Introduction to Plasmas, Part II
How do we model the behavior of a plasma

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SULI Introductory Course in Plasma Physics

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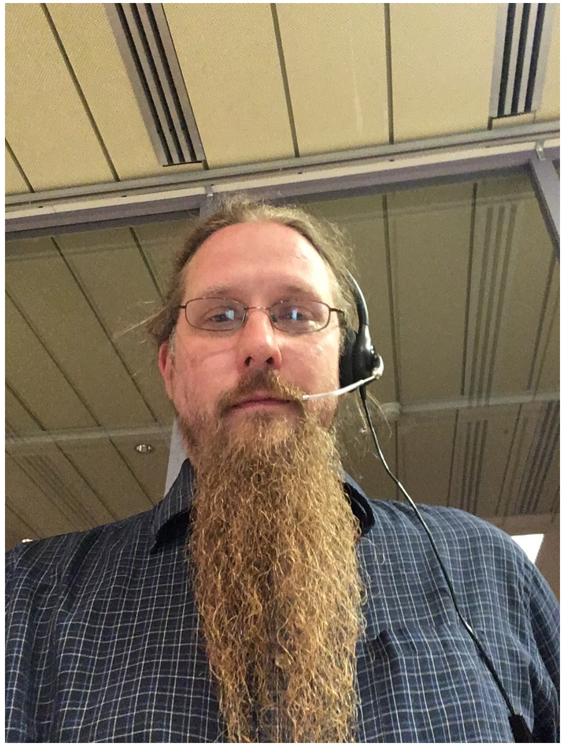
A bit about me



Recap: Plasmas – The first state of matter

- Physical systems whose intrinsic properties are governed by collective interactions of large ensembles of free charged particles

- Spans a wide densities, temperatures and spatial scales

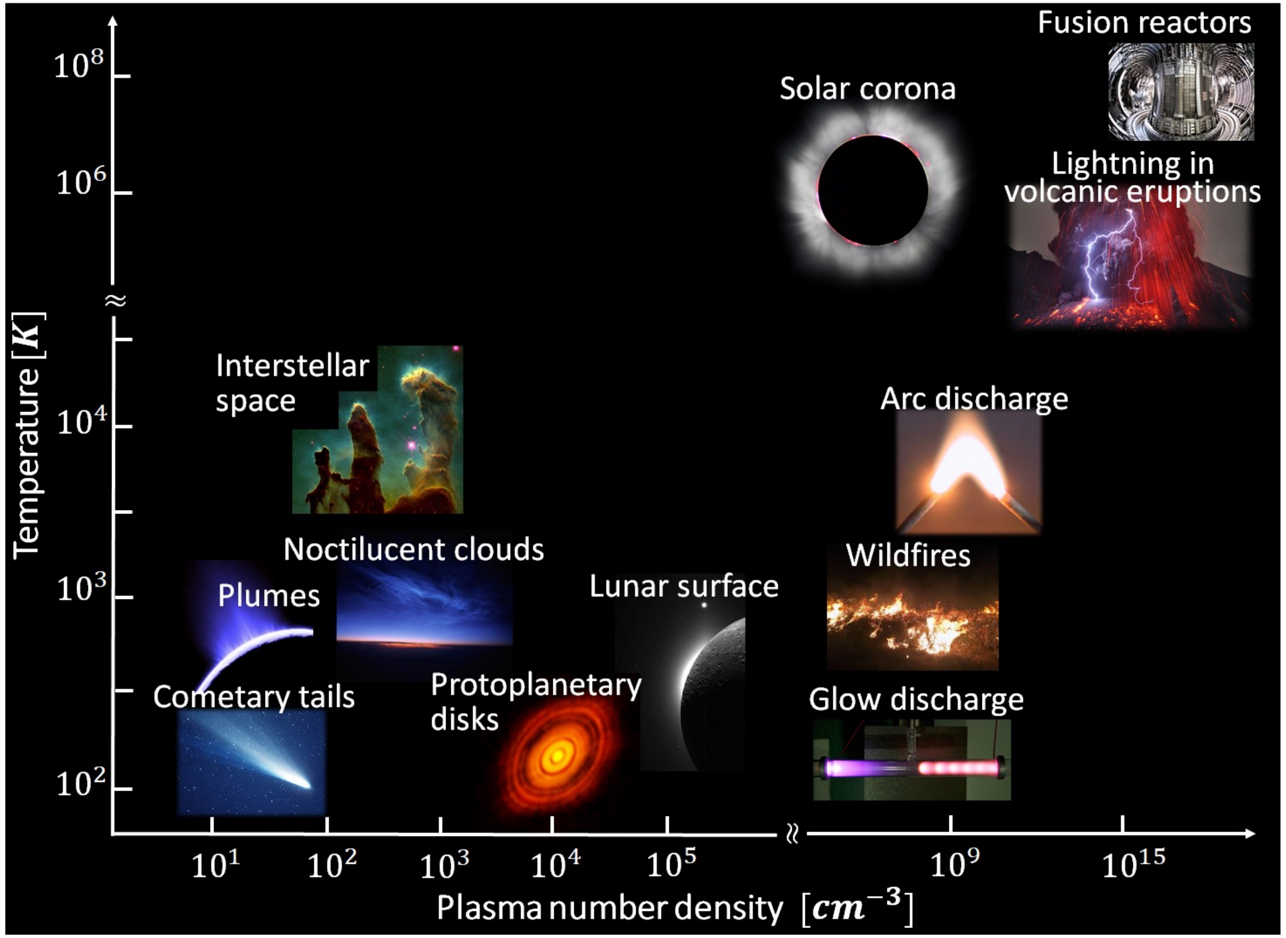


Image from L. Matthews 2021 SULI Talk, *Introduction to Plasma Physics*

Recap: Plasmas - Characteristic quantities

- Plasma Density, n_s
- Temperature, T_s
- Thermal Speed

$$v_{th,s} = \sqrt{\frac{k_B T_s}{m_s}}$$

- Debye Length

$$\lambda_D = \sqrt{\frac{k_B T_e \epsilon_o}{q_e^2 n_e}}$$

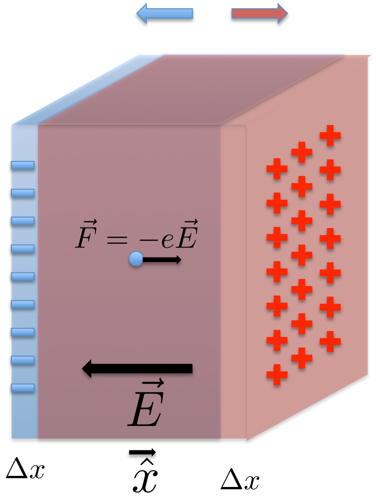
- Plasma Frequency

$$\omega_{ps} = \sqrt{\frac{q_s^2 n_s}{m_s \epsilon_o}}$$

- Collision Frequency

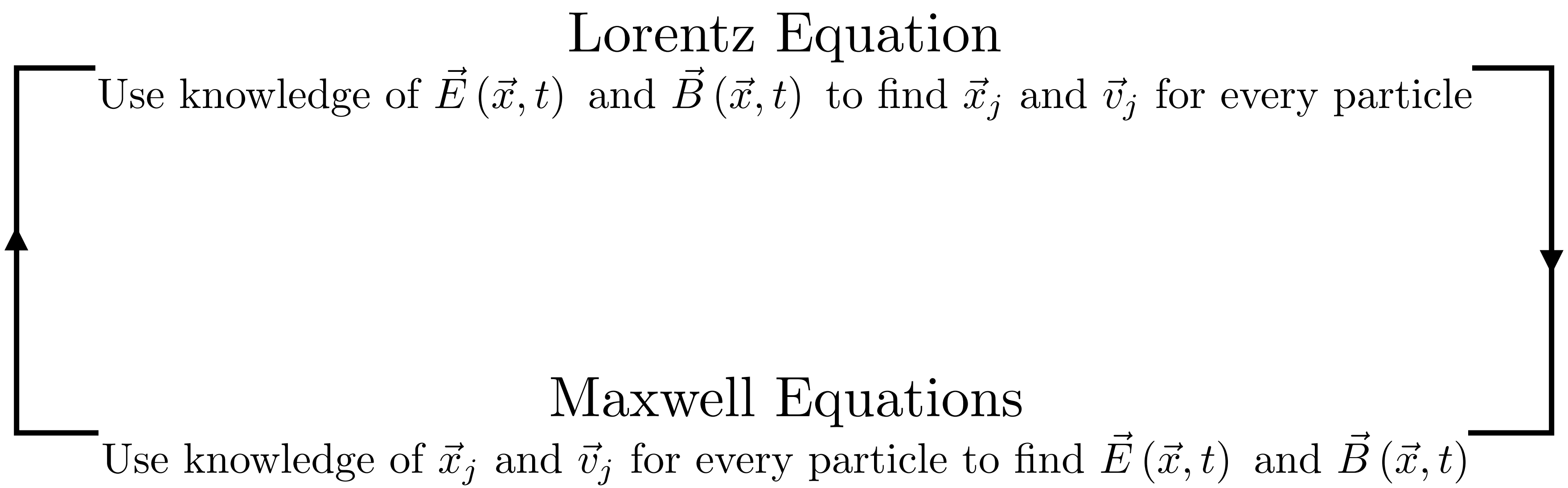
$$\nu_{ns} = n_n v_{th,s} \sigma_n$$

- Cyclotron Frequency



Today: How do we describe the behavior of a plasma

- The dynamics of a plasma is determined by the self-consistent interaction between electromagnetic fields and a statistically large number of particles.



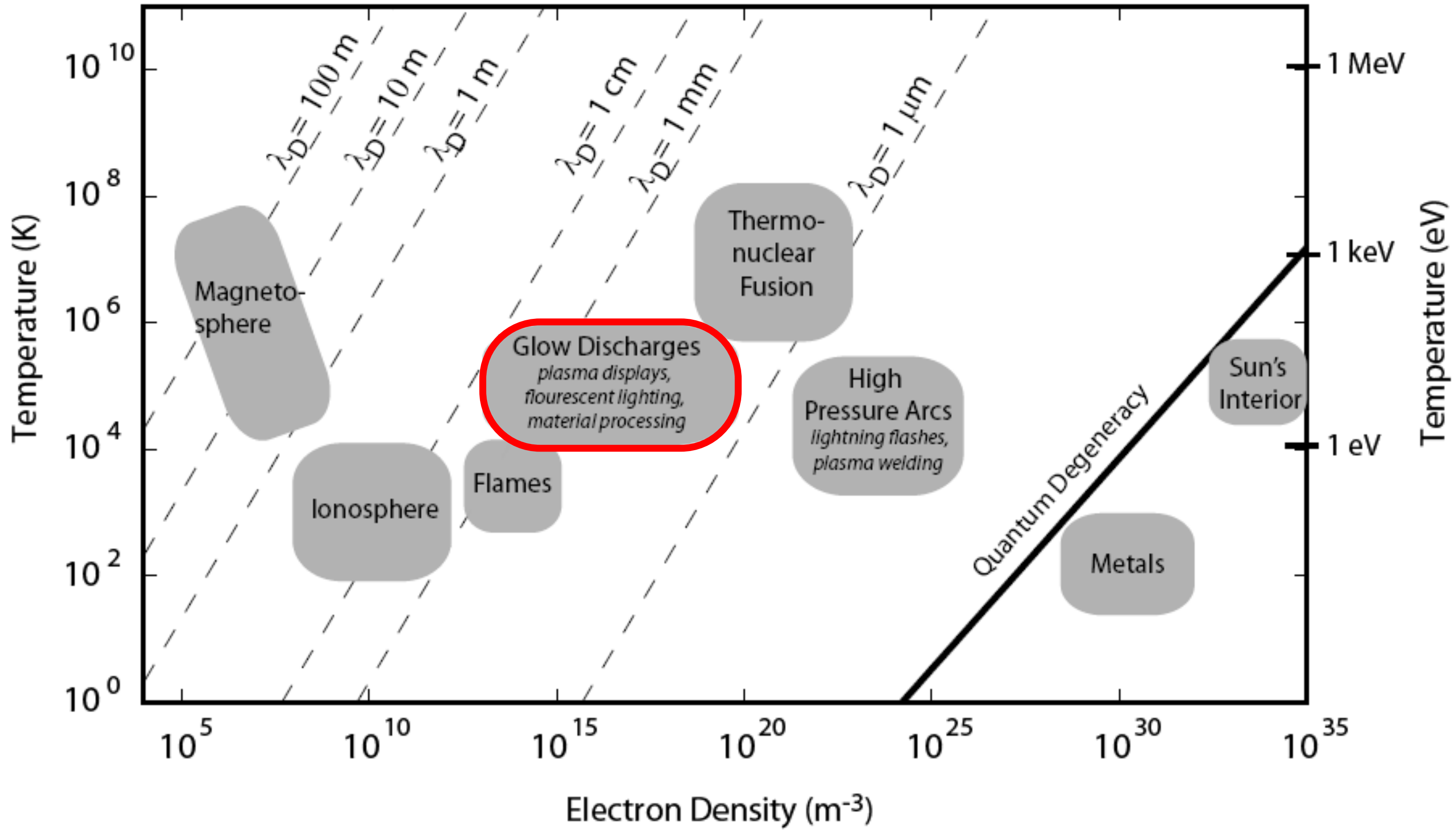
- Conceptually, this is simple but in practice quite hard

