Short bio

- I studied aerospace engineering in Madrid (‘99-’04), where I did undergraduate research on plasmas for electric space propulsion
- Moved to MIT to get a PhD in electric space propulsion (‘04-’06), and I ended up switching to a PhD magnetic confinement fusion (‘06-’09)
  - Arturo, thank you for convincing me to do fusion!
- Spent two years at the Rudolf Peierls Centre for Theoretical Physics (Oxford) as a postdoc (‘09-’11)
- Since 2011, I have held faculty positions at MIT (‘11-’13) and Oxford (‘13 – present).

Turbulence in fluids and plasmas

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Aims of this lecture

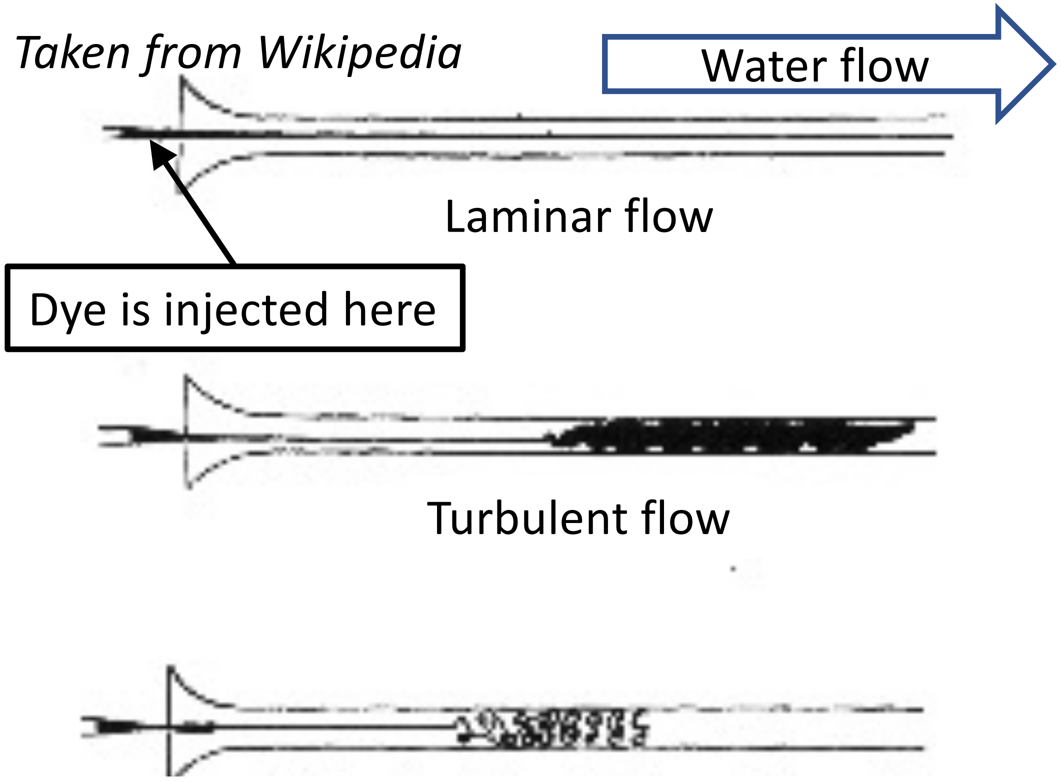
- Turbulence plays a role in many aspects of fluid and plasma physics

- In this lecture:
  - Turbulence in fluids
    - Convection and viscosity: Reynolds number
    - Turbulence energy spectrum
    - Kolmogorov theory of fluid turbulence
  - Turbulent transport in magnetic confinement fusion
    - Collisional and turbulent transport
    - Measurements
    - Simulations

Reynolds experiment

- Reynolds studied the effect of increasing the flow speed in a water pipe

Taken from Wikipedia



Turbulent flow (observed with stroboscopic light)

- Low flow
  - water flows easily
  - dye doesn't mix

$\Rightarrow$ <u>laminar flow</u>

- Large flow
  - larger drag
  - dye gets mixed

$\Rightarrow$ <u>turbulent flow</u>

Reynolds number

- Transition between laminar and turbulent flow happens at flow velocity $u_{\text{crit}} \propto 1/$(pipe diameter L)
- Can be understood by comparing terms in momentum conservation equation

density

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \rho\nu\nabla^2\mathbf{u}$$

Material time derivative

Pressure

Viscosity

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} &= \frac{\partial}{\partial t} + \boxed{\mathbf{u}\cdot\nabla} \\ &= \frac{\partial}{\partial t} + \boxed{u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3}}\end{aligned}$$

Convection

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Reynolds number

- Transition between laminar and turbulent flow happens at flow velocity $u_{\rm crit} \propto 1/$(pipe diameter L)
- Can be understood by comparing terms in momentum conservation equation

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \rho\nu\nabla^2\mathbf{u}$$

  - Ignore pressure p: for water, only ensures ρ = constant
  - Compare the other two terms ⇒ Reynolds number

$$\frac{|\mathrm{d}\mathbf{u}/\mathrm{d}t|}{\nu|\nabla^2\mathbf{u}|} \sim \frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{\nu|\nabla^2\mathbf{u}|} \sim \frac{u^2/L}{\nu u/L^2} = \frac{uL}{\nu} = \mathrm{Re}$$

  - Transition to turbulence at $\mathrm{Re} \sim 10^4 \Rightarrow u_{\rm crit} \propto 1/L$

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Why is viscosity so important?

- Viscosity dissipates fluctuations
- To prove it, get kinetic energy equation
  - Multiply momentum conservation equation by $\mathbf{u}$ and integrate over pipe volume

$$\rho \int \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{u}\, \mathrm{d}^3x = -\int \mathbf{u} \cdot \nabla p\, \mathrm{d}^3x + \nu \int (\nabla^2 \mathbf{u}) \cdot \mathbf{u}\, \mathrm{d}^3x$$

  - Use $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u^2}{2}\right)$ and divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \frac{1}{2}\rho u^2\, \mathrm{d}^3x\right) = \underbrace{\text{boundary terms}}_{\text{energy injection}} \underbrace{-\rho\nu \int \sum_i \left|\frac{\partial \mathbf{u}}{\partial x_i}\right|^2 \mathrm{d}^3x}_{\text{energy dissipation}}$$

Limit of small viscosity ($\mathrm{Re} \gg 1$)

- If one neglects viscosity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \frac{1}{2}\rho u^2 \,\mathrm{d}^3x\right) \simeq \text{injection}$$

- Does the energy grow without bound? No, because viscosity is always important

- Fluctuations force $\mathbf{u}$ to have large gradients

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \frac{1}{2}\rho u^2 \,\mathrm{d}^3x\right) = \text{injection} - \rho\nu \int \sum_i \left|\frac{\partial \mathbf{u}}{\partial x_i}\right|^2 \,\mathrm{d}^3x$$

  - $\nabla \gg 1/L$ somehow!

