Columbia | Engineering
The Fu Foundation School of Engineering and Applied Science

Applied Physics and
Applied Mathematics
with Materials Science & Engineering

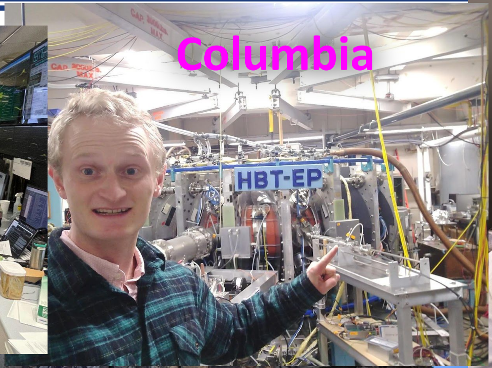
Plasma Waves

A. O. Nelson - June 7th, 2023

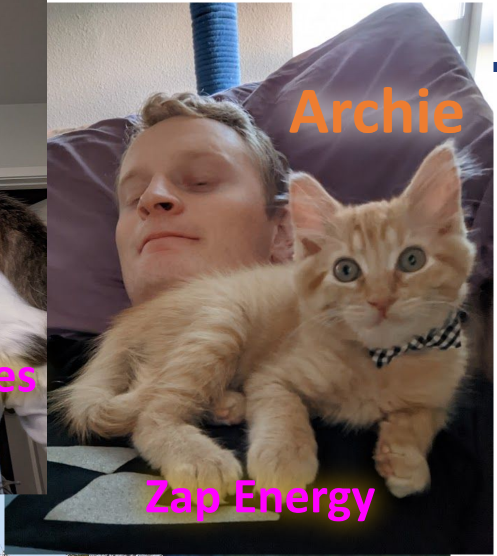
Introduction to Fusion Energy and Plasma Physics SULI Course

Who am I?


DIII-D


Columbia
HBT-EP


Tuna
TAE Technologies
I AM THE

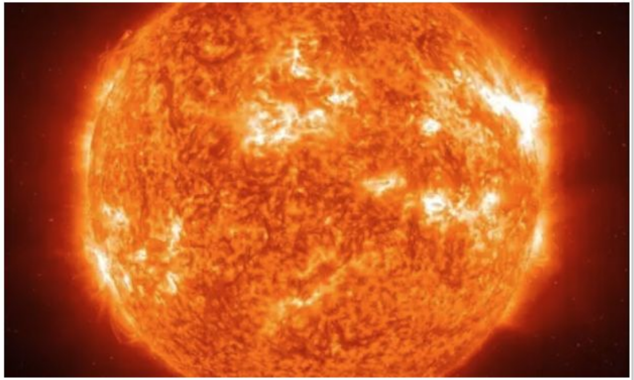

Archie
Zap Energy

My path to fusion started with some clickbait…

- **Undergraduate in Engineering Physics at CU Boulder**
  - Initial research in terahertz metrology
  - Inspired by some NIF results to check out fusion! (Thank you Facebook…)
  - No local experimental work, so studied abroad in Darmstadt, Germany!
  - Continued work in dusty plasmas at CU

- **Graduate School at Princeton**
  - Two years in plasma-wall interactions
  - Three years in H-mode tokamak plasmas

- **Finally made my way to Columbia**
  - Using remote work as opportunity to live closer to mountains ;)


This article is more than **9 years old**

Sustainable nuclear fusion breakthrough raises hopes for ultimate green energy

Scientists have moved a step closer to achieving sustainable nuclear fusion and almost limitless clean energy

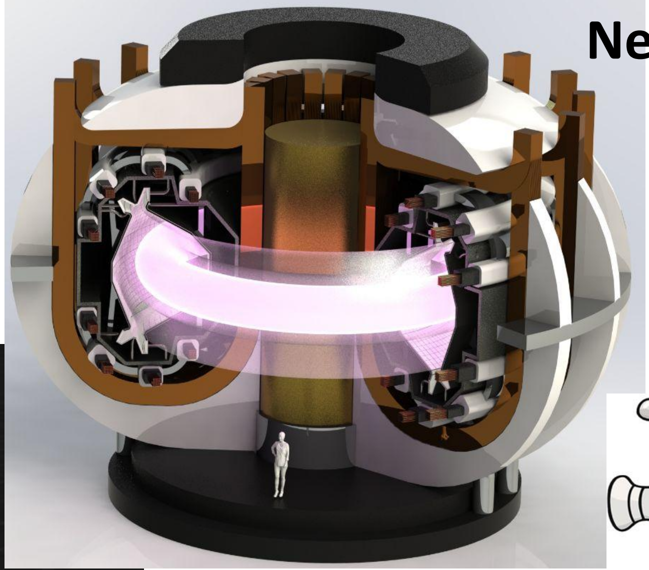
Explaining nuclear fusion: is it the way to cheap energy?

The sun is powered by nuclear fusion, which smashes hydrogen nuclei together to make helium. Photograph: EPA/Corbis

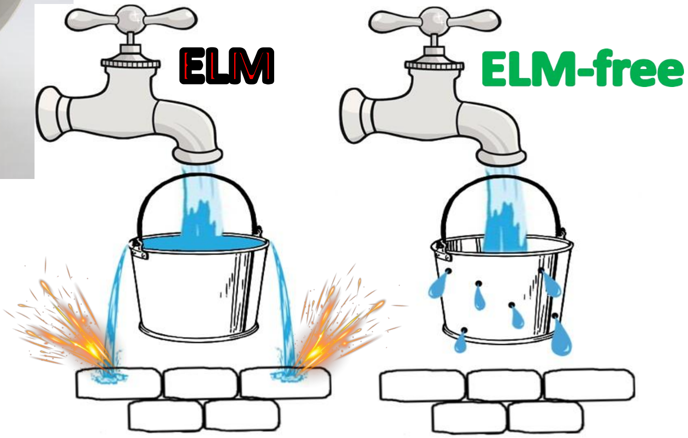
My research focuses on simplifying reactor strategies

Strong magnetic fields

Negative Triangularity



ELM-free operation

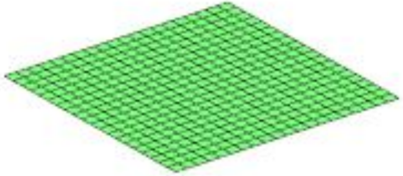


APS-DPP Student Day - join us in 2023!!!

Alright, on to the waves…

How are waves in plasmas described?

- Plasmas are a complicated, self-organizing mess…
  - MoVinG and OsCiLlaTiNg plasmas are even more so



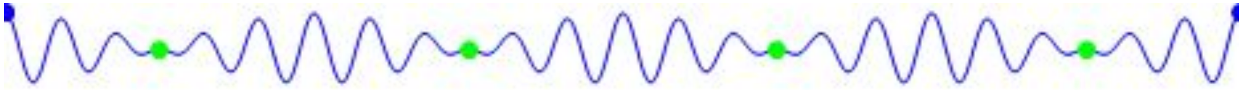
- Waves in plasmas significantly impact the plasma state!
  - modify density, electric fields and magnetic fields
  - transport and deposit energy and momentum

- The standard approach for understanding a wave in a plasmas is to determine its ***dispersion relation:*** **ω(*k*)**
  - relates the ***wavelength*** **(λ)** or ***wavenumber*** **(*k* = 2π / λ)** of a wave to its ***frequency*** **(ω = 2π *f*)**
  - can be used to calculate the ***phase velocity*** **(v_p= ω / k)** and ***group velocity*** **(v_g= ∂ω / ∂k)**

Dispersion relation examples: light, strings and water

- For electromagnetic waves in a vacuum:
  - This is a linear dispersion relation!

$$\omega = ck \qquad v = \frac{\omega}{k} = \frac{d\omega}{dk} = c$$



- For waves on a string, we get something similar:
  - Tension T and mass density μ are important

$$\omega = k\sqrt{\frac{T}{\mu}}$$

- For waves on the ocean, gravity plays a leading role…

$$\omega = \sqrt{gk}$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \qquad v_g = \frac{d\omega}{dk} = \frac{1}{2}v_p$$