Why is this so hard?

$$n_p \sim 10^{15}\mathrm{m}^{-3},\ n_d \sim 10^{10}\mathrm{m}^{-3}$$

$$T_e = 3\mathrm{eV},\ T_i = \frac{1}{40}\mathrm{eV}$$

$$V = 3 \times 10^{-3}\mathrm{m}^3$$



- Timescales

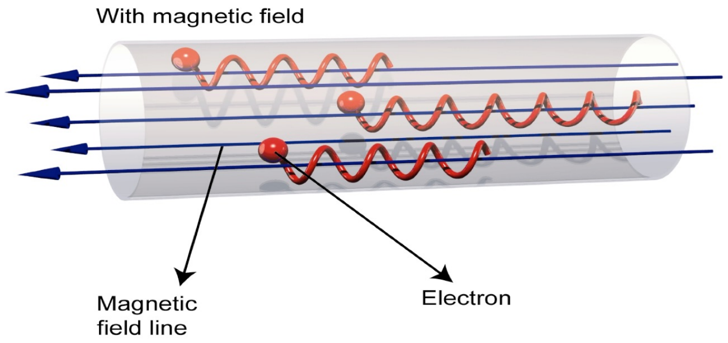
  $\tau_e \sim 10^{-9}s,\ \tau_i \sim 10^{-6}s$

  $\tau_d \sim 10^{-2}s$

- Number of particles

  $N \sim 10^9\mathrm{particles}$

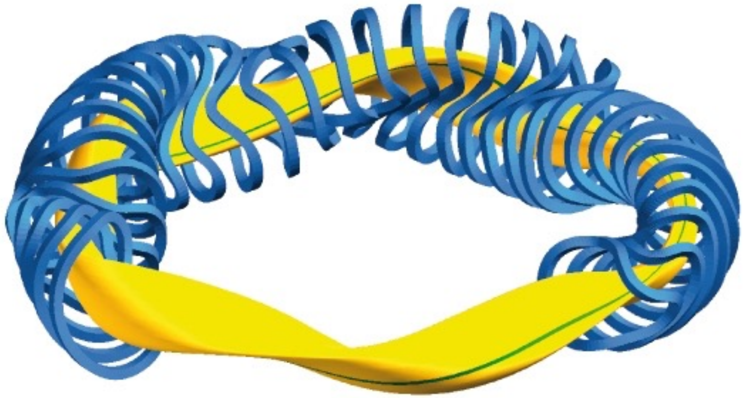
Hierarchy of Models



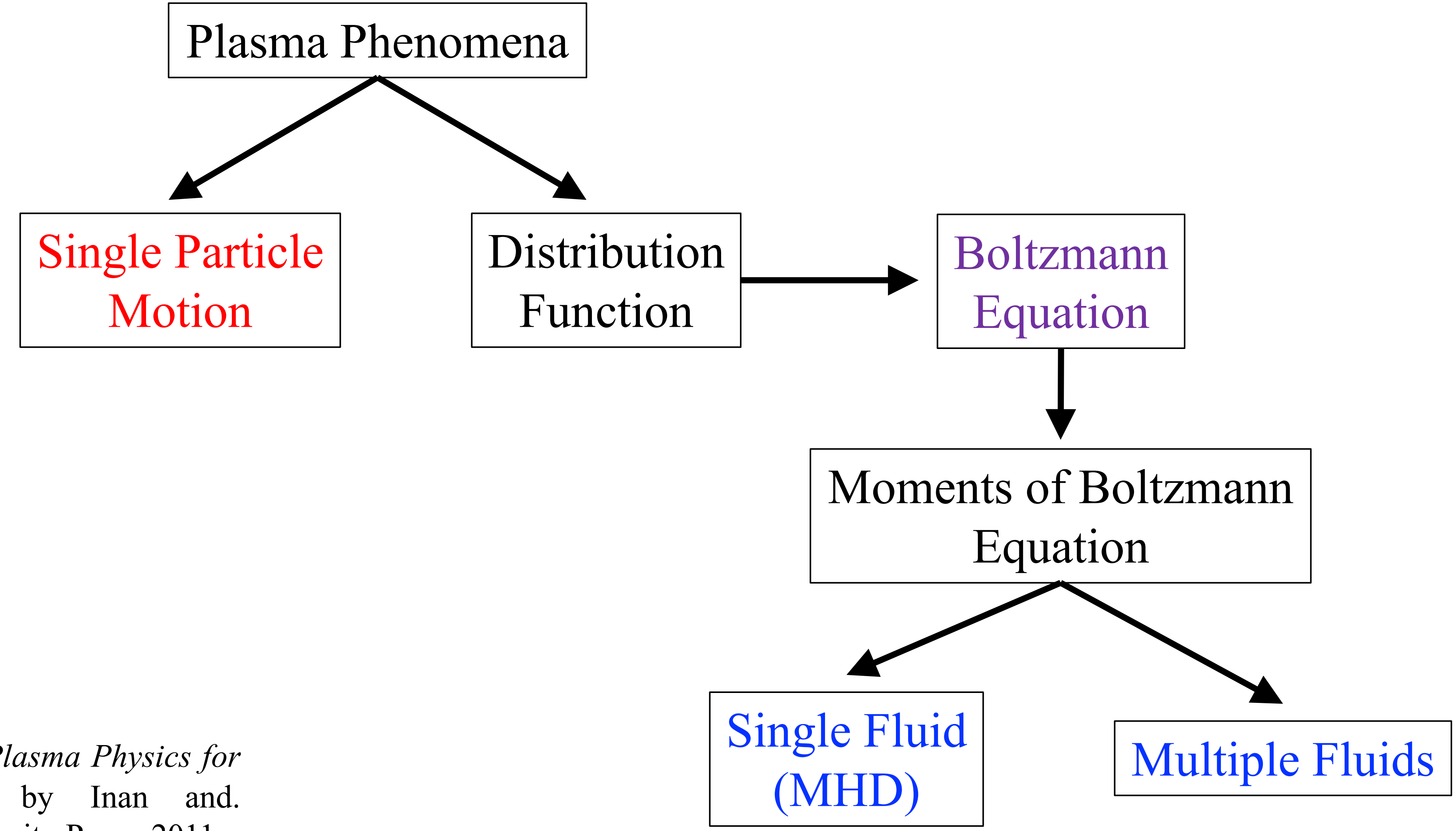
Single particle motion (particle pushing)

Kinetic Theory (distribution functions)

Fluid Model (moments, conservation eq.)

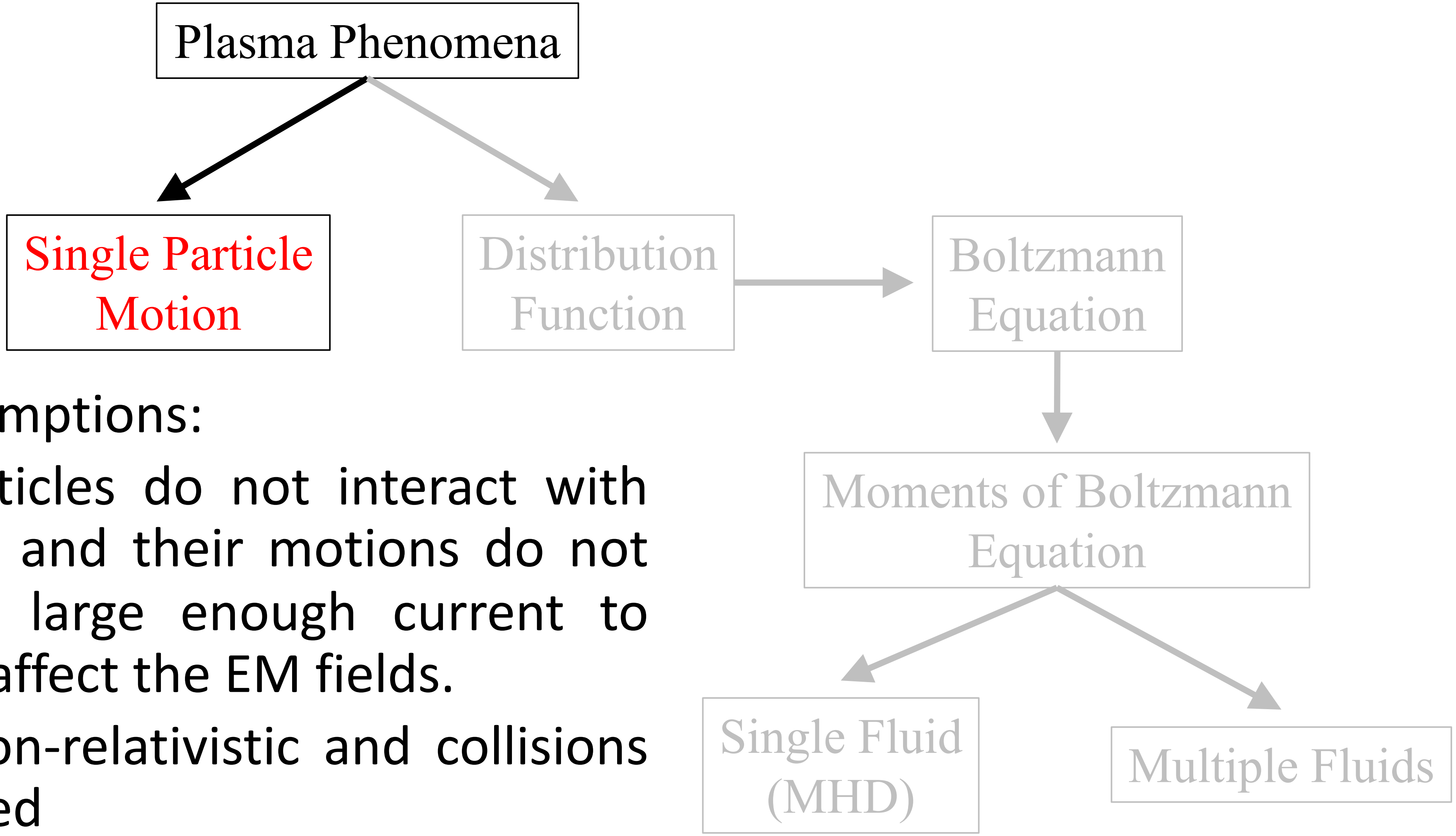


Hierarchy of Models



Adapted from *Principles of Plasma Physics for Engineers and Scientists* by Inan and. Gołkowski, Cambridge University Press, 2011.