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The many length scales of turbulence

- Turbulence contains a continuum of scales
  - Diagnostics: energy spectrum and correlation function
- From here on, $\mathbf{u}$ has zero time average, $\langle \mathbf{u} \rangle_t = 0$
  - E.g. in a pipe, move with the flow of the pipe
- Spectrum: use Fourier transform (= sum of waves)

$$\mathbf{u}(x,t) = \int \mathbf{U}(\mathbf{k},t)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\,\mathrm{d}^3k$$

  - $\mathbf{U}\,\mathrm{d}^3k$ is the amplitude of a wave with wavelength $2\pi/k$
  - Since $\mathbf{u}$ is real, $\mathbf{U}(-\mathbf{k}) = \mathbf{U}^*(\mathbf{k})$.
- Need time averaged information because $\mathbf{U}$ fluctuates
  - $\langle \mathbf{U} \rangle_t = 0$, so instead use $\langle U^2 \rangle_t$

Relation between $\langle U^2 \rangle_t$ and energy

- Replacing Fourier transform into energy averaged over time and volume V

$$\frac{1}{2}\rho\left\langle u^2\right\rangle_{V,t} = \frac{1}{2}\rho\left\langle \mathbf{u}\cdot\mathbf{u}\right\rangle_{V,t}$$
$$= \frac{1}{2}\rho\left\langle \iint \mathbf{U}(\mathbf{k})\cdot\mathbf{U}(\mathbf{k}')\exp(\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x})\,\mathrm{d}^3k\,\mathrm{d}^3k'\right\rangle_{V,t}$$

- For $\mathbf{k} + \mathbf{k}' \neq 0$, the average of $\exp(\mathrm{i}(\mathbf{k} + \mathbf{k}')\cdot\mathbf{x})$ over V vanishes because it is very oscillatory $\Rightarrow \mathbf{k}' = -\mathbf{k}$
  - Recall that $\mathbf{U}(-\mathbf{k}) = \mathbf{U}^*(\mathbf{k})$

$$\frac{1}{2}\rho\left\langle u^2\right\rangle_{V,t} = \frac{(2\pi)^3}{V}\int \frac{1}{2}\rho\langle U^2\rangle_t\,\mathrm{d}^3k$$

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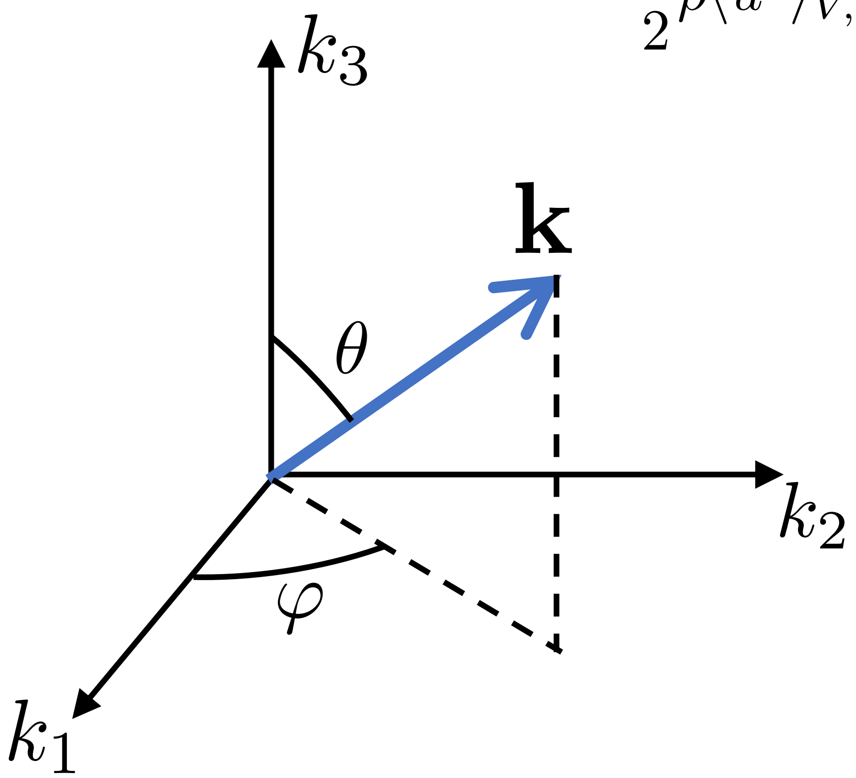
Turbulence energy spectrum

- To sketch in 1D plot, use energy spectrum $E(k)$
  - Use spherical coordinates



$$\frac{1}{2}\rho\langle u^2\rangle_{V,t} = \frac{(2\pi)^3}{V}\int \frac{1}{2}\rho\langle U^2\rangle_t \,\mathrm{d}^3k$$

$$= \frac{(2\pi)^3}{V}\int\!\!\int\!\!\int \frac{1}{2}\rho\langle U^2\rangle_t k^2 \sin\theta \,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}k$$