Plasmas can be described by many models

- Many plasma wave derivations start with the MHD equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{(Ampere's Law)}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0 \varepsilon_r} \qquad \text{(Gauss's Law)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \qquad \text{(continuity equation)}$$

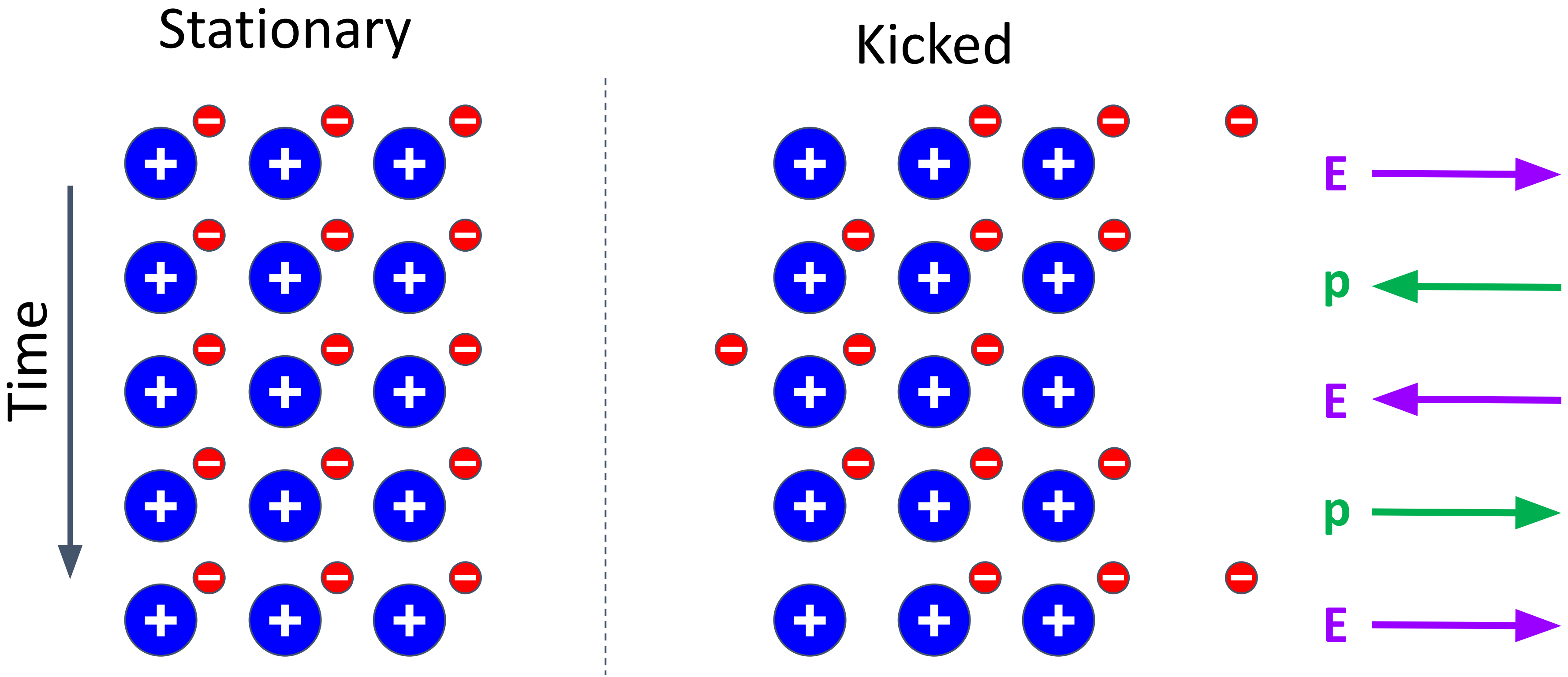
$$\rho \left(\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right) = \mathbf{J} \times \mathbf{B} - \nabla P + \rho g \qquad \text{(momentum equation)}$$

$$\mathbf{E} + \mathbf{U} \times \mathbf{B} = \eta \mathbf{J} \qquad \text{(resistive Ohm's Law)}$$

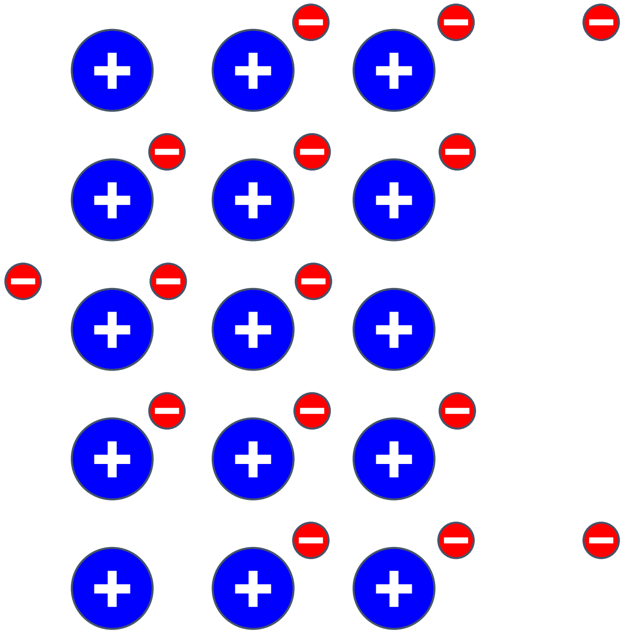
$$\left(\frac{\partial}{\partial t} + (\mathbf{U} \cdot \nabla) \right) \left(\frac{P}{\rho^\gamma} \right) = 0 \qquad \text{(equation of state)}$$

- The hunt for waves is the hunt for oscillatory modes in these equations

A simple picture: electron motion in a plasma



A simple picture: electron motion in a plasma



Assumptions:

- Plasma is homogeneous
- Time scale is short; ions are "cold"
- No thermal effects

Equations:

$$m_e n_e \left(\frac{\partial}{\partial t} \mathbf{u}_e + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e \right) = e n_e \mathbf{E} - \nabla p_e$$

$$\frac{\partial}{\partial t} n_e + \nabla \cdot (n_e \mathbf{u}_e) = 0$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = e(n_i - n_e)$$

What do these equations mean??

$$m_e n_e \left(\frac{\partial}{\partial t} \mathbf{u}_e + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e \right) = e n_e \mathbf{E} - \nabla p_e$$

Momentum Equation:
Force balance given by density * [F = ma]

$$\frac{\partial}{\partial t} n_e + \nabla \cdot (n_e \mathbf{u}_e) = 0$$

Continuity Equation:
Electrons are neither created nor destroyed!

$$\epsilon_0 \nabla \cdot \mathbf{E} = e(n_i - n_e)$$

Gauss's Law:
Potential caused by an electric charge

"Linearization" allows us to continue the derivation…

Consider the continuity equation:

$$\frac{\partial}{\partial t} n_e + \nabla \cdot (n_e \mathbf{u}_e) = 0$$

Assume "wave like" variations:

$$\begin{cases} n = n_0 + n_1 e^{i(kx - \omega t)} \\ u = u_1 e^{i(kx - \omega t)} \end{cases}$$

$$n_1(-i\omega \cancel{e^{i(kx-\omega t)}}) + n_0 u_1 (ik \cancel{e^{i(kx-\omega t)}}) + \cancel{\mathcal{O}(n_1 u_1)} = 0$$

$$-i\omega n_1 + ikn_0 u_1 = 0$$

"Linearization" allows us to continue the derivation…

$$m_e n_e \left(\frac{\partial}{\partial t} \mathbf{u}_e + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e \right) = e n_e \mathbf{E} - \nabla p_e$$

$$\nabla p_e = 3 T_e \nabla n$$

$$\frac{\partial}{\partial t} n_e + \nabla \cdot (n_e \mathbf{u}_e) = 0$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = e(n_i - n_e)$$