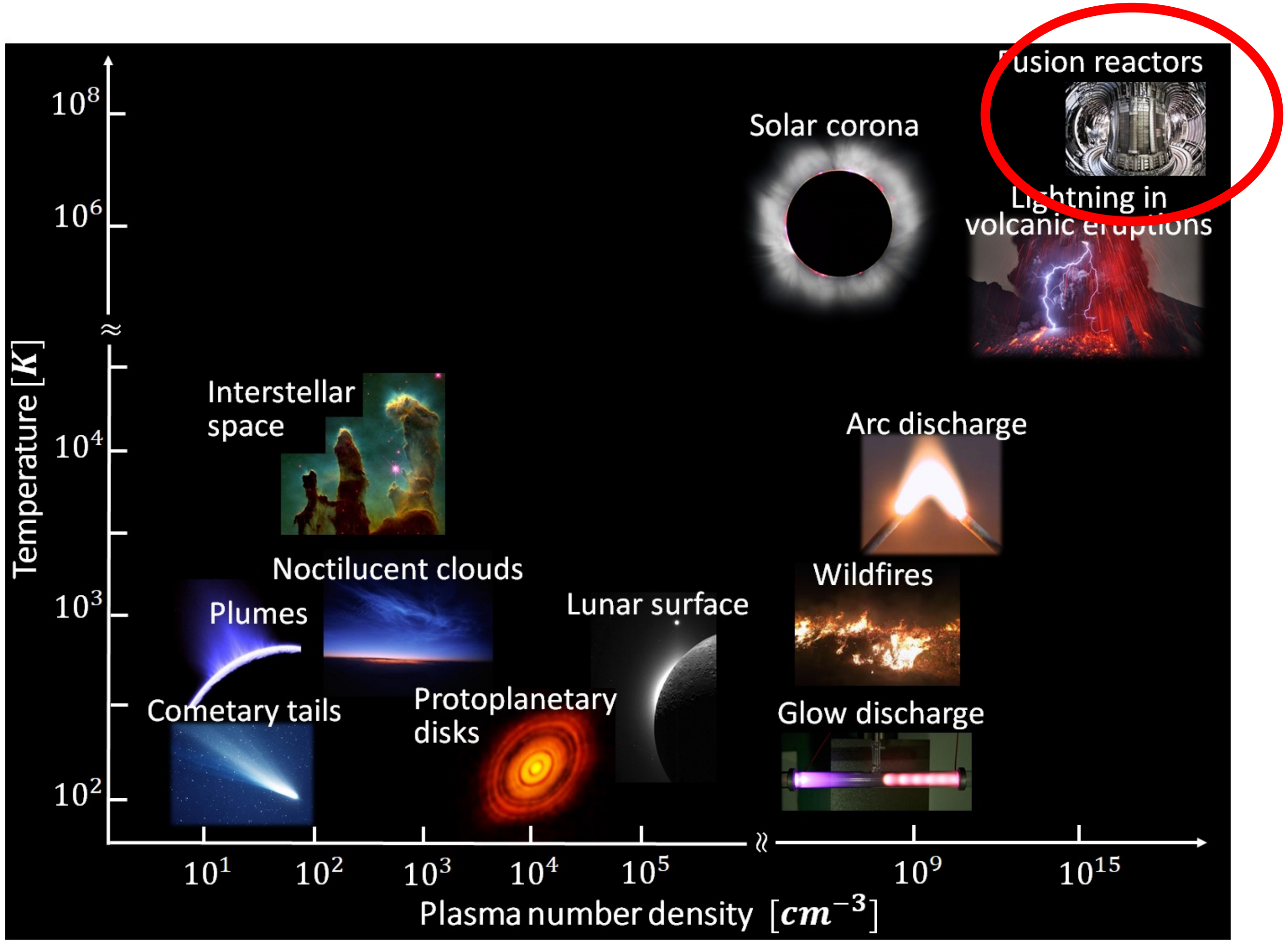
Single particle motion



- Simplifying assumptions:
  - Charged particles do not interact with one another and their motions do not constitute a large enough current to significantly affect the EM fields.
  - Motion is non-relativistic and collisions can be ignored

Single particle motion

- Consider an ion in fusion reactor



$$n_i \sim 10^{20}\,\frac{\text{ions}}{\text{m}^3},\; T_i = 10^8\ \text{K}$$

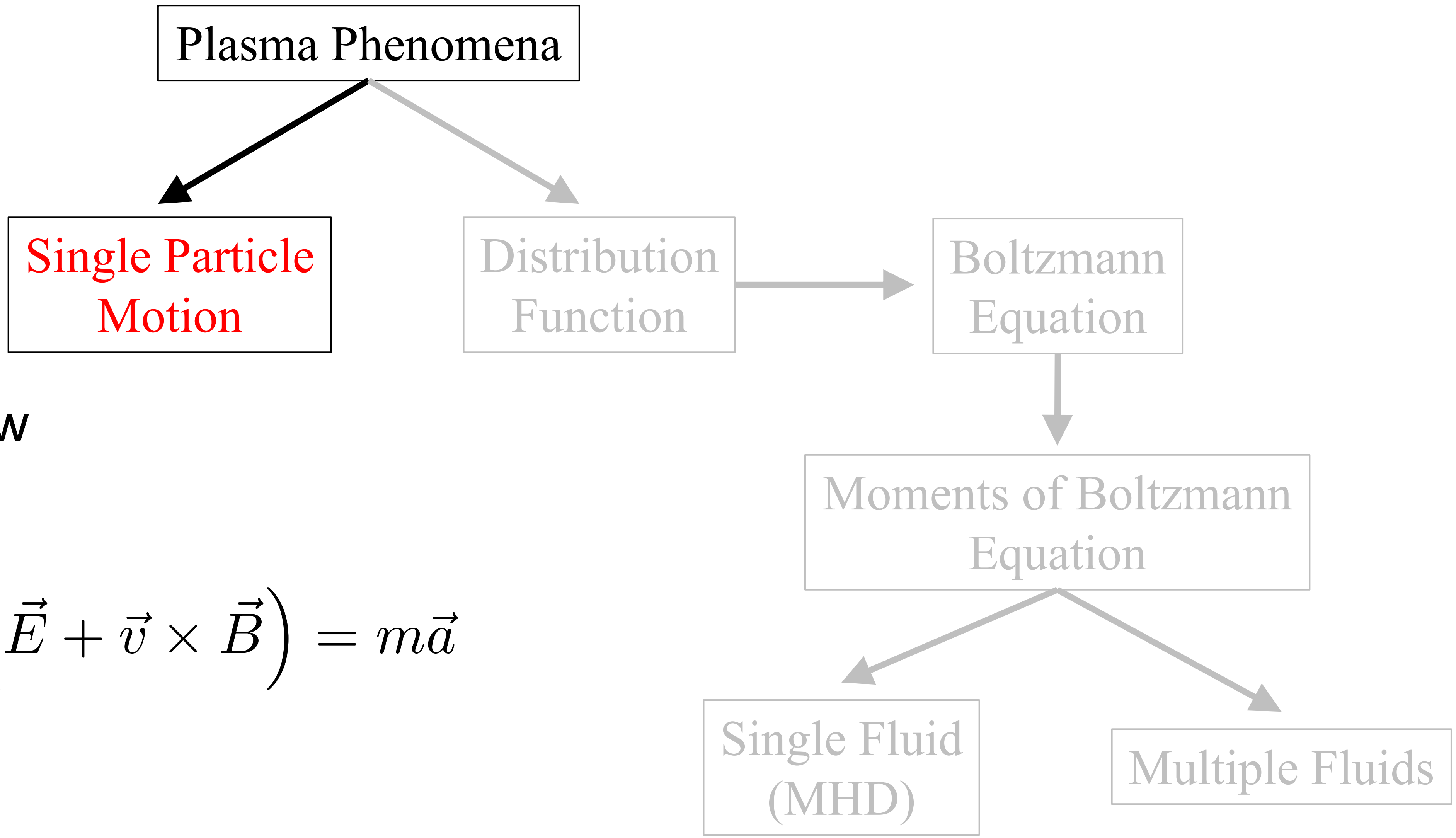
$$k_B T = \frac{1}{2} m v_{th}^2$$

$$v_{th} \sim 6 \times 10^5\,\frac{\text{m}}{\text{s}}$$

➢ ion would travel ~10 km before colliding with another ion

Image from L. Matthews 2021 SULI Talk, *Introduction to Plasma Physics*

Hierarchy of Models



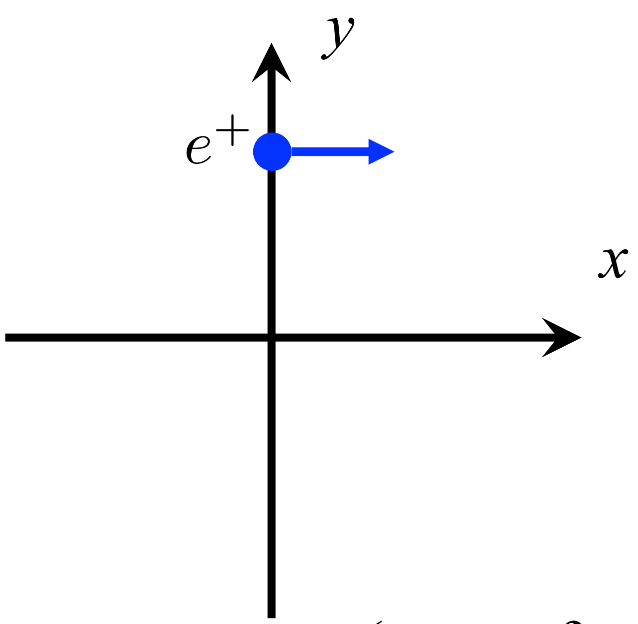
- Newton's 2nd Law

$$\vec{F}_{net} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = m\vec{a}$$

Single particle motion

- Consider a charged particle (proton) in a constant, uniform magnetic field

$$\vec{B} = B_o \hat{z} \qquad \vec{E} = 0$$



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$$\vec{F}_{net} = q\left(\cancel{\vec{E}} + \vec{v} \times \vec{B}\right) = m\vec{a}$$

$$qvB = m\frac{v^2}{R}$$

Single particle motion

- Consider a charged particle in a constant, uniform magnetic field

$$\vec{B} = B_z \hat{z} \qquad \vec{E} = 0$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \frac{q}{m}\vec{v} \times \vec{B}$$

Dot represents a time derivative

Single particle motion

$$\dot{\vec{v}} = \frac{q}{m}\vec{v} \times \vec{B}$$

- In component form

$$\dot{v}_x = \frac{q v_y B_z}{m} \qquad \dot{v}_y = -\frac{q v_x B_z}{m} \qquad \dot{v}_z = 0$$

Motion in the *x*- and *y*-direction is coupled

Motion is constant in the *z*-direction

$$v_z = \text{constant}$$

Single particle motion

$$\dot{v_x} = \frac{q v_y B_z}{m}$$

$$\dot{v_y} = -\frac{q v_x B_z}{m}$$

- These can be decoupled by taking a time derivative

$$\frac{d}{dt}\dot{v_x} = \ddot{v_x}$$

$$\ddot{v_x} = -\frac{q^2 B_z^2}{m^2} v_x$$

$$\ddot{v_y} = -\omega_c^2 v_y$$

$$\omega_c \equiv \frac{|q| B_z}{m}$$

Single particle motion

$$\ddot{v_x} = -\omega_c^2 v_x \qquad \ddot{v_y} = -\omega_c^2 v_y$$

- This can be solved (and we leave it to you to verify that it works)!

$$v_x = v_\perp \cos\left(\omega_c t + \phi_o\right) \qquad v_y = \mp v_\perp \sin\left(\omega_c t + \phi_o\right)$$

$$v_\perp = \sqrt{v_x^2 + v_y^2}$$

arbitrary phase to match the initial velocity

accounts for a positive or negatively charged particle

$$\omega_c \equiv \frac{|q| B_z}{m}$$

Single particle motion

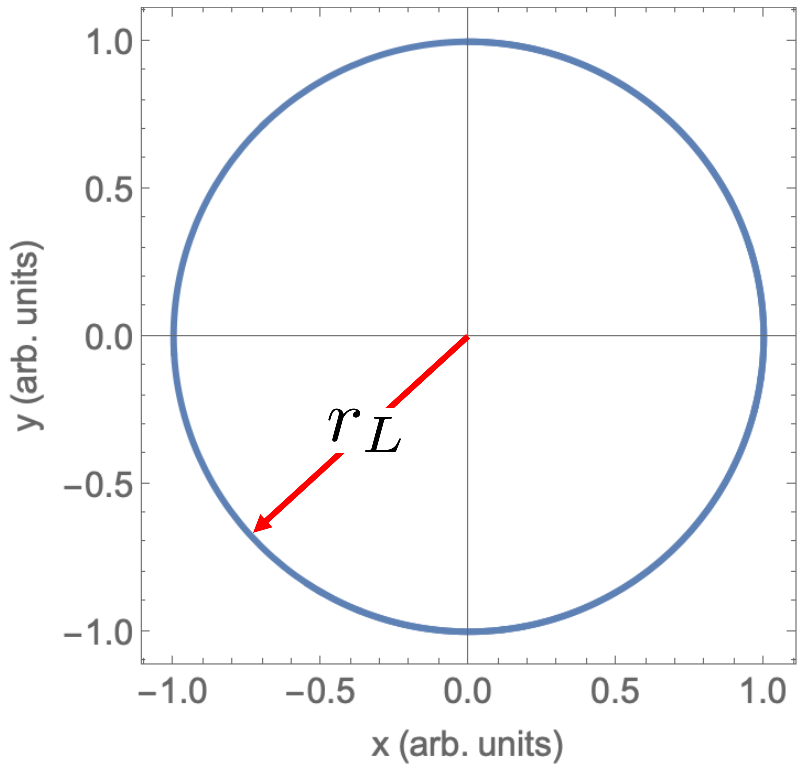
$$v_x = v_\perp \cos\left(\omega_c t + \phi_o\right) \qquad v_y = \mp v_\perp \cos\left(\omega_c t + \phi_o\right)$$