- Define

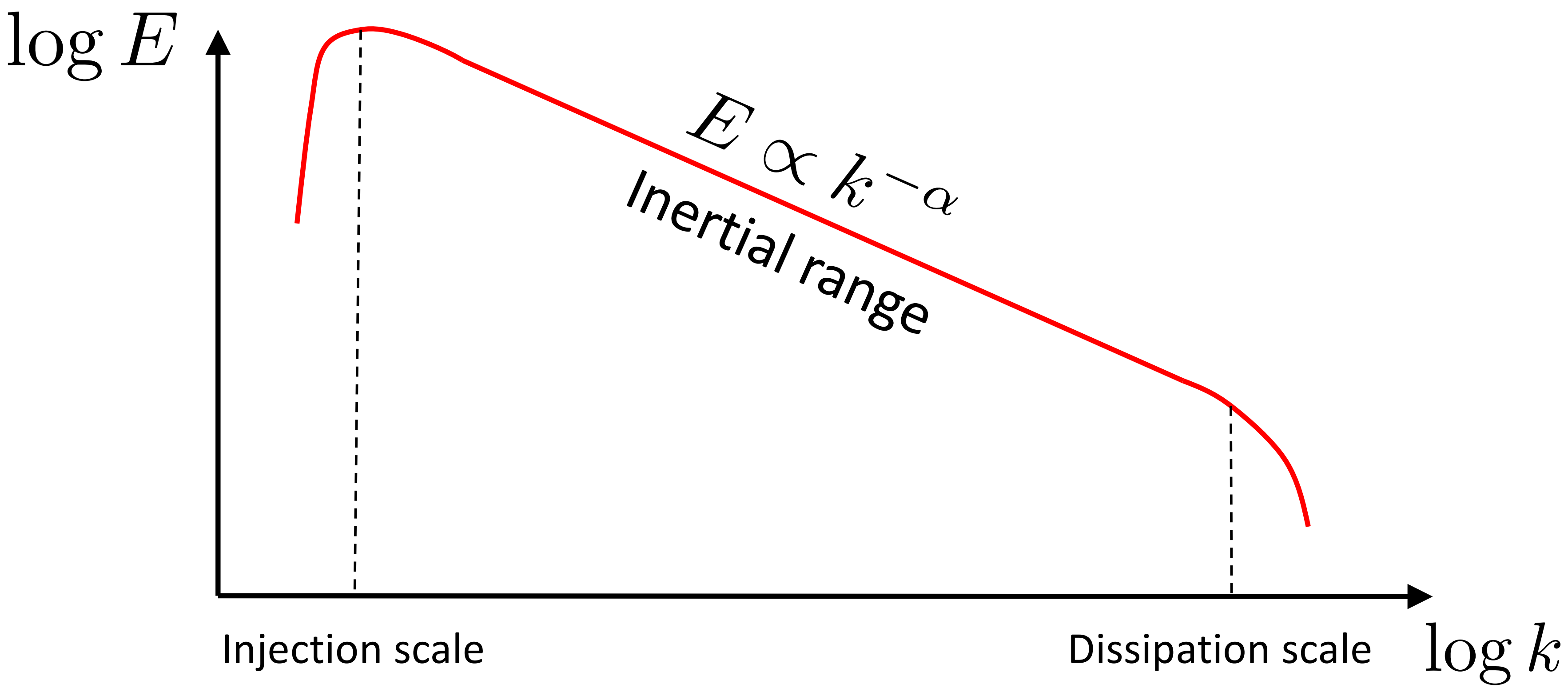
$$E(k) = \frac{(2\pi)^3}{V}\int\!\!\int \frac{1}{2}\rho\langle U^2\rangle_t k^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta$$

$$\Rightarrow \frac{1}{2}\rho\langle u^2\rangle_{V,t} = \int E(k)\,\mathrm{d}k$$

Turbulence energy spectrum

- To sketch in 1D plot, use energy spectrum $E(k)$

$$\frac{1}{2}\rho\langle u^2\rangle_{V,t} = \int E(k)\,\mathrm{d}k$$



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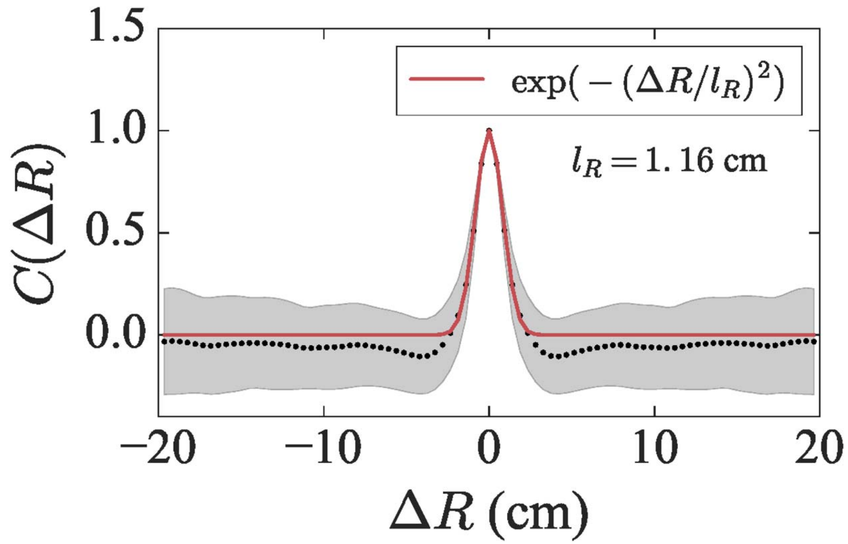
Correlation function

- Correlation function

$$C(\Delta\mathbf{x}) = \frac{1}{\langle u^2 \rangle_{V,t}} \langle \mathbf{u}(\mathbf{x}+\Delta\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \rangle_{V,t}$$

  - $C(\Delta\mathbf{x} = 0) = 1$
  - $C(\Delta\mathbf{x})$ vanishes for large Δx because $\mathbf{u}$ at two very different locations fluctuates in an uncorrelated manner



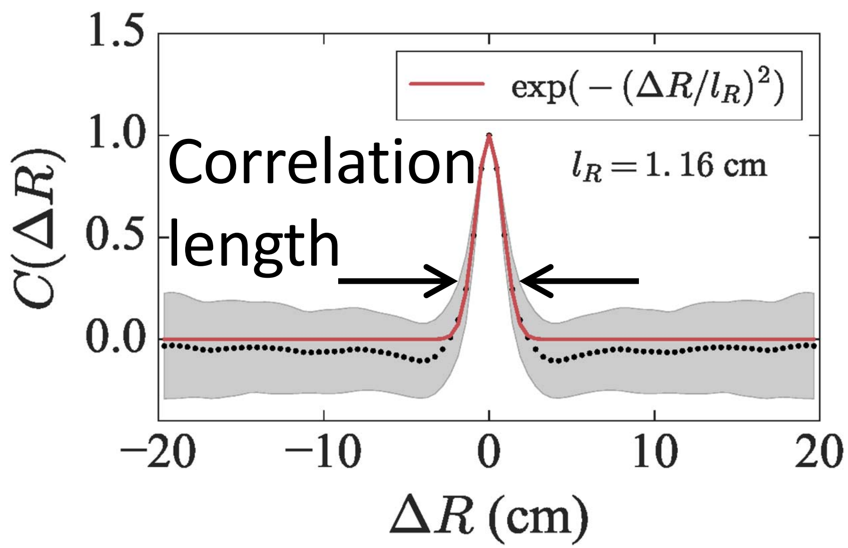
Correlation function of the density fluctuations in MAST plasma [Van Wyck PPCF 2017]