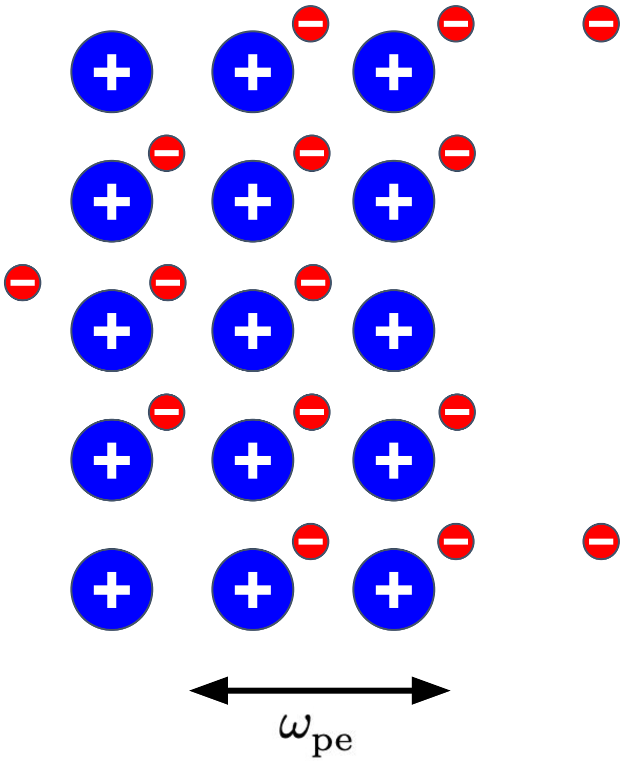
Three equations!
Three perturbed quantities!

$$i\omega m_e n_0 u_1 = e n_0 E_1 + 3ikTn_1$$

$$-i\omega n_1 + ikn_0 u_1 = 0$$

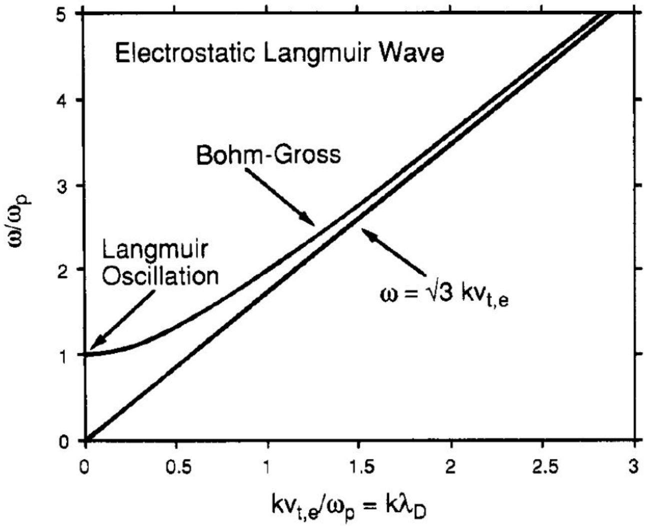
$$ik\epsilon_0 E_1 = -en_1$$

The dispersion relation for Langmuir Waves is simple!



Dispersion Relation:

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_{t,e}^2 \qquad \omega_{pe}^2 = \frac{n_e e^2}{\epsilon_0 m_e}$$



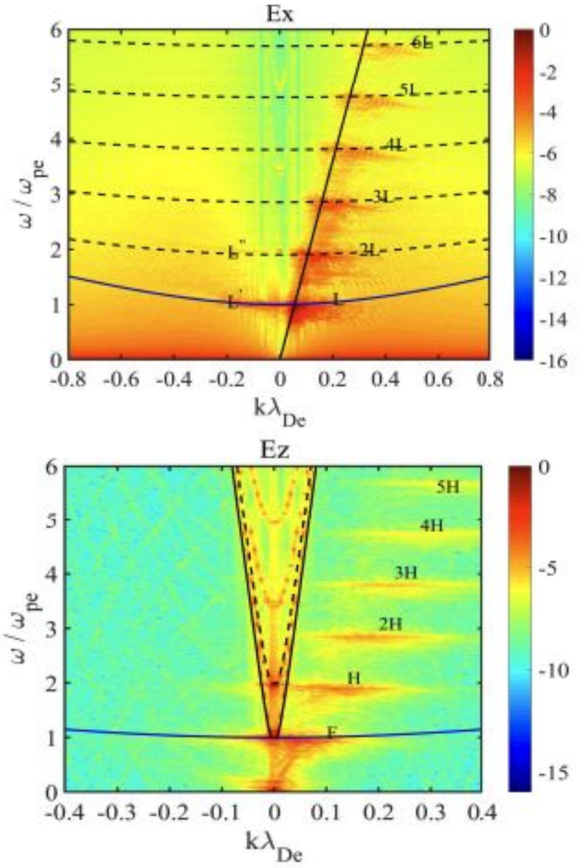
These things are real! Example: solar radio bursts

- Relativistic electron beam injected into magnetized background plasmas
  - Langmuir waves excited by beam-plasma interaction!

$$\omega_n^2 = n^2 + 3k^2\lambda_{\mathrm{De}}^2 - \frac{3(n-1)}{2}\left(\frac{k}{k_0} - \frac{n}{2}\right)\frac{k_0^2 v_{\mathrm{e}}^2}{\omega_{\mathrm{pe}}^2}$$

- Langmuir wave decay produces backward Langmuir wave and fundamental EM wave

- Coalescence of forward and backward Langmuir waves produces the second harmonic EM emission



T. M. Li, *et. al.*, *Astronomy & Astrophysics* 653 (2021)

Three main species of non-magnetized waves

Langmuir Waves

- Electrostatic ($\mathbf{k} \parallel \mathbf{E}_1$)
- Electron oscillations around stationary ions

$$\omega^2 = \omega_{\mathrm{pe}}^2 + 3k^2 v_{\mathrm{t,e}}^2 \qquad \omega_{\mathrm{pe}}^2 = \frac{n_{\mathrm{e}} e^2}{\epsilon_0 m_{\mathrm{e}}}$$

Ion Sound Waves

- Electrostatic ($\mathbf{k} \parallel \mathbf{E}_1$)
- Allow ions to participate in the motion to establish force balance

$$\omega^2 = k^2 \left(\frac{T_{\mathrm{e}}/m_{\mathrm{i}}}{1 + k^2 \lambda_{\mathrm{D}}^2} + \gamma_{\mathrm{i}} T_{\mathrm{i}}/m_{\mathrm{i}} \right)$$

Electromagnetic Waves

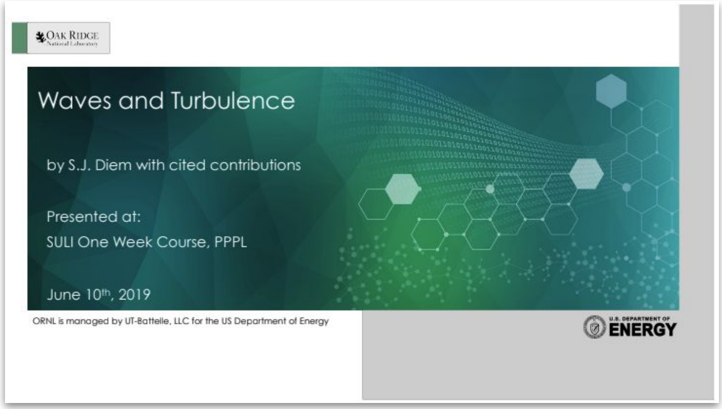
- Transverse ($\mathbf{k} \perp \mathbf{E}_1$)
- EM radiation "shorted" by electron oscillations

$$\omega^2 = \omega_{\mathrm{p}}^2 + c^2 k^2$$

The next step in complexity: “cold” magnetized waves

- “Cold” – assume that $\omega / k \gg v_{th}$, such that all particles see the same electromagnetic fields
- Also assume:
  - homogeneous, steady state, collisionless plasma
  - background magnetic field ***B***
  - no background electric field ***E*** (can be eliminated by reference frame switch)