- Integrate with respect to time to find the position

$$x\left(t\right) = \frac{m v_\perp}{|q| B_o} \sin\left(\omega_c t + \phi_o\right) + x_o$$

$$y\left(t\right) = \pm \frac{m v_\perp}{|q| B_o} \cos\left(\omega_c t + \phi_o\right) + y_o$$

$$\omega_c \equiv \frac{|q| B_z}{m} \qquad r_L \equiv \frac{m v_\perp}{|q| B_o}$$



Single particle motion

- What if there is an electric field?

$$\vec{B} = B_z \hat{z} \qquad \vec{E} = E_z \hat{z}$$

$$\vec{F}_{net} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = m\vec{a}$$

- In component form

$$\dot{v_x} = \frac{q}{m}\left(v_y B_z\right) \qquad \dot{v_y} = -\frac{q v_x B_z}{m} \qquad \dot{v_z} = \frac{qE_z}{m}$$

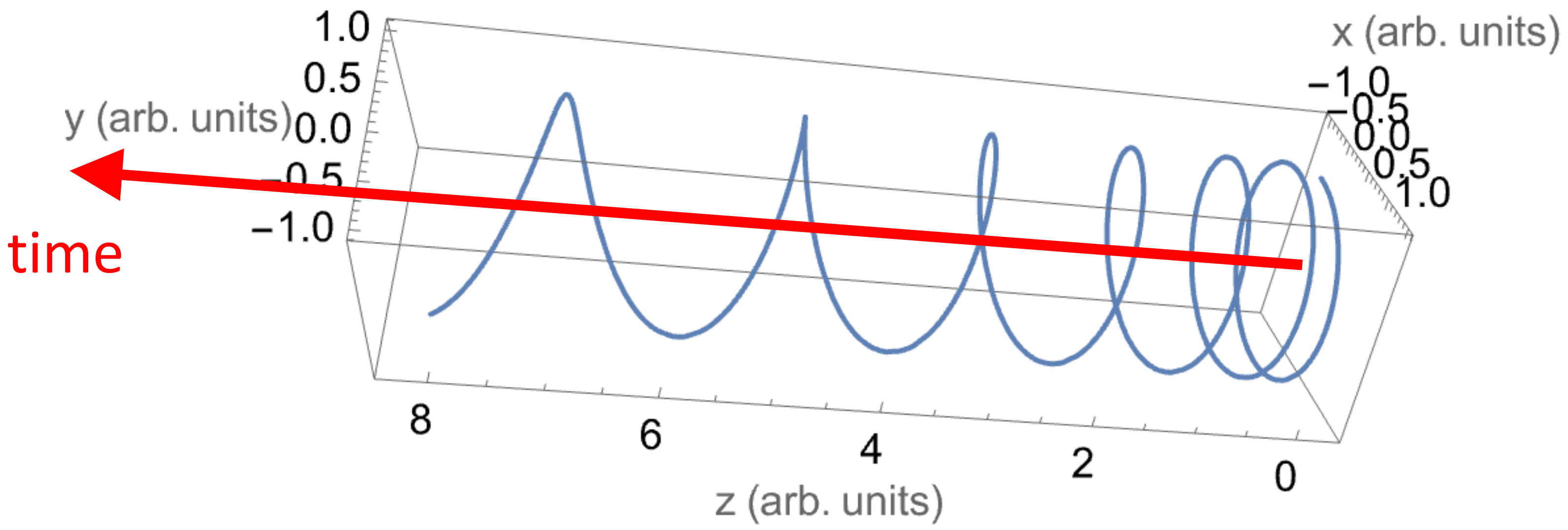
Same motion as before

constant acceleration in the *z*-direction

Single particle motion

- What if there is an electric field?

$$\vec{B} = B_z \hat{z} \qquad \vec{E} = E_z \hat{z}$$



Single particle motion

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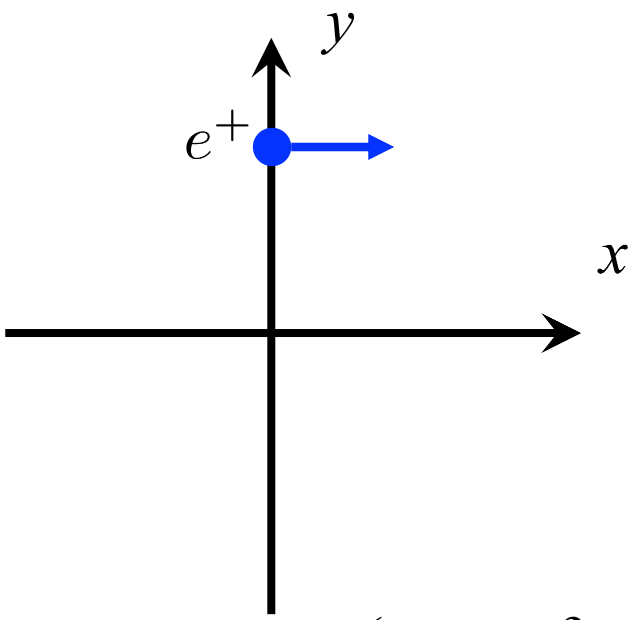

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Single particle motion

- What if there is an electric field?

$$\vec{B} = B_z\hat{z} \qquad \vec{E} = E_x\hat{x}$$

$$\vec{F}_{net} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = m\vec{a}$$

- In component form

$$\dot{v_x} = \frac{q}{m}\left(v_y B_z + E_x\right) \qquad \dot{v_y} = -\frac{q v_x B_z}{m} \qquad \dot{v_z} = 0$$

Single particle motion

$$\omega_c \equiv \frac{|q|B_z}{m}$$

$$\dot{v_x} = \frac{q}{m}\left(v_y B_z + E_x\right) \qquad \dot{v_y} = -\frac{q v_x B_z}{m}$$

- These can be decoupled by taking a time derivative

$$\ddot{v_x} = -\frac{q^2 B_z^2}{m^2} v_x = -\omega_c^2 v_x \qquad \ddot{v_y} = -\omega_c^2 \left(v_y + \frac{E_x}{B_z}\right)$$

- Which we can solve

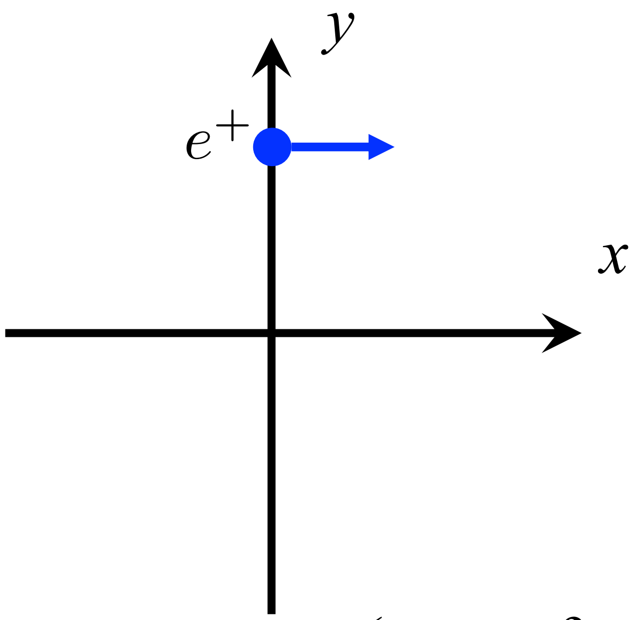
$$v_x = v_\perp \cos\left(\omega_c t + \phi_o\right) \qquad v_y = \mp v_\perp \sin\left(\omega_c t + \phi_o\right) - \frac{E_x}{B_z}$$

Guiding center moves at a constant speed in the –*y* direction

Single particle motion

- What if there is an electric field?

$$\vec{B} = B_z \hat{z} \qquad \vec{E} = E_x \hat{x}$$



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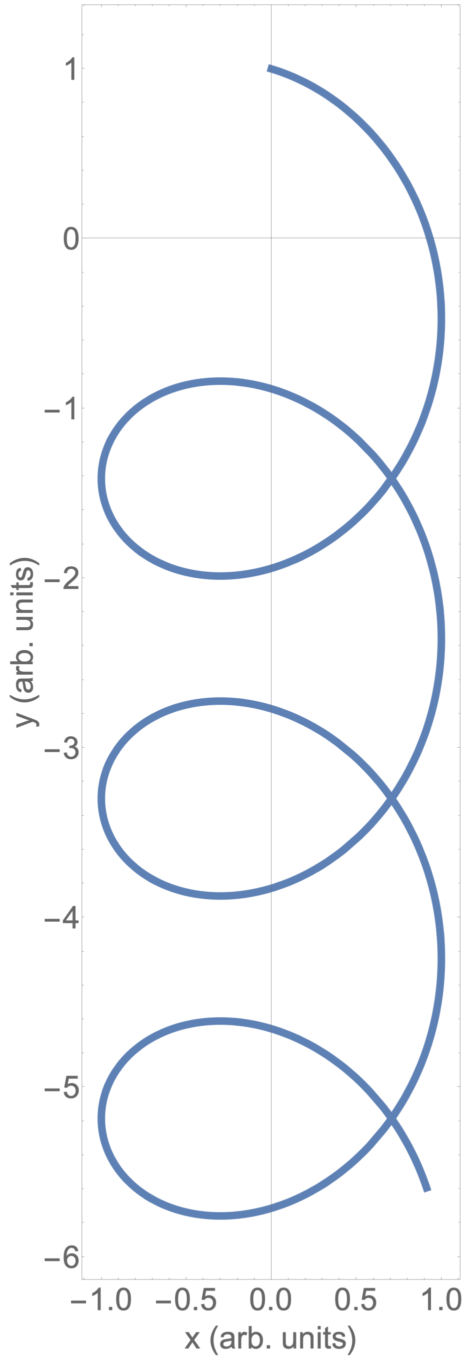
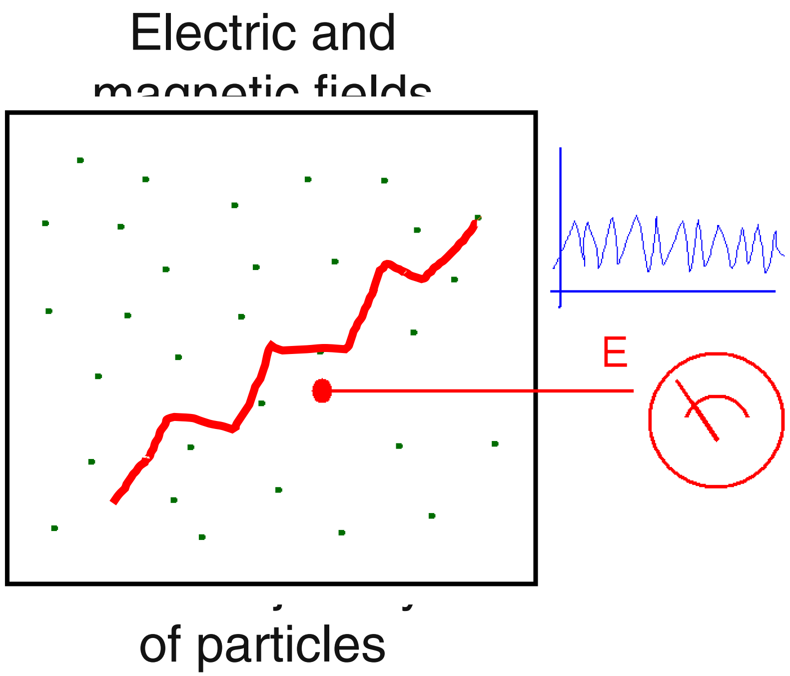