Correlation function and spectrum

- Using Fourier transforms and the fact that volume average implies $\mathbf{k}' = -\mathbf{k}$

$$C(\Delta\mathbf{x}) = \frac{(2\pi)^3}{V\langle u^2\rangle_{V,t}} \int \langle U^2\rangle_t \exp(\mathrm{i}\mathbf{k}\cdot\Delta\mathbf{x})\,\mathrm{d}^3k$$

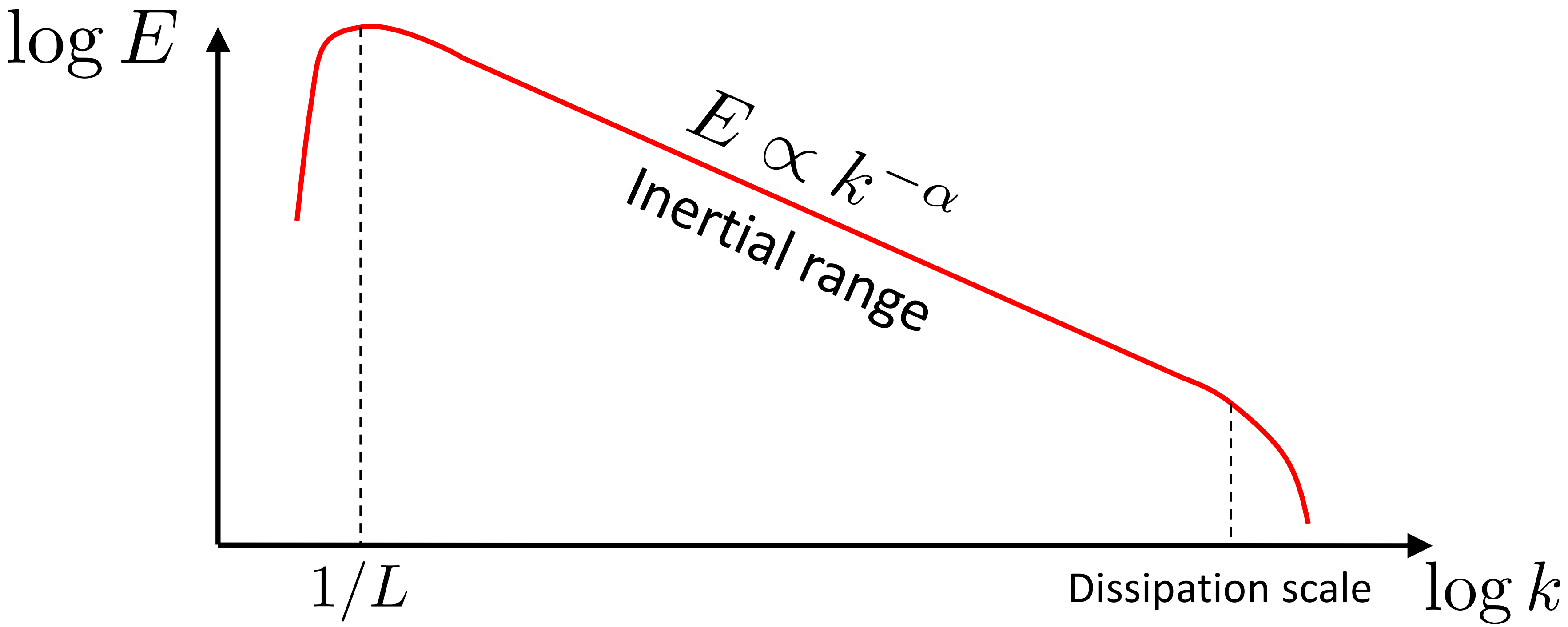
$\Rightarrow$ correlation function is Fourier transform of spectrum



- Correlation length = width of correlation function
  - Correlation length ~ injection scale

Determining the spectrum

- Given the injection scale L (pipe diameter, size of propeller, etc), we try to determine
  - Exponent α of inertial range
  - Dissipation scale



Kolmogorov's cascade

- At the injection scale, instabilities or active stirring create structures of size L

- For $\mathrm{Re} >> 1$, viscosity is too small to dissipate energy at the <u>injection scale</u>

- Energy must go somewhere, and if it goes to smaller scales (larger $\mathbf{k}$), it can be dissipated by viscosity

$$\nabla \left[\mathbf{U}\,\mathrm{d}^3k\ \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\right] = \mathrm{i}\mathbf{k}\mathbf{U}\,\mathrm{d}^3k\ \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})$$

$\Rightarrow$ <u>cascade</u> = big turbulent structures (= <u>eddies</u>) break up into smaller eddies that in turn break into smaller eddies and so on

Transfer of energy between scales

- Equation for $\mathbf{u}(\mathbf{x}, t)$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \text{pressure} + \text{viscosity}$$

- For $\mathrm{Re} >> 1$, only possible typical time given by

$$\Delta t \sim \frac{1}{\mathbf{u} \cdot \nabla} \sim \frac{\Delta x}{\Delta u}$$

  - For each scale Δx, there is a corresponding velocity Δu

- Only viscosity dissipates energy ⇒ energy flow ε to small scales must be constant within inertial range

$$\varepsilon = \frac{\text{energy}/\text{volume}}{\text{time}} \sim \frac{\rho \Delta u^2}{\Delta t} \sim \frac{\rho \Delta u^3}{\Delta x}$$

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Velocity increment

- Need to determine the velocity Δu for each scale Δx
  - Average velocity of a patch of size Δx is irrelevant, so we need the velocity increment

$$\Delta \mathbf{u}(\Delta \mathbf{x}, \mathbf{x}, t) = \mathbf{u}(\mathbf{x} + \Delta \mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t)$$

- $\Delta \mathbf{u}$ fluctuates $\Rightarrow$ use structure function $\langle \Delta u^2 \rangle_{V,\, t}$
- $\mathbf{u}(\mathbf{x} + \Delta \mathbf{x},\, t) \simeq \mathbf{u}(\mathbf{x},\, t) + \Delta \mathbf{x} \cdot \nabla \mathbf{u}(\mathbf{x},\, t) \Rightarrow \langle \Delta u^2 \rangle_{V,\, t} \propto \Delta x^2$ for small Δx, BUT in the inertial range of turbulence

$$\langle \Delta u^2 \rangle_{V,t} \propto \Delta x^\beta$$

  - Constant energy flow ε from one scale to another:

$$\varepsilon \sim \frac{\rho \Delta u^3}{\Delta x} \Rightarrow \Delta u \sim \left(\frac{\varepsilon}{\rho} \Delta x \right)^{1/3} \Rightarrow \langle \Delta u^2 \rangle_{V,t} \propto \Delta x^{2/3} \Rightarrow \boxed{\beta = \frac{2}{3}}$$