- For a more mathematical derivation, see this excellent talk by Professor Steffi Diem: https://suli.pppl.gov/2019/course/index.html

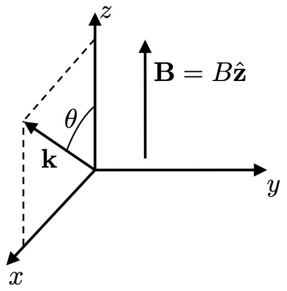


The cold plasma dispersion relation

- Eventually can get the dispersion relation into the form:

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

- Where $n = \dfrac{kc}{\omega}$ is the index of refraction and the other quantities are:



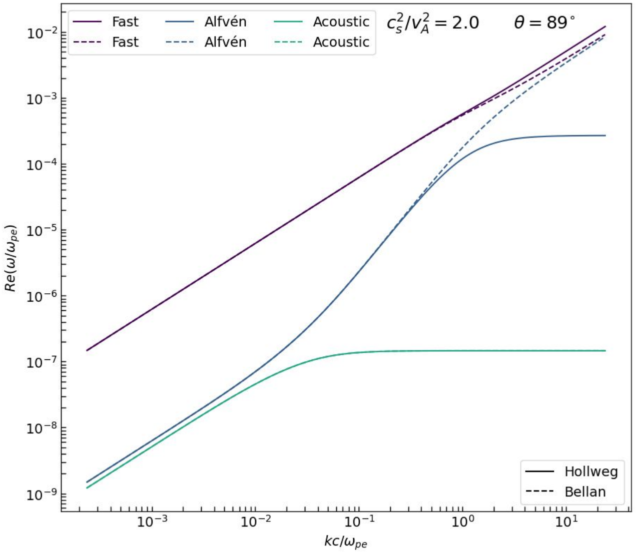
$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_{cs}^2} \qquad R = S + D = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\left(\omega + \frac{q_s}{|q_s|}\Omega_{cs}\right)} \qquad P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

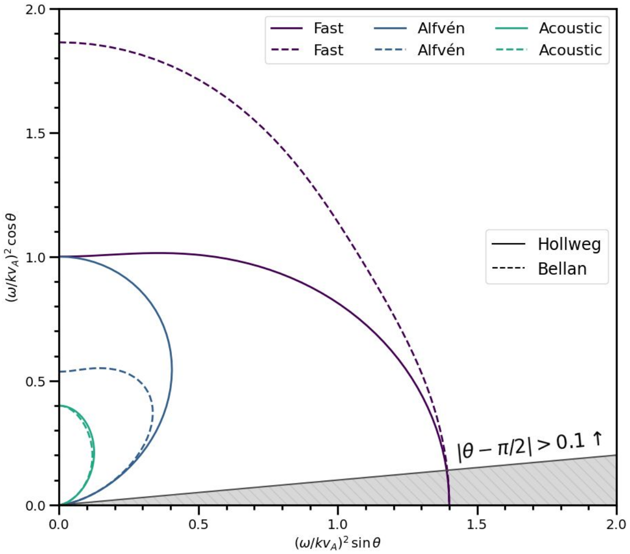
$$D = \sum_s \frac{\Omega_{cs}\omega_{ps}^2}{\omega\left(\omega^2 - \Omega_{cs}^2\right)} \qquad L = S - D = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\left(\omega - \frac{q_s}{|q_s|}\Omega_{cs}\right)} \qquad \omega_{ps}^2 = \frac{q^2 n}{\varepsilon_0 m}$$

Solving dispersion relations with plasmapy

- Plasmapy is a [free] python package with many amazing tools
  - https://docs.plasmapy.org/

- A portion of the package is dedicated to solving dispersion relations!





Special cases: cutoffs, resonances and solutions

$$n = \frac{kc}{\omega}$$

$$\tan^2\theta = \frac{-P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

- ***Cutoff***: $n \to 0$ and thus $\lambda \to \infty$
  - waves will not propagate
  - occurs when $PRL = 0$
- ***Resonance***: $n^2 \to \infty$ and thus $\lambda \to 0$
  - waves will resonate with the particle motion
  - occurs when $\tan^2\theta = -P/S$
- Propagation parallel to ***B*** ($\theta = 0$): $P = 0$, $n^2 = R$, $n^2 = L$
- Propagation perpendicular to ***B*** ($\theta = \pi/2$): $n^2 = P$, $n^2 = RL/S$

Damping and propagation from the dispersion relation

- In general, ω and *k* can be complex! $\omega = \omega_{Re} + i\omega_{Im}$ $k = k_{Re} + ik_{Im}$

$$e^{i(kx-\omega t)} \rightarrow e^{i(k_{\mathrm{Re}}x-\omega_{\mathrm{Re}}t)-(k_{\mathrm{Im}}x-\omega_{\mathrm{Im}}t)}$$

- If *k* is imaginary, the wave will not propagate…
  - Implications when coupling a wave to a plasma
  - Power can be lost between an antenna and the start of the plasma

- If ω has an imaginary component, the wave can grow or damp!
  - Damping: wave amplitude diminishes (transfers energy to particles)
  - Growth: instabilities can form and take over!

What else is in the Zoo?

A few common types of cold magnetized waves

L Wave - Circularly polarized along ion gyration

R Wave - Circularly polarized along electron gyration

Ordinary (O) Waves - Same as EM wave in unmagnetized plasma

Extraordinary (X) Waves - Elliptically polarized EM wave

Alvén Waves - Ions oscillate in response to magnetic field line tension

Shear Alvén Waves - Magnetic field lines twist to store energy

Compressional Alvén Waves - Field lines compress (magnetosonic)

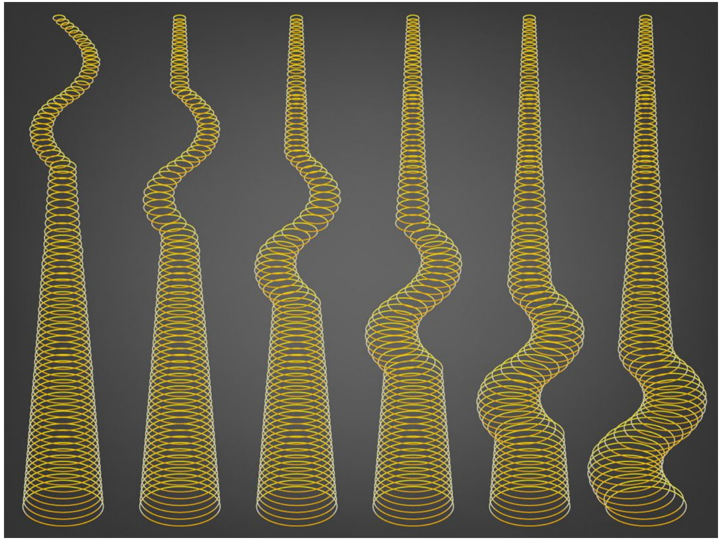
Whistler Waves - Special case of R Wave observed in the Magnetosphere

Alfvén Waves: fundamental in all of plasma physics

- Low frequency MHD instabilities where ions oscillate to a restoring force provided by an **effective tension on the magnetic field lines**