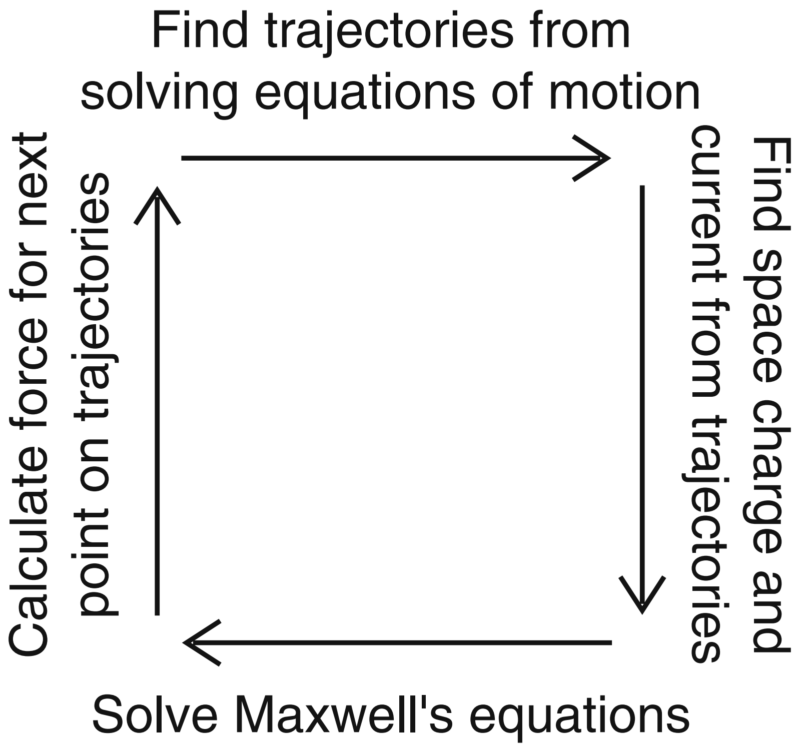
Single particle motion - Key takeaways

- Particle motion can be broken into two types of motion
  - Gyration
    - Particles orbit the applied magnetic field
    - Radius of this orbit is the Larmor radius, $r_L \equiv \frac{mv_\perp}{|q|B_o}$
    - Frequency of the orbit is the cyclotron frequency, $\omega_c \equiv \frac{|q|B_o}{m}$
  - Guiding center (what is left when you average over the gyration)

Single particle motion

- Can we extend this?
  - Charged particles do ~~not~~ interact with one another and their motions do ~~not~~ constitute a large enough current to significantly affect the EM fields.



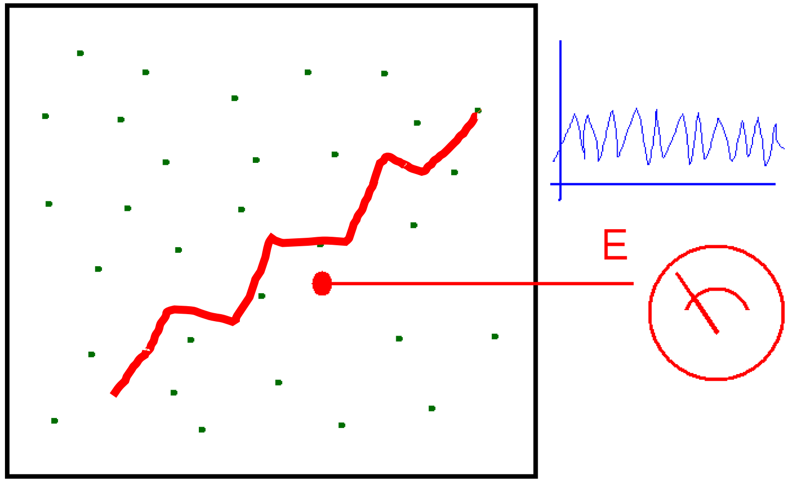


Cartoon figures by G. Lapenta in “Particle In Cell Methods With Application to Simulations in Space Weather”

Single particle motion:

- But. . .
  - There are many particles that are interacting
  - The interaction depends on the position and velocity of all of the other particles.

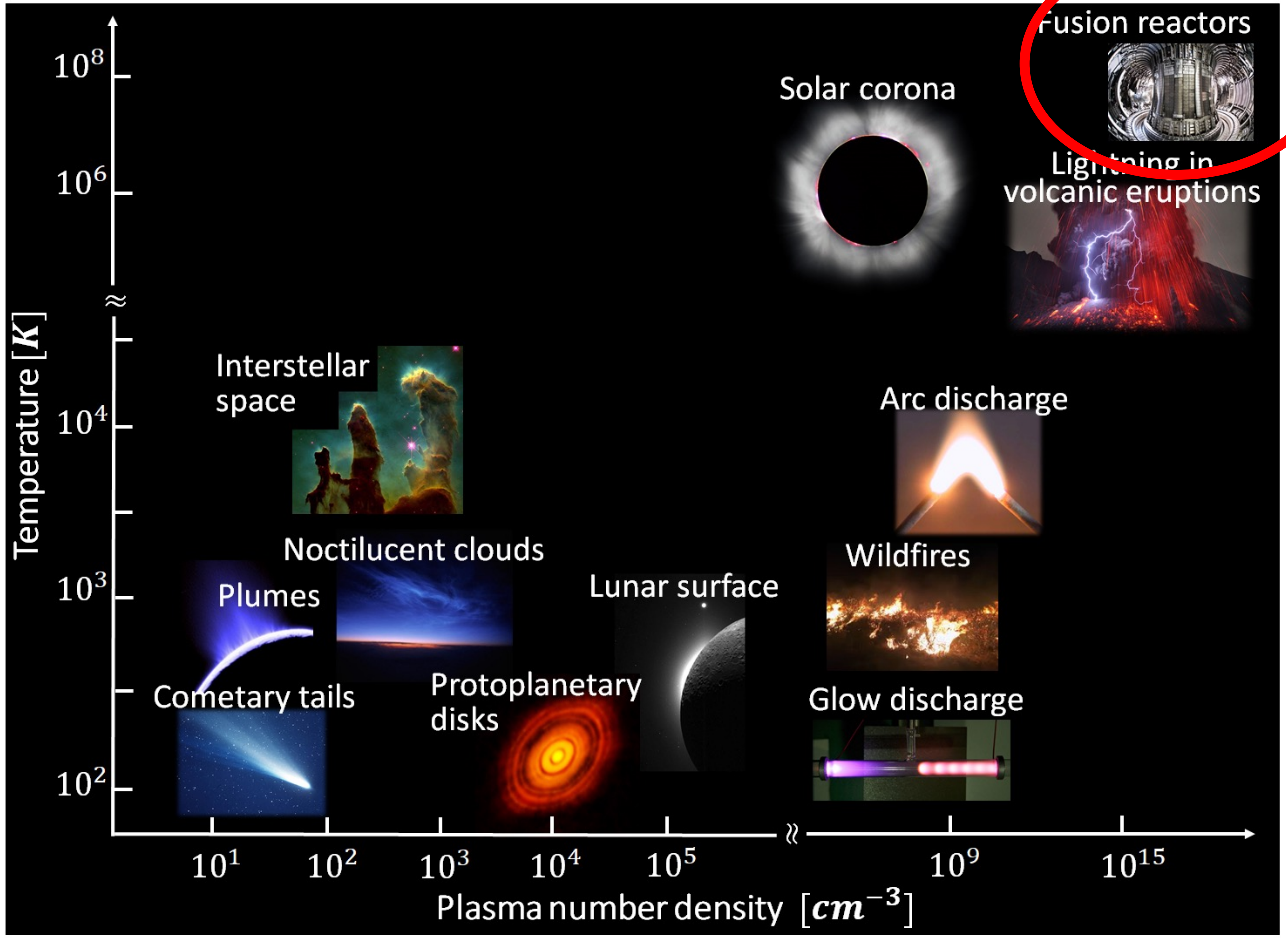
$$F_p = q_p \sum_{j=1}^{N} \frac{1}{4\pi\epsilon_o} \frac{q_j}{|\vec{r_i} - \vec{r_j}|^2}$$



Cartoon figures by G. Lapenta in "Particle In Cell Methods With Application to Simulations in Space Weather"

Single particle motion

$$\lambda_D = \sqrt{\frac{\epsilon_o T}{q^2 n}}$$



$$n_i \sim 10^{20} \frac{\text{ions}}{\text{m}^3}, \; T_i = 10^8 \text{ K}$$

$$N \sim 10^{23} \text{ions}$$

- Is there any hope?

$$\lambda_D \sim 70 \times 10^{-6} \text{m}$$

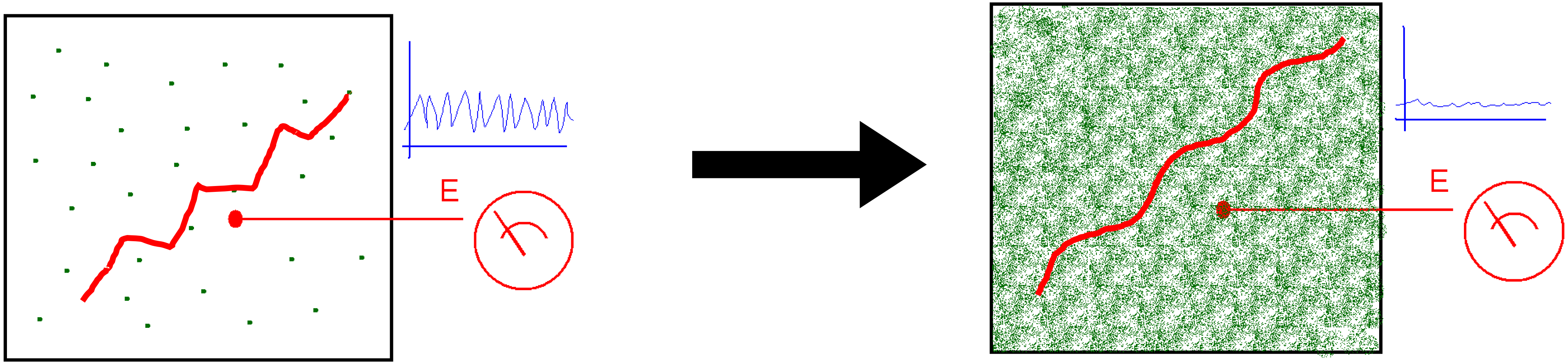
$$N_{3\lambda_D} \sim 10^8$$

- Is there another way?

Image from L. Matthews 2021 SULI Talk, *Introduction to Plasma Physics*

Kinetic Description

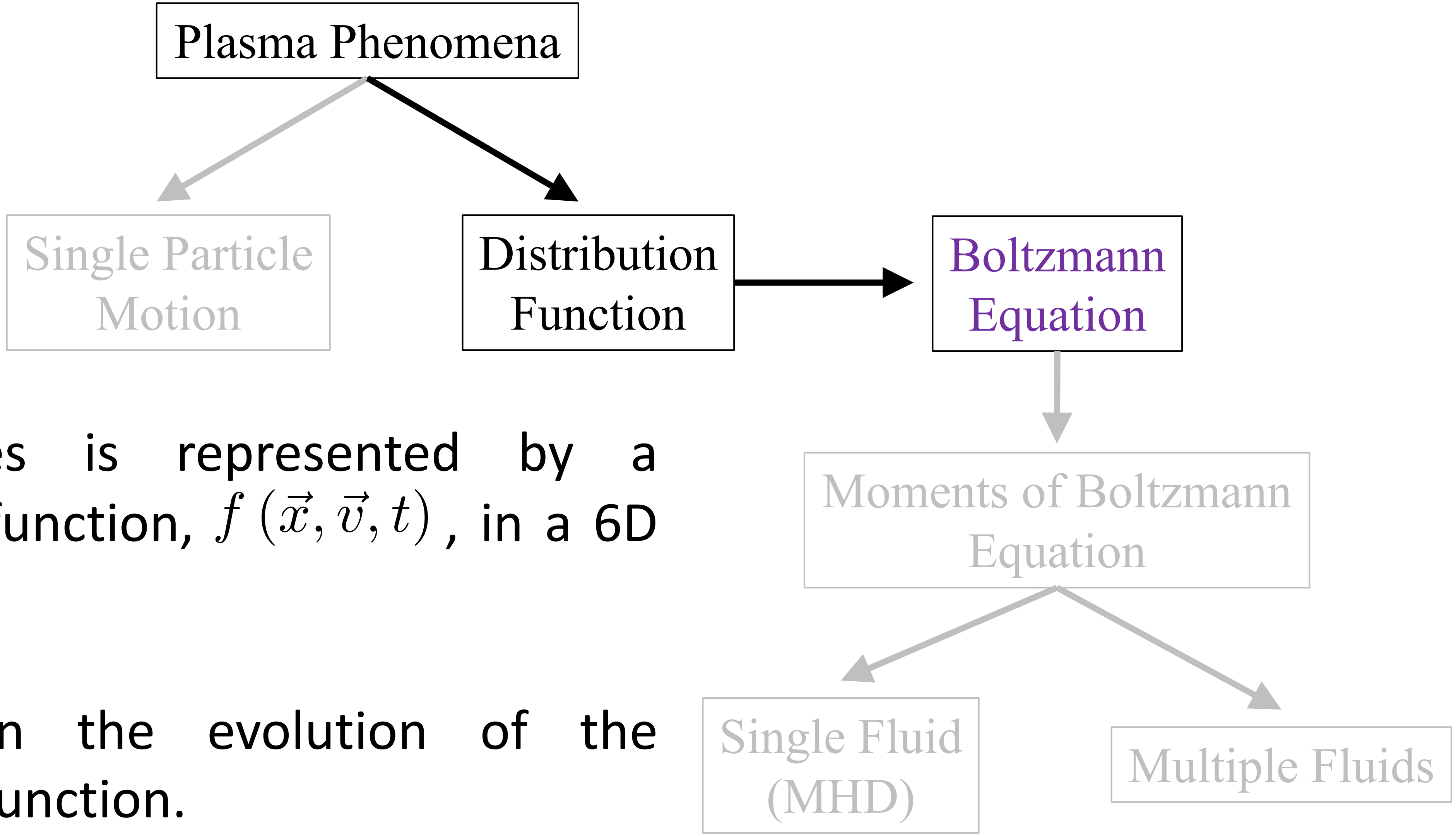
- There is a large number of particles in a volume of size λ_D^3