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Velocity increment and spectrum

- Expand square inside structure function

$$\langle \Delta u^2 \rangle_{V,t} = \underbrace{\langle u^2(\mathbf{x}+\Delta\mathbf{x})\rangle_{V,t}}_{=\langle u^2\rangle_{V,t}} + \underbrace{\langle u^2(\mathbf{x})\rangle_{V,t}}_{=\langle u^2\rangle_{V,t}} - 2\underbrace{\langle \mathbf{u}(\mathbf{x}+\Delta\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\rangle_{V,t}}_{=\langle u^2\rangle_{V,t}C(\Delta\mathbf{x})}$$

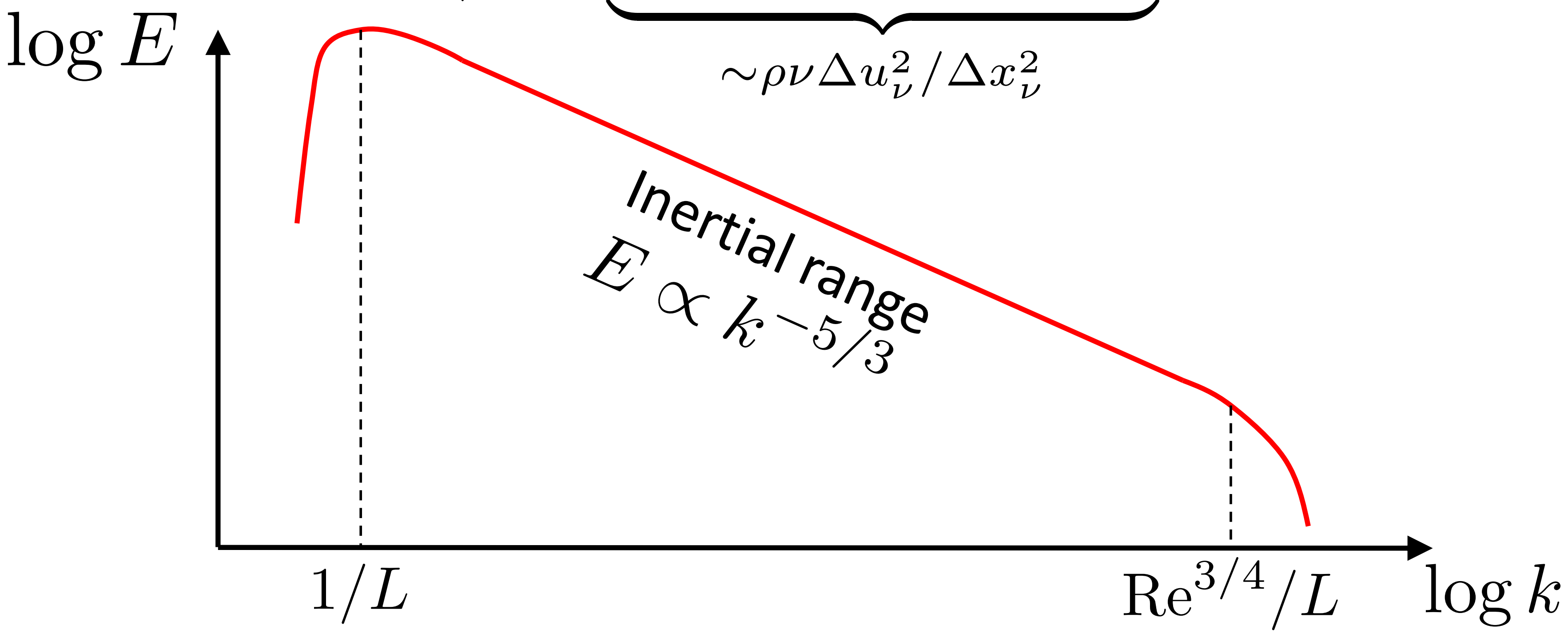
$$\Rightarrow \langle \Delta u^2 \rangle_{V,t} = 2\langle u^2\rangle_{V,t}[1 - C(\Delta\mathbf{x})] = \boxed{\frac{2(2\pi)^3}{V}\int \langle U^2\rangle_t \,[1 - \exp(\mathrm{i}\mathbf{k}\cdot\Delta\mathbf{x})]\,\mathrm{d}^3k}$$

- Effectively, integral is between
  - Lower limit: $k \sim 1/\Delta x$ ($1 - \exp(\mathrm{i}\mathbf{k}\cdot\Delta\mathbf{x})$ is small for smaller k)
  - Upper limit: the dissipation scale
- For Δx in inertial range,

$$\langle \Delta u^2 \rangle_{V,t} \propto \int_{1/\Delta x}^{\text{dissipation}} k^{-\alpha}\,\mathrm{d}k \simeq \frac{\Delta x^{\alpha-1}}{\alpha - 1} \Rightarrow \boxed{\beta = \alpha - 1}$$