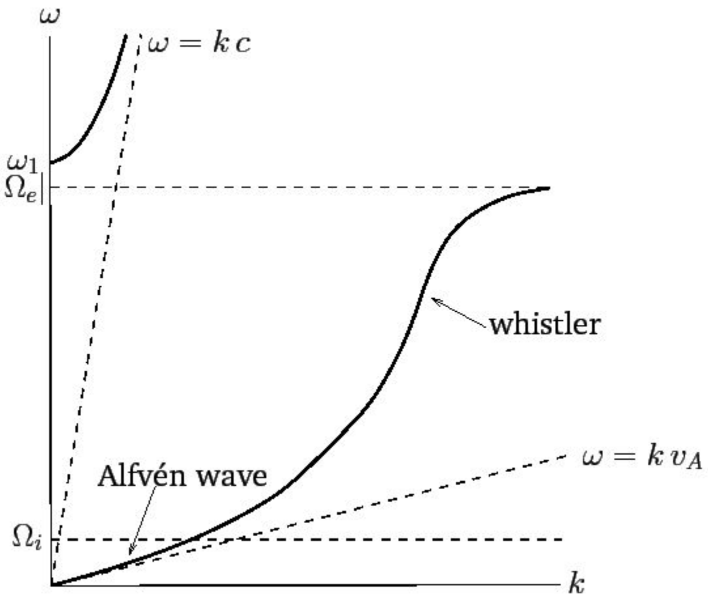
$$\omega = k_{||} v_A \qquad v_A = \frac{B}{\sqrt{\mu_0 \rho}}$$

- Winner of 1970 Nobel Prize in Physics!
  - carry energy from the sun's surface (~1,000 K) to heat up the corona (~1,000,000 K) and the solar wind.
- Black holes can trigger Alfvén waves too!
  - carries energy in the jet…

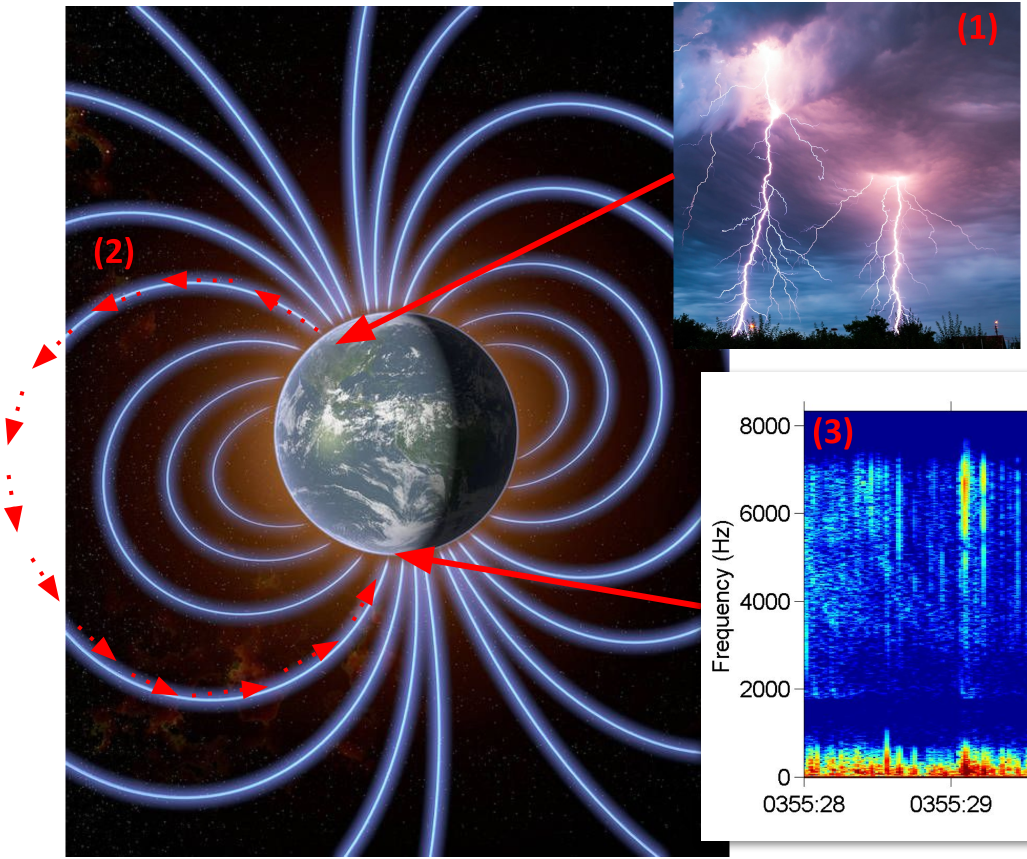


Whistler Waves: an audible plasma mystery story

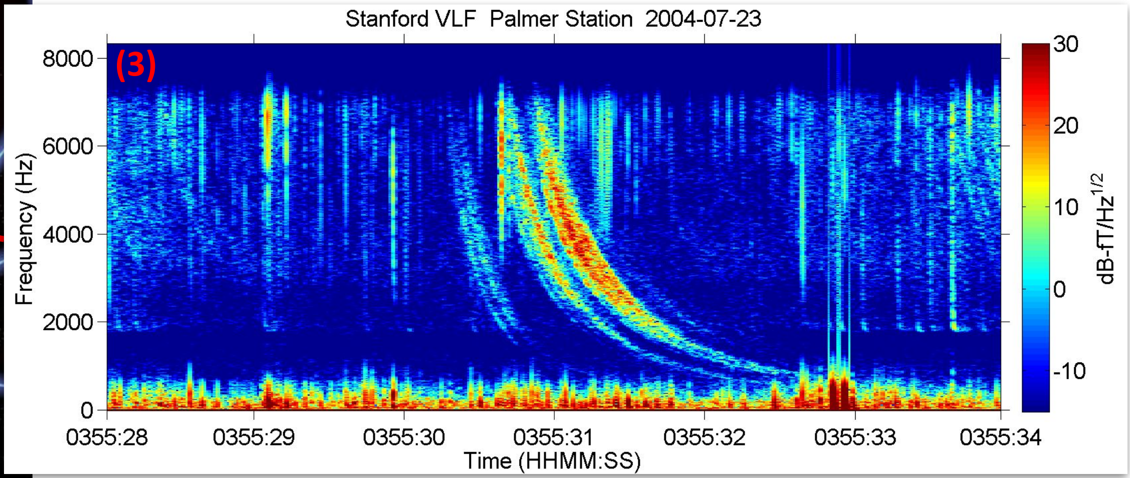
- Heard as early as 1886 on long telephone lines, then eventually linked to lightning in his 1953
  - Helliwell: "weird, strange and unbelievable as flying saucers"
  - named by WWI radio operators because they sound like an incoming grenade
- Now understood to be plasma waves!
- At very low frequency (audible) the R-wave $v_g = \partial\omega / \partial k$ increases with ω
  - high frequencies travel faster than low ones!



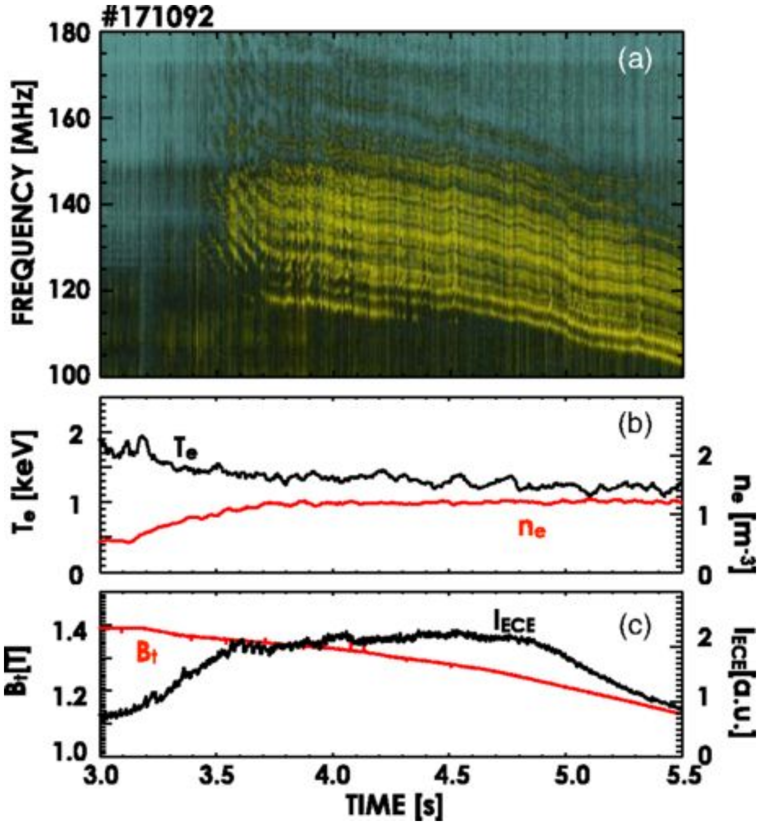
What makes a Whistler?