- Replace the discrete particles with a (smooth) distribution function, $f\left(\vec{x}, \vec{v}, t\right)$

Cartoon figures by G. Lapenta in "Particle In Cell Methods With Application to Simulations in Space Weather"

Hierarchy of Models



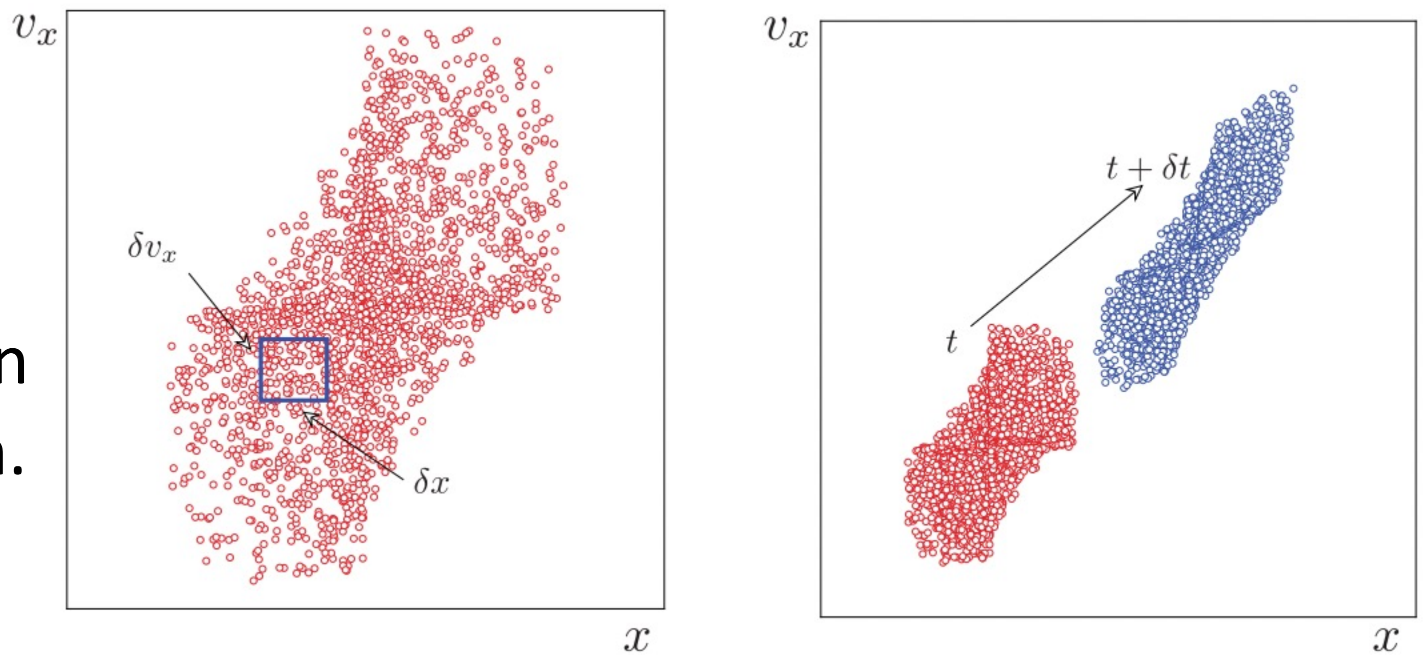
- Each species is represented by a distribution function, $f\left(\vec{x}, \vec{v}, t\right)$, in a 6D phase space
- Interested in the evolution of the distribution function.

Kinetic Description

- Each species is represented by a distribution function, $f\left(\vec{x}, \vec{v}, t\right)$, in a 6D phase space

$$f\left(\vec{x}, \vec{v}, t\right) d\vec{x} d\vec{v} = \#\ \text{of particles in a 6D phase-space volume}\ d\vec{x} d\vec{v}$$

- Interested in the evolution of the distribution function.



Cartoon figures by G. Colonna in “Plasma Modeling: Methods and Applications”

Kinetic Description

- If we track the distribution function along the trajectories that the particles follow in phase space, the distribution function is conserved.

$$0 = \frac{d}{dt}\left(f\left(\vec{x}, \vec{v}, t\right)\right)$$

$$\vec{v} = \frac{d\vec{x}}{dt} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \nabla_v f = 0$$

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \nabla_v f = \left(\frac{\partial f}{\partial t}\right)_c$$

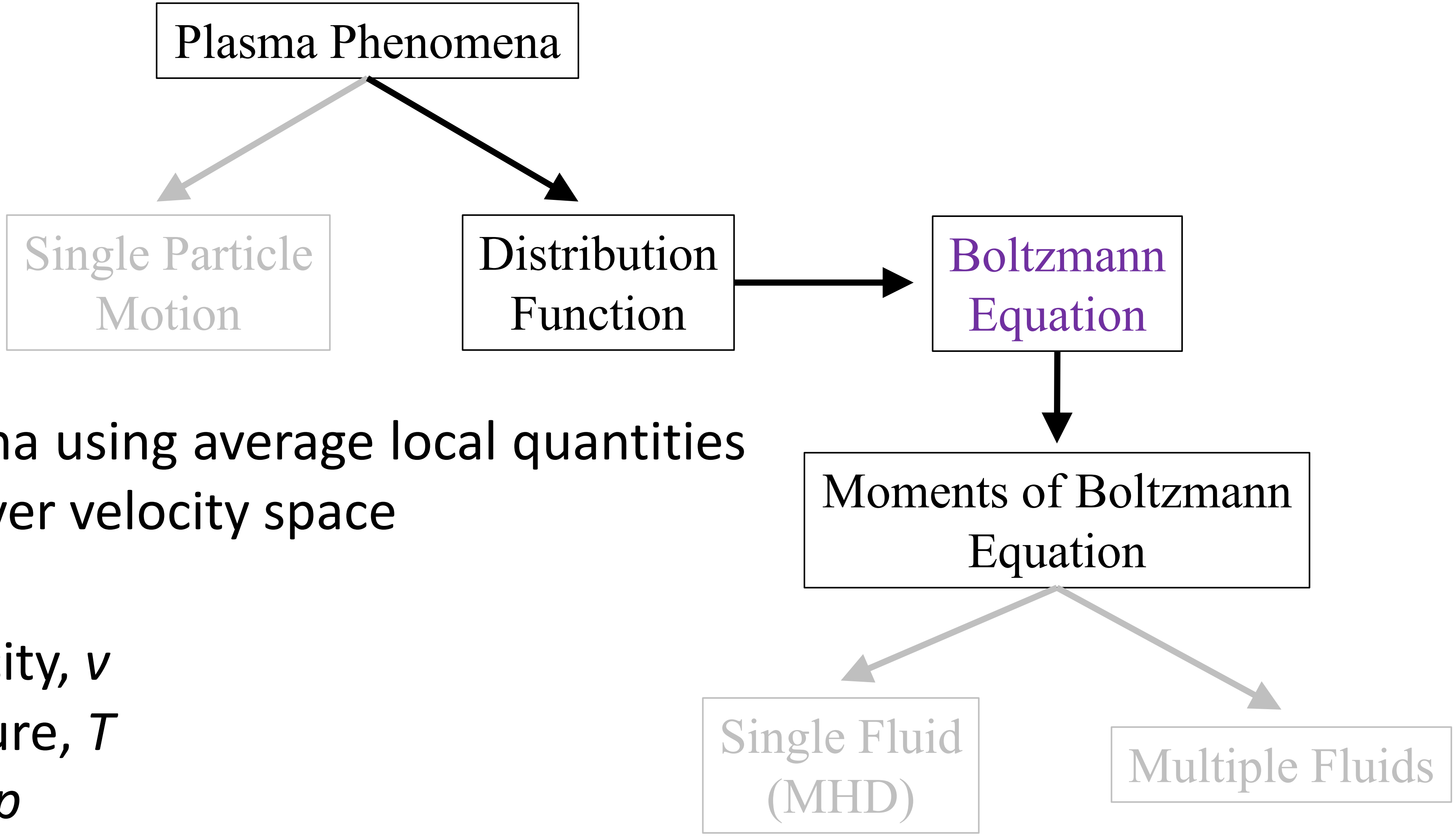
Kinetic Description

- Vlasov Equation: $$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \nabla_v f = 0$$
- Boltzmann Equation: $$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \nabla_v f = \left(\frac{\partial f}{\partial t} \right)_c$$

- Maxwell Equations: $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_o \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o} \qquad \nabla \cdot \vec{B} = 0$$

➢ This is better but it is still computationally/analytically and intuitively really hard.

Hierarchy of Models



Model the plasma using average local quantities by integrating over velocity space

- density, *n*
- fluid velocity, *v*
- temperature, *T*
- pressure, *p*

Moment Approach

- Macroscopic (fluid) quantities in physical space are found by taking velocity moments of the distribution function

  - Density: $n\left(\vec{x},t\right)=\int f\left(\vec{x},\vec{v},t\right)d\vec{v}$
  - Mean Flow: $n\vec{V}\left(\vec{x},t\right)=\int \vec{v}f\left(\vec{x},\vec{v},t\right)d\vec{v}$
  - Pressure: $\overleftrightarrow{P}\left(\vec{x},t\right)=m\int\left(\vec{v}-\vec{V}\right)\left(\vec{v}-\vec{V}\right)f\left(\vec{x},\vec{v},t\right)d\vec{v}$

Moment Approach

- If we take moments of the Boltzmann Equation, we get the exact fluid equations.