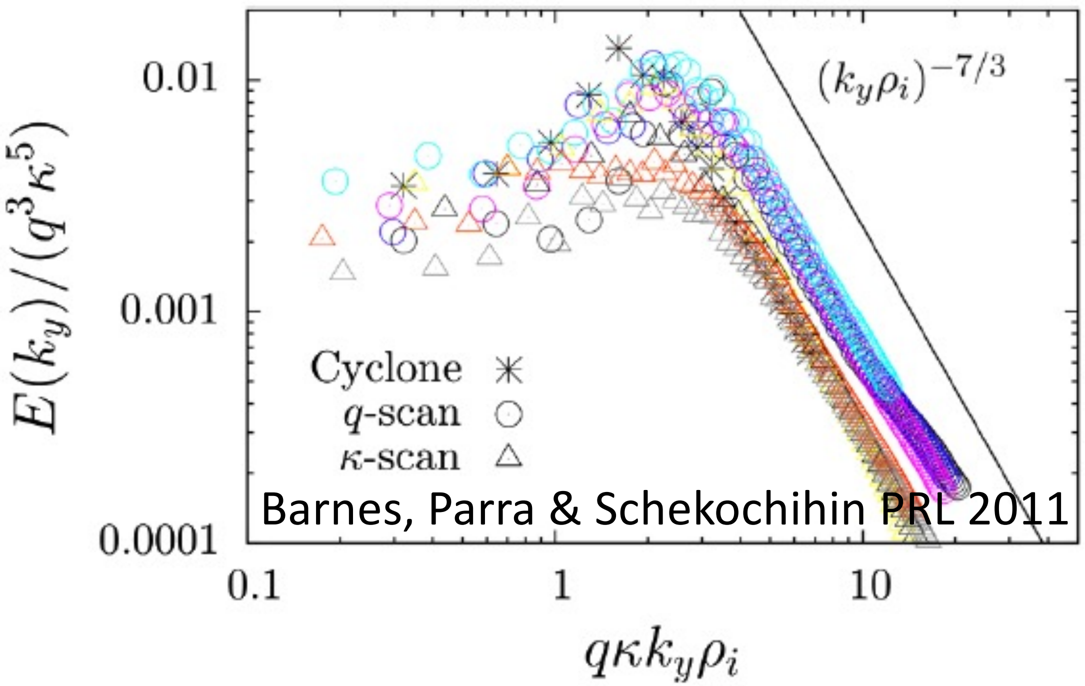
Kolmogorov's spectrum

- $\langle \Delta u^2 \rangle_{V,t} \propto \Delta x^{2/3} \Rightarrow \beta = 2/3 \Rightarrow \alpha = 5/3$
- Dissipation: $\underbrace{\varepsilon}_{\sim \rho u^3/L} \sim \underbrace{\rho\nu \left\langle \sum_i \left| \frac{\partial \mathbf{u}}{\partial x_i} \right|^2 \right\rangle_{V,t}}_{\sim \rho\nu \Delta u_\nu^2 / \Delta x_\nu^2} \Rightarrow \Delta x_\nu \sim \frac{L}{\mathrm{Re}^{3/4}}$



More on cascades

- Far more rigorous treatment in U. Frisch's book *Turbulence: The legacy of A.N. Kolmogorov*
  - Kolmogorov's results rest on several symmetry assumptions, such as isotropy, that I have not emphasized
- Power-law spectra observed in plasma turbulence
- Theories in plasmas must use assumptions different from those of Kolmogorov
  - E.g. magnetized plasmas are anisotropic

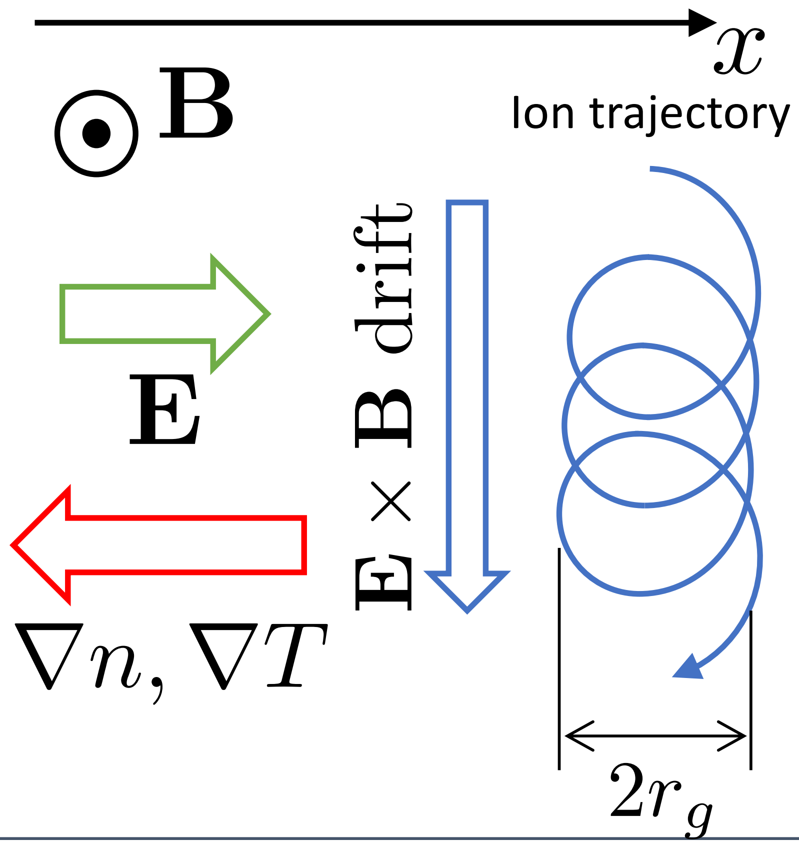


Barnes, Parra & Schekochihin PRL 2011

Transport in magnetized plasmas

Particle density — Temperature

- In fusion devices, electric field $\mathbf{E}$ aligns with ∇n, ∇T



- Seemingly, no average motion parallel to ∇n, ∇T
  - Particle only moves a gyroradius $r_g << L$