- White radio noise produced by lightning can propagate along B to the other side of the world!



Whistler waves also observed in tokamaks!

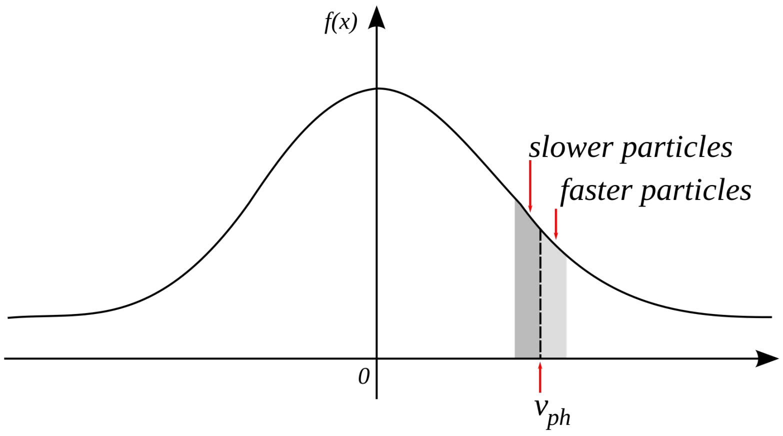


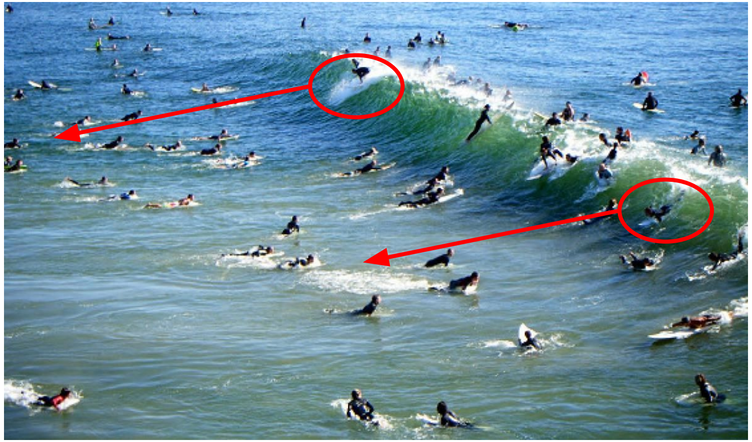
- Runaway electrons (multi-MeV) can provide driving energy for whistlers
- Modes can be stabilized with increasing B (wave-particle resonance)
- In common with ionospheric whistlers, tokamak whistlers lead to pitch-angle scattering of energetic electrons
- Runaways in a tokamak might be mitigated by the intentional launching of whistlers to scatter particles

D. A. Spong, *et. al.*, *PRL* ***120***, *155002* (2018)

More wave-particle interactions: Landau damping

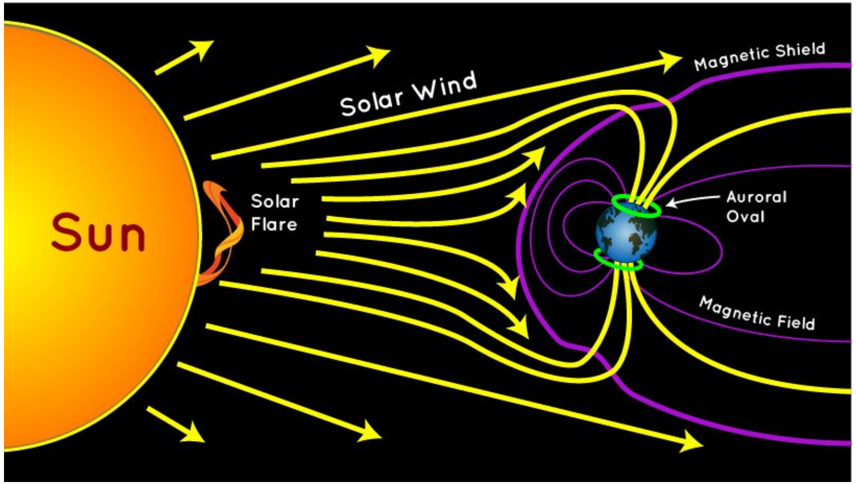
- Imagine we have a distribution of particles in a plasma with a wave:





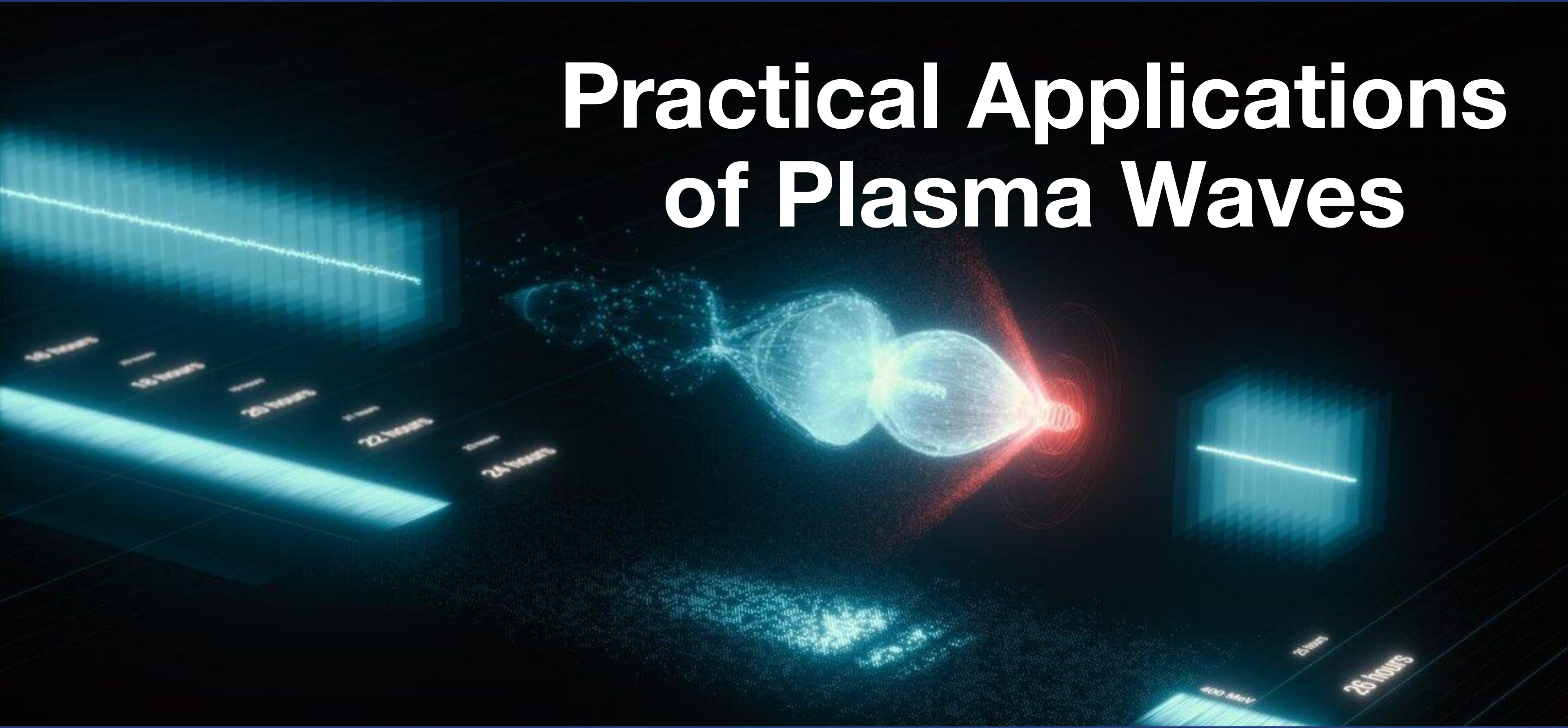
- Particles with velocities close to the v_{ph} will interact!
  - slower particles will be accelerated, faster particles will be slowed
  - on average, the wave will be damped!