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \nabla_v f = \left(\frac{\partial f}{\partial t}\right)_c$$

- Example: 0th moment (integrate over all of the velocities, $\int v^0 \, [\;] \, d\vec{v}$)

$$\int \frac{\partial f}{\partial t} d\vec{v} + \int \vec{v} \cdot \nabla f d\vec{v} + \int \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \nabla_v f d\vec{v} = \int \left(\frac{\partial f}{\partial t}\right)_c d\vec{v}$$

$$\frac{\partial n_s}{\partial t} + \nabla \cdot \left(n_s \vec{V}_s\right) = 0$$

Moment Approach

- Continuity Equation, $\int [\ ]\, d\vec{v}$

$$\frac{\partial n_s}{\partial t} + \nabla \cdot \left(n_s \vec{V}_s \right) = 0$$

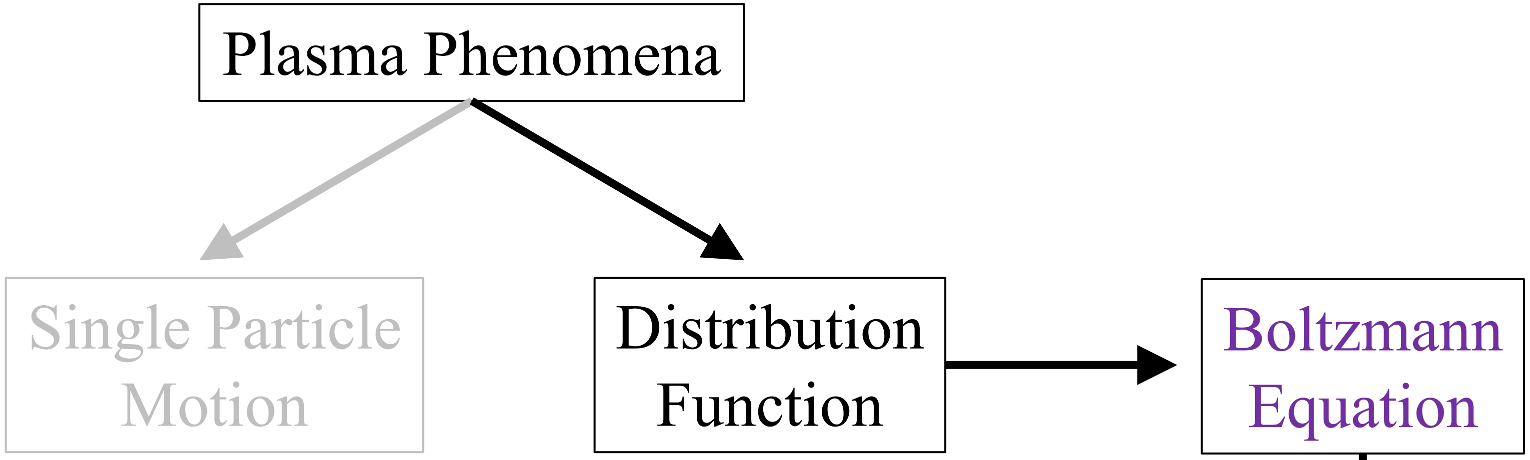
- Momentum Equation, $\int m\vec{v}\, [\ ]\, d\vec{v}$

$$m_s n_s \left(\frac{\partial \vec{v}_s}{\partial t} + \vec{v}_s \cdot \nabla \vec{v}_s \right) + q_s n_s \left(\vec{E} + \vec{v}_s \times \vec{B} \right) - \nabla \cdot \overleftrightarrow{P}_s + \vec{R}_s$$

- Energy Equation, $\int \frac{m\vec{v}^2}{2}\, [\ ]\, d\vec{v}$

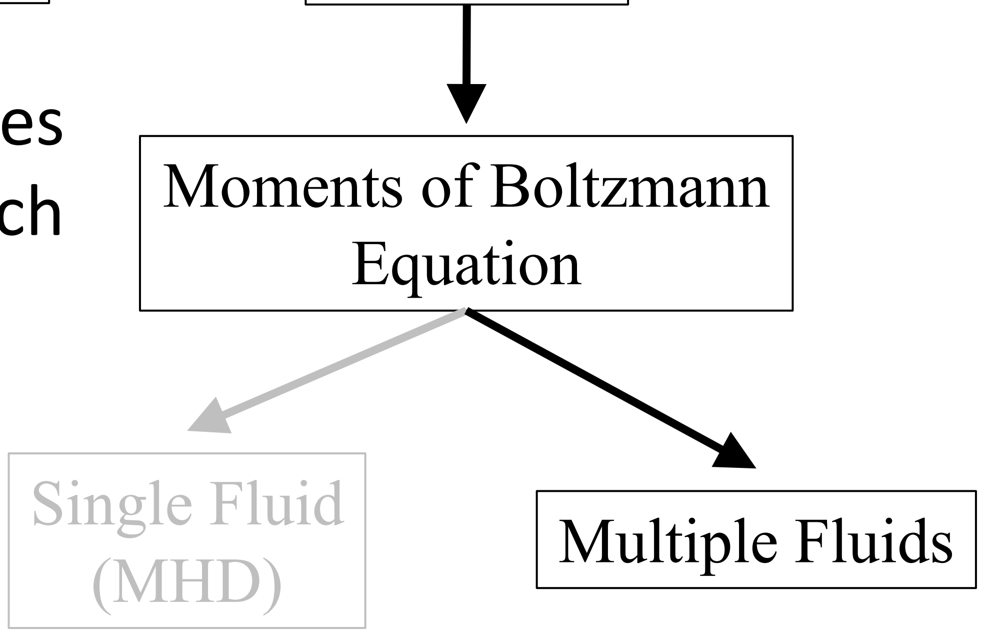
➢ Closure problem: for each moment, we introduce a new unknown
  - Need to make approximations to close the moment hierarchy

Hierarchy of Models



Model the plasma using average local quantities by integrating over velocity space for each plasma species

- density, n_s
- fluid velocity, v_s
- temperature, T_s
- pressure, p_s



Two-fluid Model

- Momentum Equation

$$m_s n_s \left(\frac{\partial \vec{v}_s}{\partial t} + \vec{v}_s \cdot \nabla \vec{v}_s\right) + q_s n_s \left(\vec{E} + \vec{v}_s \times \vec{B}\right) - \nabla \cdot \overleftrightarrow{P}_s + \vec{R}_s$$

- Continuity Equation,

$$\frac{\partial n_s}{\partial t} + \nabla \cdot \left(n_s \vec{V}_s\right) = 0$$

- Space charge and current

$$\rho = n_i e - n_e e$$

$$\vec{j} = n_i e \vec{u}_i - n_e e \vec{u}_e$$

- Maxell's Equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o}$$

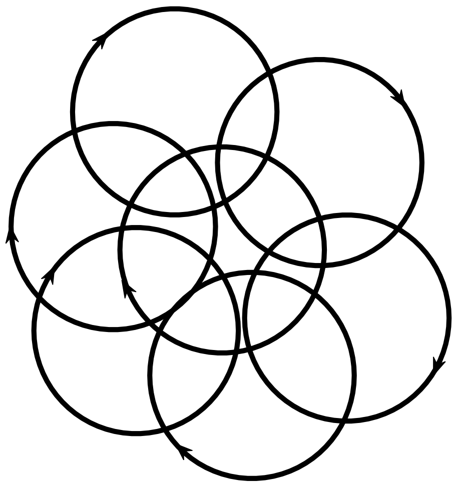
$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

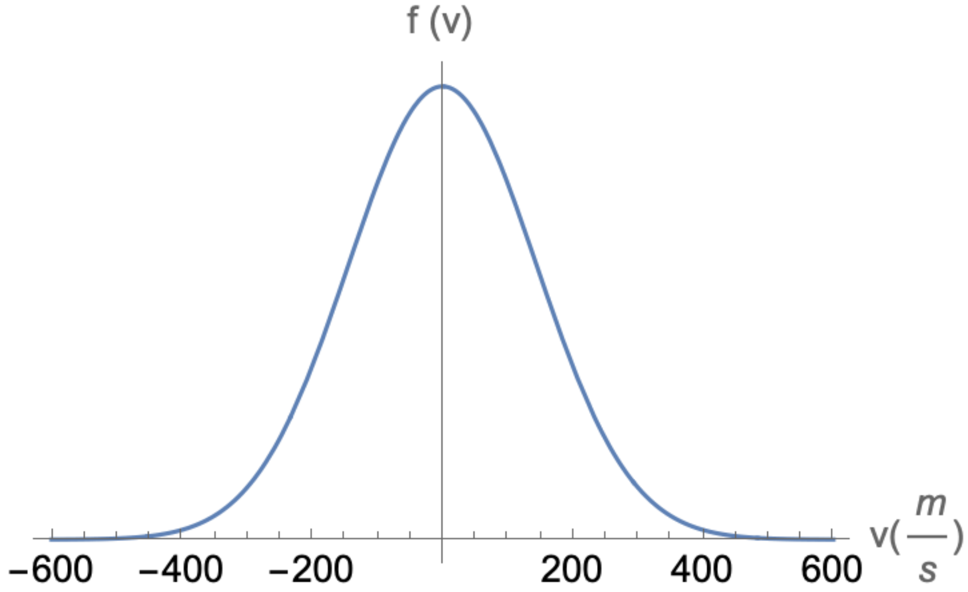
Fluid Approach – Is it reasonable?

- Particles to move together during the evolution of the plasma
  - High collisionality
  - Low temperature
  - Strong magnetic field

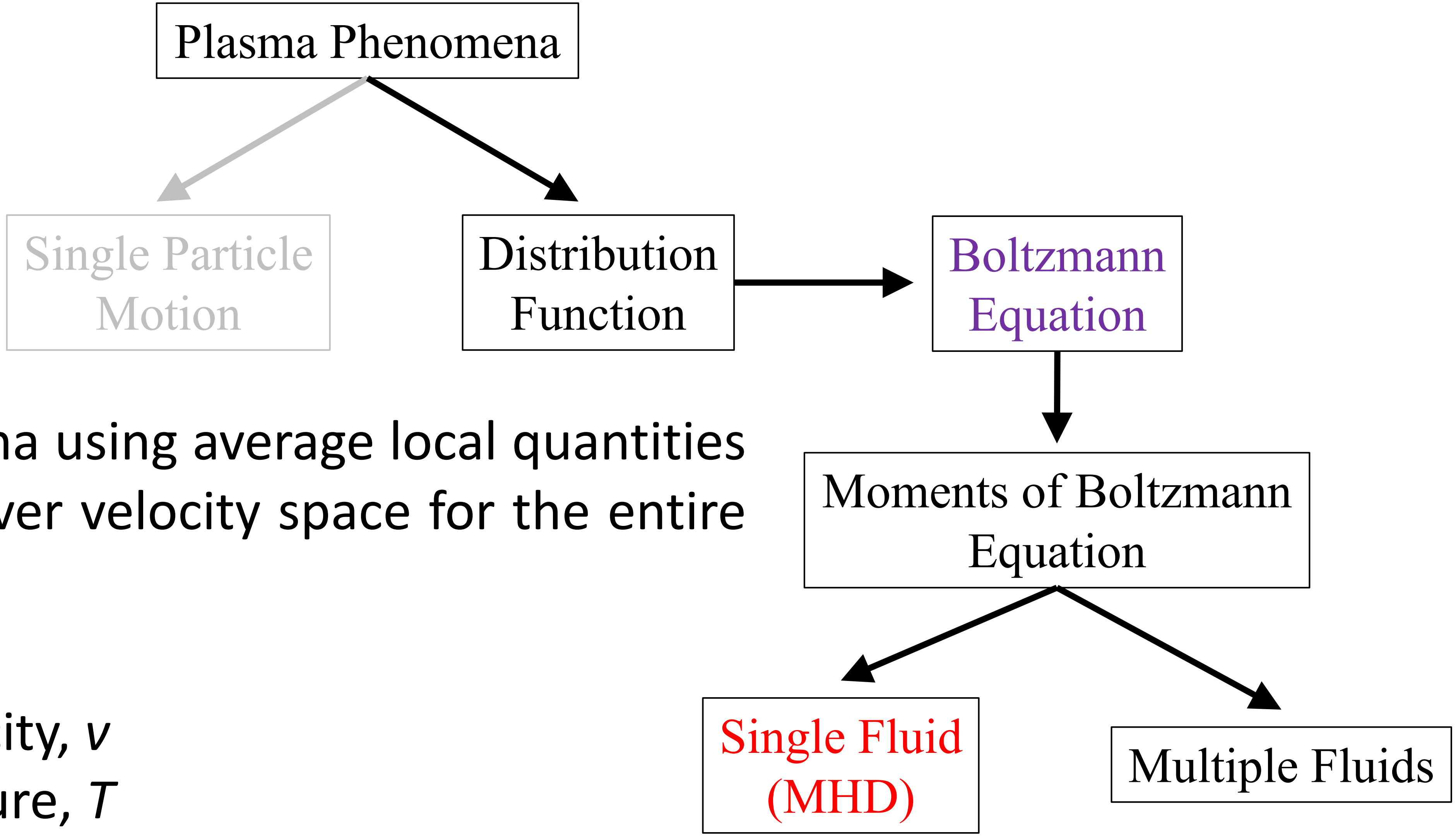


Weak magnetic field

Strong magnetic field



Hierarchy of Models



Model the plasma using average local quantities by integrating over velocity space for the entire plasma

- density, n
- fluid velocity, v
- temperature, T
- pressure, p

(Ideal) Magnetohydrodynamics

- One fluid model of a plasma that treats the plasma as a nonrelativistic, highly collisional, electrically conductive fluid in an electromagnetic field
  - Highly collisional, so the the electron and ion temperatures have equilibrated and their particle distributions are Maxwellian
  - Gyroradius is small compared to any spatial scales of interest
  - Frequencies faster than the electron plasma frequency are ignored
  - Non-relativistic means we can ignore the displacement currents in Ampere's Law

(Ideal) Magnetohydrodynamics

- Assumptions:
  - the plasma is quasi-neutral, $n_e = n_i$
  - model the electron behavior by assuming that $m_e \to 0$
  - treat the plasma as a perfect conductor

- Key parameters
  - Mass density, ρ
  - Center of mass fluid velocity, $\vec{u}$
  - Current density, $\vec{J}$
  - Species summed pressure, *p*
  - Electric field, $\vec{E}$
  - Magnetic field, $\vec{B}$

(Ideal) Magnetohydrodynamics

- Assumptions:
  - the plasma is quasi-neutral, $n_e = n_i$
  - model the electron behavior by assuming that $m_e \to 0$
  - treat the plasma as a perfect conductor .