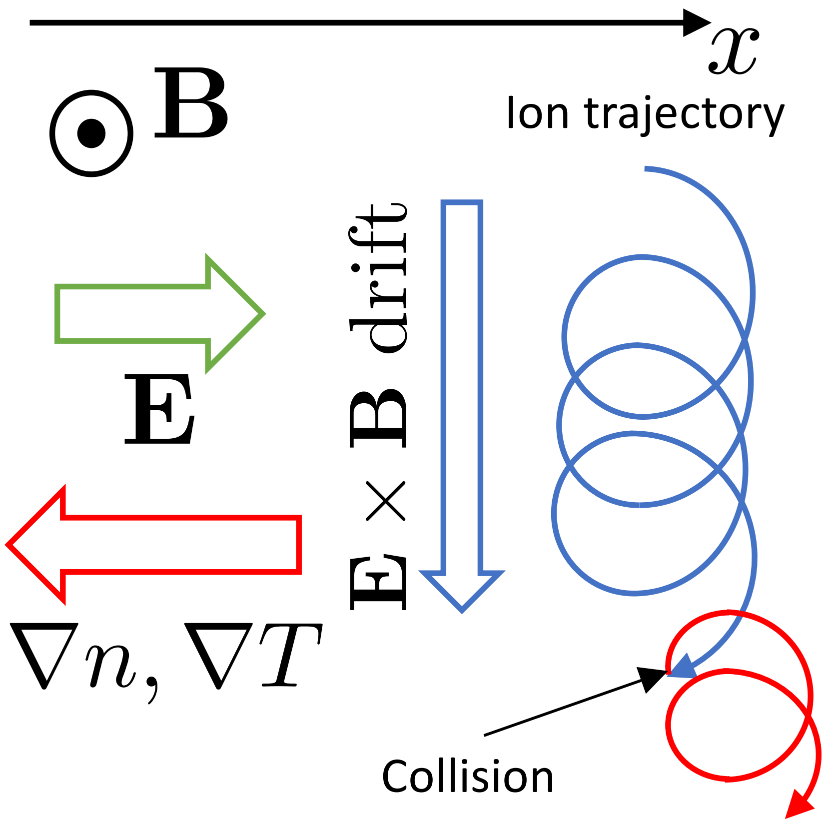
Mass — Thermal speed — Charge

$$r_g = \frac{m v_t}{qB}$$

- True except for collisions
  - Rare at high temperatures: need frontal collisions to change large energies

Transport in magnetized plasmas

- In fusion devices, electric field $\mathbf{E}$ aligns with ∇n, ∇T



- After a collision, particles move ~ gyroradius r_g
- Time between collisions: τ_c

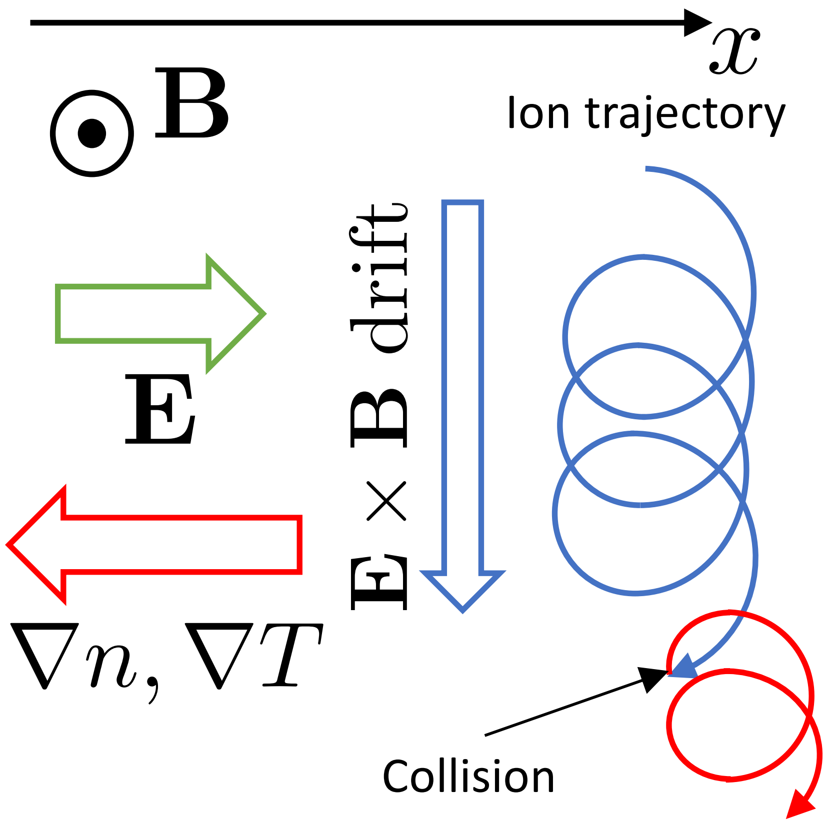
$$T(x, t+\tau_c)$$
$$= \frac{1}{2}T(x+r_g, t) + \frac{1}{2}T(x-r_g, t)$$

- After Taylor expansions

$$\boxed{\frac{\partial T}{\partial t} = \frac{r_g^2}{2\tau_c}\frac{\partial^2 T}{\partial x^2}}$$

Transport in magnetized plasmas

- In fusion devices, electric field $\mathbf{E}$ aligns with ∇n, ∇T



- Random walk estimate
$$\frac{\partial T}{\partial t} = \frac{r_g^2}{2\tau_c} \frac{\partial^2 T}{\partial x^2}$$
- Energy confinement time
$$\tau_E \sim \frac{L^2}{r_g^2} \tau_c$$
- r_g small for large B, and τ_c large at large $T \Rightarrow$ fusion!
- Not quite how it works…

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Turbulent transport

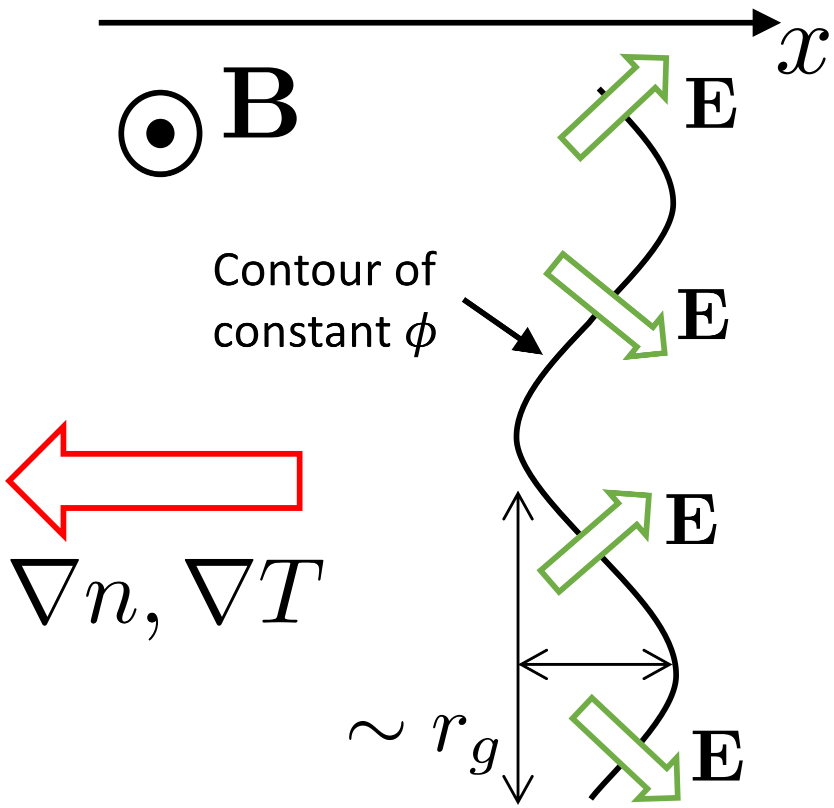
- τ_E is much shorter than the collisional estimate

$$\tau_E = \frac{\text{energy in the plasma volume}}{\text{power leaving the plasma}}$$