Landau damping is key to the Aurora Borealis!



J. W. R. Schroeder, *et. al.*, *Nature Communications* 12 (2021)

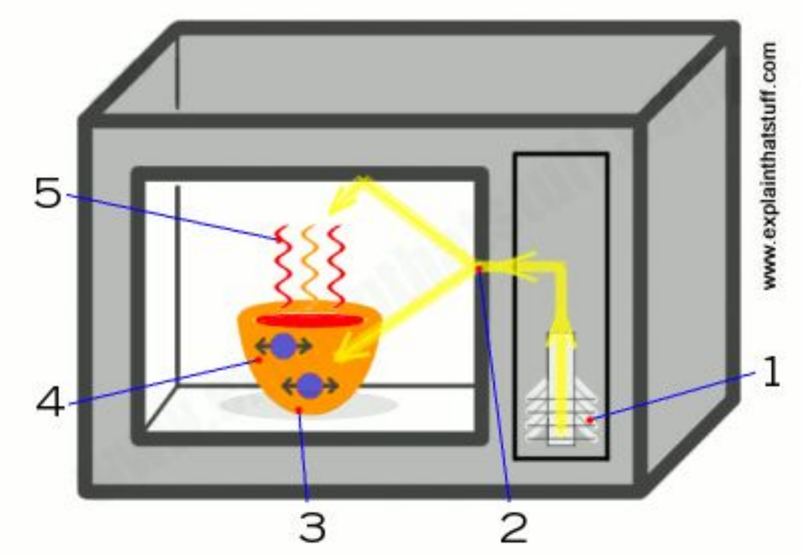
Practical Applications of Plasma Waves



Using waves to heat plasmas (and to measure T!)

- Because particles and waves can interact:
  - You can inject waves to heat up particles!
  - You can measure waves to learn about the particles!

- Very similar to microwaves:
  1. Microwave generator launches waves
  2. Waves propagate through a waveguide
  3. Waves are absorbed in your food
  4. Food molecules vibrate faster due to wave interaction
  5. Vibrating molecules heat up the rest of the food!



Heating via electron cyclotron wave injection (ECH)

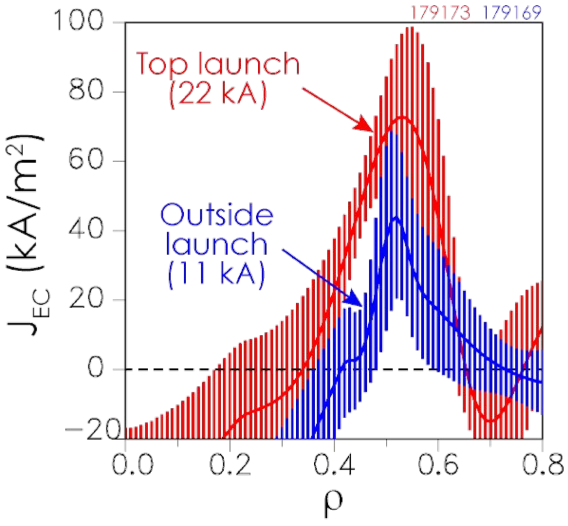


Figure 1: Measured ECCD current density profiles via top and outside launch with 0.5MW absorbed power from a single 110GHz gyrotron

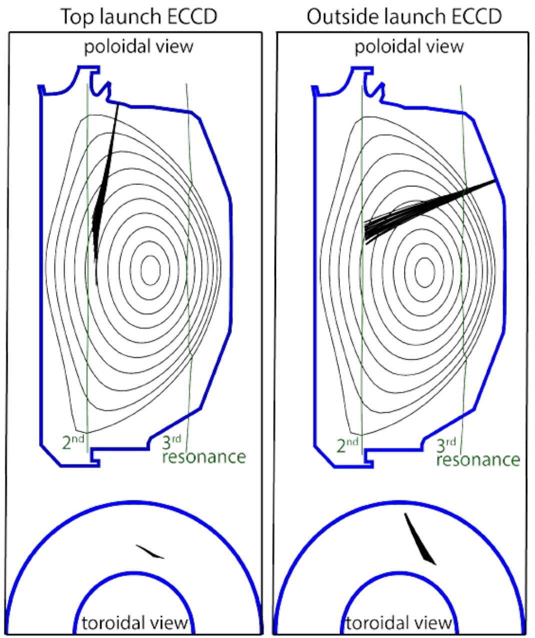


Figure 2: EC trajectories via top launch ECCD compared to outside launch in DIII-D

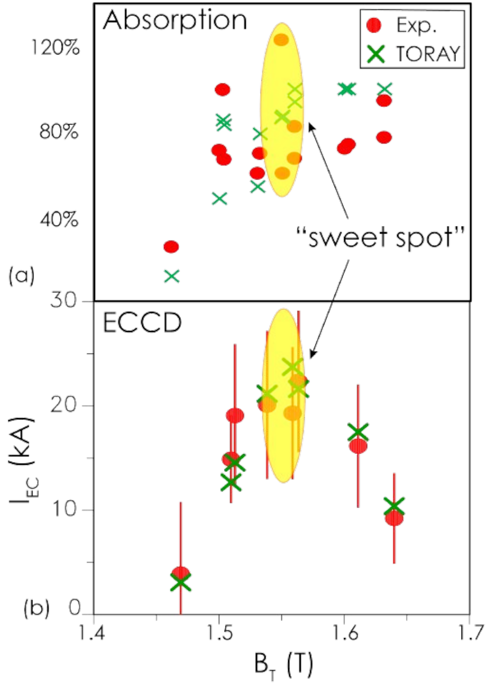
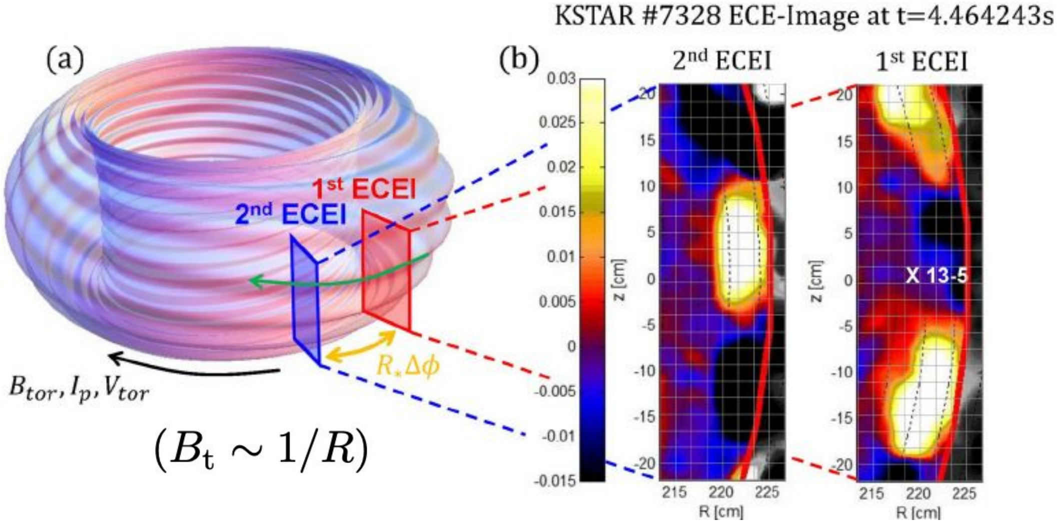