- Continuity Equation

$$\rho = \sum_s m_s n_s$$

$$\frac{\partial n_s}{\partial t} + \nabla \cdot \left(n_s \vec{V}_s \right) = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) = 0$$

(Ideal) Magnetohydrodynamics

- Assumptions:
  - the plasma is quasi-neutral, $n_e = n_i$
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- Add the Momentum Equation for ions and electrons

$$m_s n_s \left(\frac{\partial \vec{v}_s}{\partial t} + \vec{v}_s \cdot \nabla \vec{v}_s\right) + q_s n_s \left(\vec{E} + \vec{v}_s \times \vec{B}\right) - \nabla \cdot \overset{\leftrightarrow}{P}_s + \vec{R}_s$$

$$m_i n_i \left(\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i \cdot \nabla \vec{v}_i\right) + q_i n_i \left(\vec{E} + \vec{v}_s \times \vec{B}\right) - \nabla \cdot \overset{\leftrightarrow}{P}_i$$

$$q_e n_e \left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \nabla \cdot \overset{\leftrightarrow}{P}_e$$

(Ideal) Magnetohydrodynamics

$\rho = \sum_s m_s n_s = m_i n_i$

- Assumptions:
  - the plasma is quasi-neutral, $n_e = n_i$
  - model the electron behavior by assuming that $m_e \to 0$
  - Treat the plasma as a perfect conductor .

- Add the Momentum Equation for ions and electrons

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = \vec{J} \times \vec{B} - \nabla \cdot \left(\overset{\leftrightarrow}{P}_i + \overset{\leftrightarrow}{P}_e \right)$$

$\vec{J} = \sum_s q_s n_s \vec{v}_s$

(Ideal) Magnetohydrodynamics

- Assumptions:
  - the plasma is quasi-neutral, $n_e = n_i$
  - model the electron behavior by assuming that $m_e \to 0$
  - Treat the plasma as a perfect conductor .

- From the Momentum Equation for electrons

$$m_s n_s \left(\frac{\partial \vec{v}_s}{\partial t} + \vec{v}_s \cdot \nabla \vec{v}_s\right) + q_s n_s \left(\vec{E} + \vec{v}_s \times \vec{B}\right) - \nabla \cdot \overset{\leftrightarrow}{P}_s + \vec{R}_s$$

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

(Ideal) Magnetohydrodynamics- Governing Equations

- Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0$$

- Momentum Equation

$$\rho \frac{\partial \vec{V}}{\partial t} = \vec{J} \times \vec{B} - \nabla p$$

- Energy Equation

$$\frac{d}{dt}\left(\frac{p}{\rho^{\frac{5}{3}}}\right) = 0$$

- Ohm's Law

$$\vec{E} + \vec{V} \times \vec{B} = 0$$

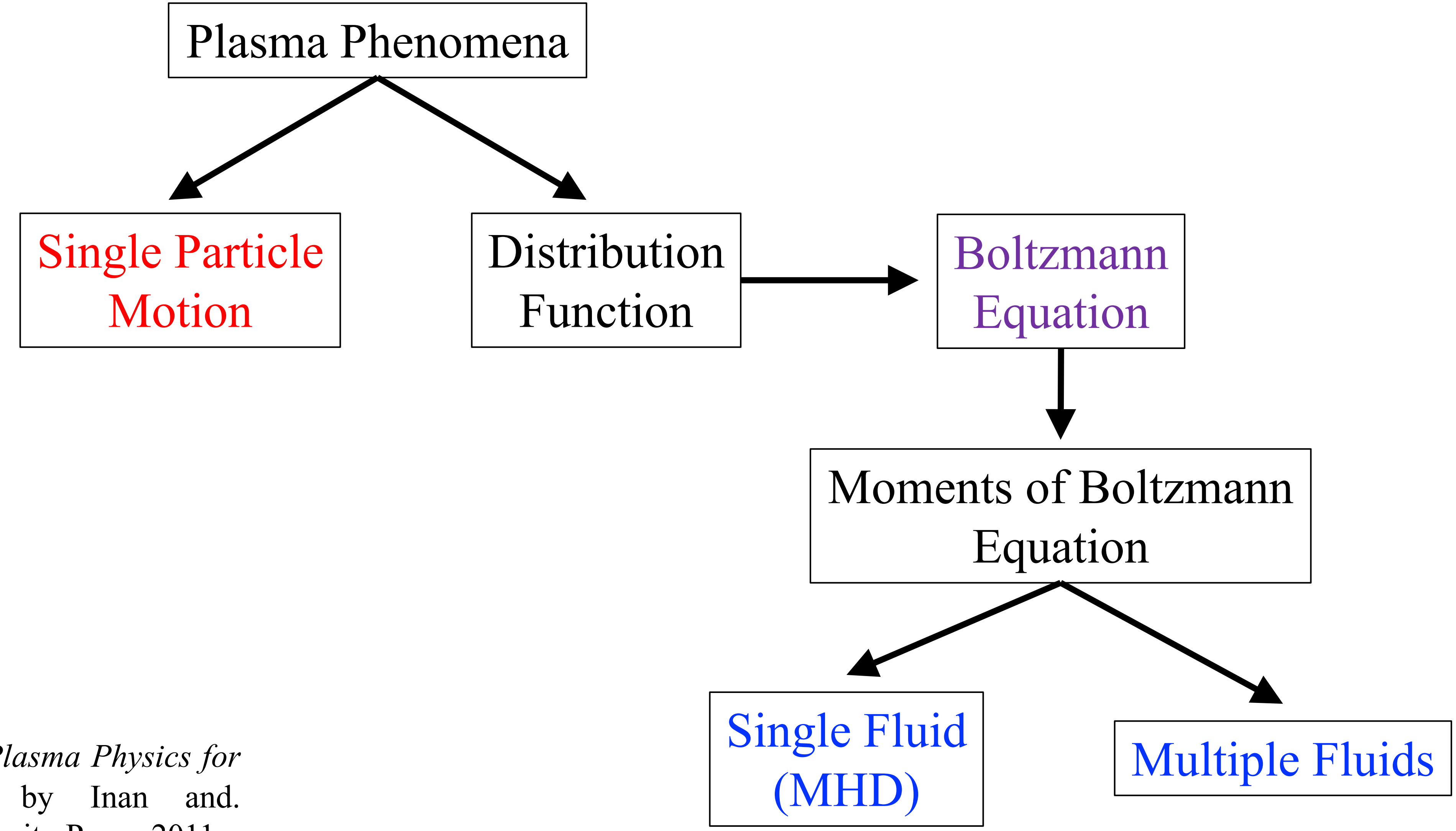
- Maxwell's Equations

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_o \vec{J}$$

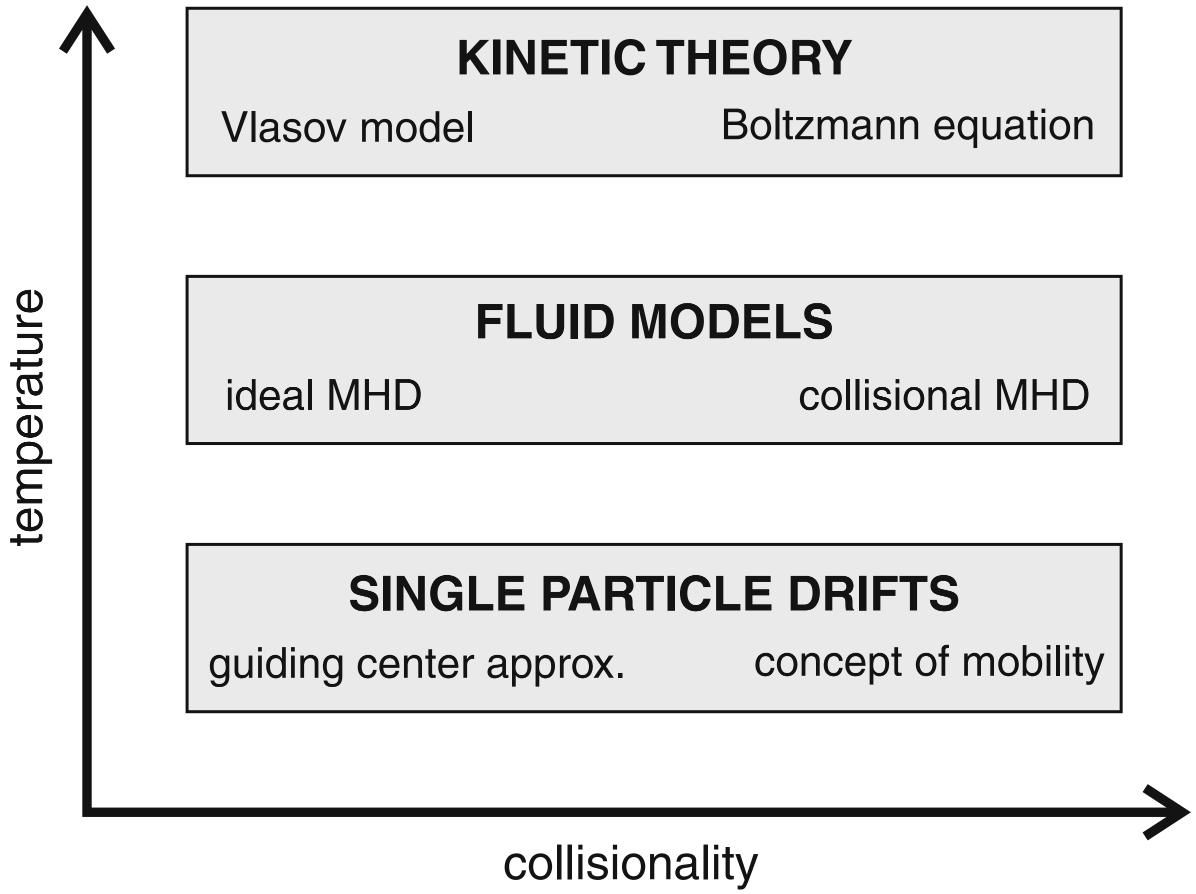
Key Takeaways



Adapted from *Principles of Plasma Physics for Engineers and Scientists* by Inan and. Gołkowski, Cambridge University Press, 2011.

Key Takeaways

- In general, each of these approaches are useful in the appropriate regime



Key Takeaways

- Each method can provide valuable insight, allowing us to build intuition and to interpret result.
  - Insights can be amplified by using multiple models.

	Magnetohydrodynamics	Two Fluids	Gyrokinetics	Kinetics	Everything
Description	The plasma is one continuous fluid - ions have all the mass, but electron carry all the current.	Break the ions & electrons into two continuous, mingling fluids.	Only track superparticles' straight motion - and ignore the corkscrewing.	Assign particles a speed and location based on a distribution. Track super particles through space.	Track every particle, at all times.
Strengthens	Easily solved.	Simple bulk effects like drift waves & reconnection can be understood.	Captures most of kinetic model, but much easier to solve - can model an entire Tokamak.	Many things captured, can get powerful results like the linear velocity-space instabilities.	Most accurate model possible.
Weakness	Most things not captured: most plasma waves, leakage, kinetic instabilities, structures etc.	Many things not captured: plasma instabilities, large effects & non-equilibrium effects. Assumes bell curves.	Non-physical behavior over long times: resonances & adiabatic invariants can be lost.	Tough to solve: hard to apply to full-size reactors. Loses some effects: like plasma microdensity and collective thomson scattering.	Typically impossible to solve.
Mathematics	Navier-stokes, Lorentz force, Maxwells' equations.	Navier-stokes, Lorentz force, Maxwells' equations.	Vlasov-Maxwell Expansion Equation	Vlasov-Maxwell Equation	Klimontovich Model

Plasma as a fluid (Chalkboard) — Plasma as a gas (Computer Required)

S i m p l i c i t y — D e t i a l

Thank you for your attention.
Enjoy your summer project.

Questions? Comments?

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