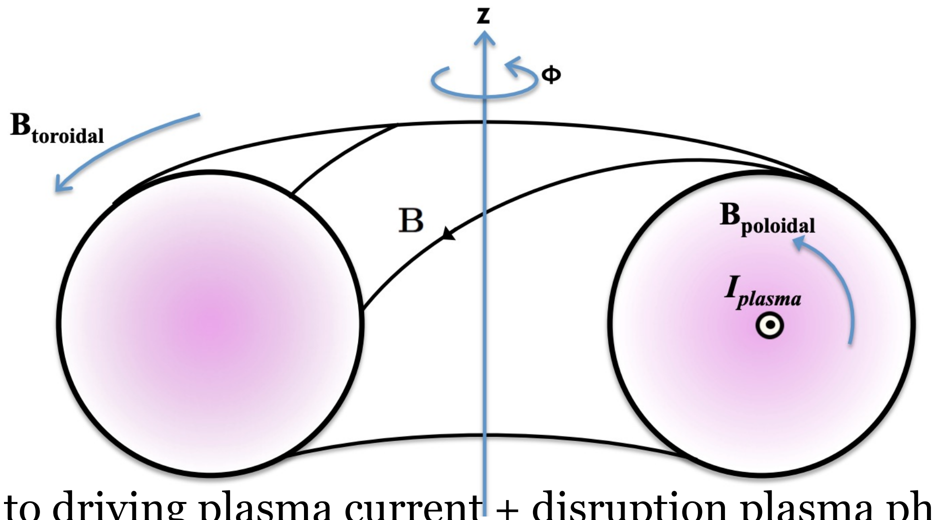
Tokamak solution: Add poloidal magnetic fields

Need to twist field lines helically to compensate particle drifts

Toroidal: long way around

Poloidal : short way around

1. Use external coils to produce toroidal magnetic field
2. Drive toroidal current in plasma to generate a poloidal magnetic field

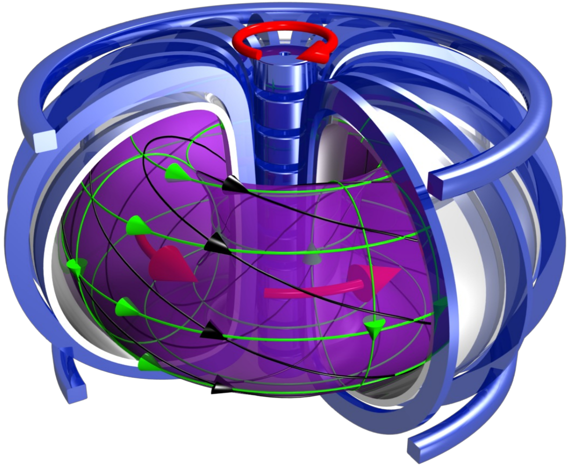


Issues related to driving plasma current + disruption plasma phenomenon a major problem!

Courtesy: Dr. Cami Collins

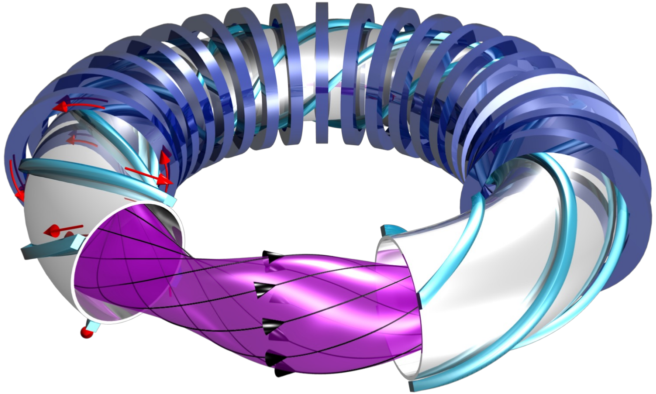
Difference between two leading magnetic confinement concepts

Tokamak (axisymmetric)



- Toroidal Field generated by external coils
- Poloidal field generated by plasma current
- Total Field has a twisted field structure

Stellarator (3D nonaxisymmetric)



- Twisted magnetic field is created by external twisted magnetic coils

Conclusions

- Charged particles undergo gyromotion about magnetic fields, and are free to move along the magnetic field line

- Depending on magnetic field geometry or the presence of other forces like electric fields, particles can drift across field lines (and even leave the system and hit the walls)

- Toroidal confinement devices have magnetic fields in both the poloidal and toroidal direction to compensate for these drifts