Figure 3: (a) Absorption and (b) ECCD via top launch at different B_T for H-mode plasmas

X. Chen, *et. al.*, 28th IAEA Fusion Energy Conference (2020)

Measuring via electron cyclotron wave emission

- Electron cyclotron emission (ECE): a primary diagnostic for the electron temperature (Te) profile
- Electrons gyrate at a set frequency $\omega_{\mathrm{ce}} = \frac{eB}{m_{\mathrm{e}}}$, allowing for localization!
- With a fancy-enough system, you can get 2D images!
  - very fast time resolution
  - great for measuring turbulence in the electron channel



$(B_{\mathrm{t}} \sim 1/R)$

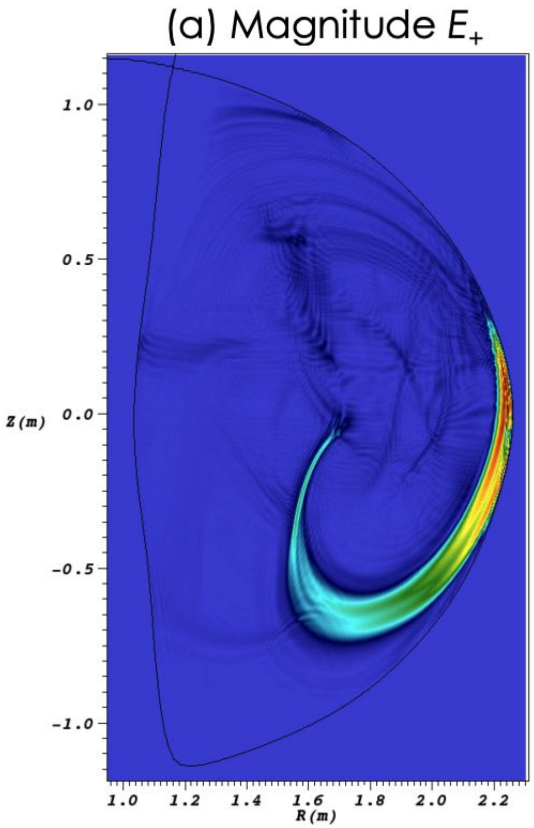
J. Lee, *et. al.*, Rev. Sci. Instrm. **85**, 063505 (2014)

Alternative options for heating: helicon injection

- Helicons (whistlers) are **fast waves** at a high harmonic of ω_{ic}
  - high frequency improves damping
  - primarily follow field lines, with some small radial component
  - spiral trajectory leads to off-axis CD
  - could lead to efficiency improvements

- More complicated trajectories…
  - requires ray tracing or full wave codes for deposition analysis

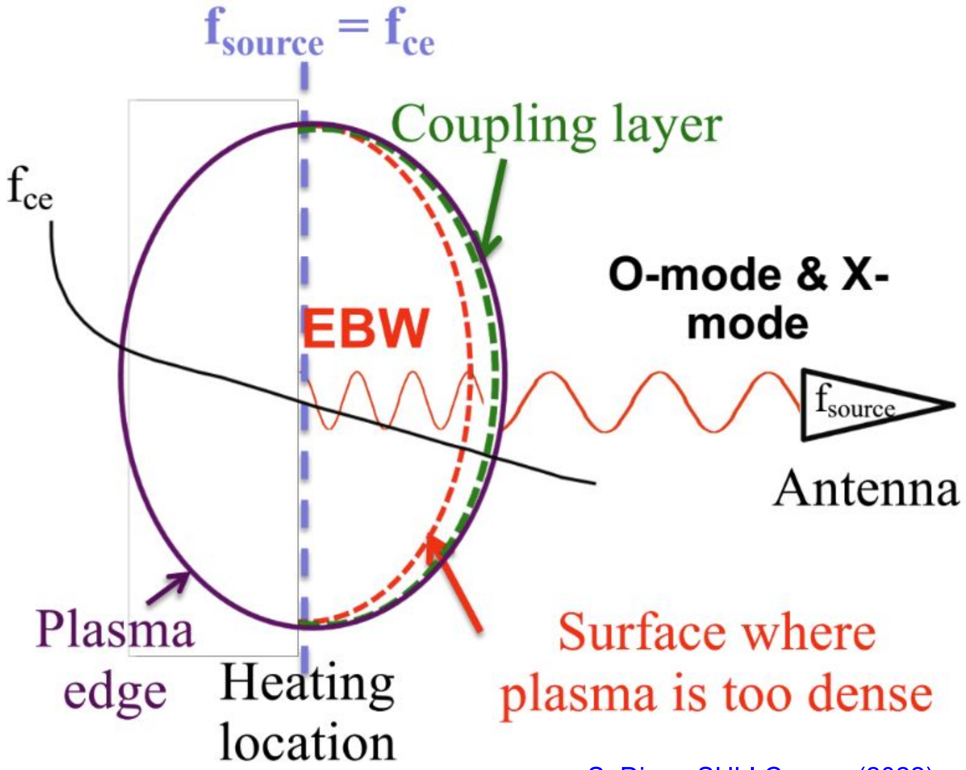
- Currently being commissioned on DIII-D



R. Prater, *et. al.*, IAEA Technical Committee Meeting (2015)

Electron Bernstein waves

- Electron Bernstein Waves (EBW) can only travel inside the plasma
  - Wave moves due to coherent motion of charged particles
  - Can couple to EBW by launching O- or X-modes
- Difficult to do, but no density cutoff and strong absorption!



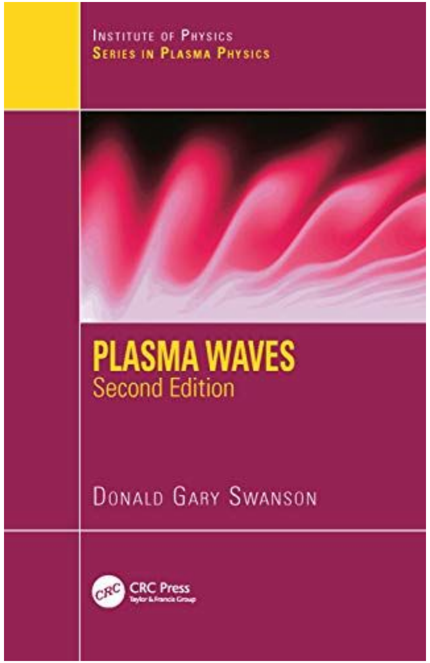
S. Diem, SULI Course (2022)

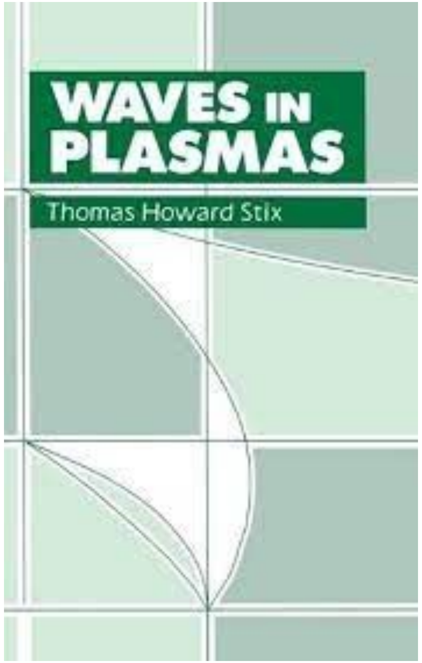
Summary

- Plasmas are incredibly complicated!
- Many degrees of freedom supports many "species" of waves
  - non-magnetized waves
  - cold magnetized waves
  - kinetic plasmas
  - nonlinear waves

- Resonant interactions with particles can lead to wave damping or instabilities
- Plasma waves encompass awesome natural phenomena and practical uses!

Further resources are available!!





plasmapy

And SULI talks from previous years!!
https://suli.pppl.gov/