- Electric field fluctuates and is not aligned with ∇n, ∇T
  - We can measure fluctuations of density, temperature and magnetic field that result from fluctuating electric field
- Theory predicts microinstabilities driven by the temperature and density gradients
  - These are 'micro' because the fastest growing structures have a characteristic length of the order of $r_g << L$
- Need to solve the injection scale problem

Turbulent transport estimate

- Assume electrostatic electric field: $\mathbf{E} = -\nabla\phi$



- Instability typical length r_g $\Rightarrow$ fluctuations $\sim r_g$
  - Injection scale $\sim r_g$
- Structures only last for a turbulent time τ_t

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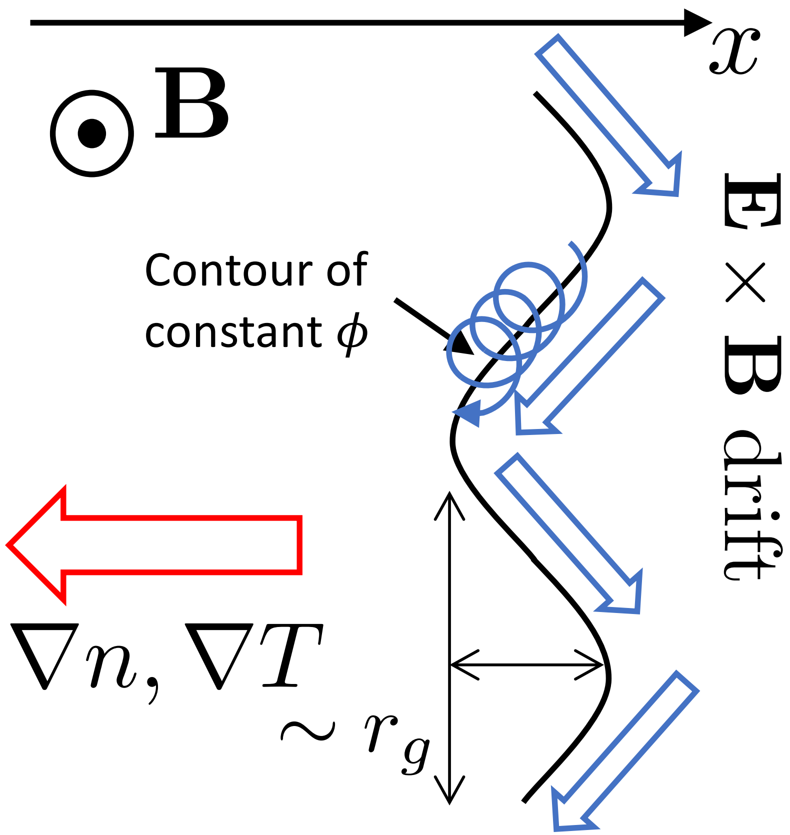
Turbulent transport estimate

- Assume electrostatic electric field: $\mathbf{E} = -\nabla\phi$

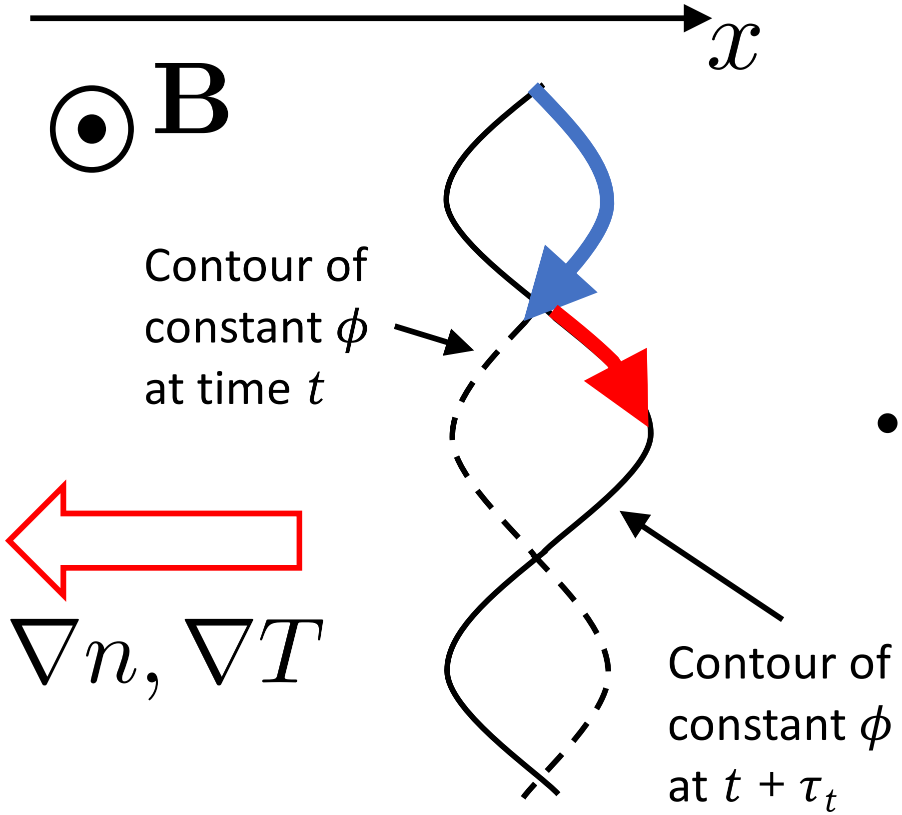


- Instability typical length r_g $\Rightarrow$ fluctuations $\sim r_g$
  - Injection scale $\sim r_g$
- Structures only last for a turbulent time τ_t
- After particles move $\sim r_g$, n and T have been redistributed $\Rightarrow$ different $\mathbf{E}$

$$\tau_t \sim \frac{r_g}{v_{\mathbf{E}\times\mathbf{B}}}$$

Turbulent transport estimate

- Assume electrostatic electric field: $\mathbf{E} = -\nabla\phi$



- Similar to collisional transport: after time τ_t, particles move $\sim r_g$

$$\tau_t \sim \frac{r_g}{v_{\mathbf{E}\times\mathbf{B}}}$$

- Use random walk estimate with τ_c replaced by τ_t

$$\tau_E \sim \frac{L^2}{r_g^2}\tau_t \sim \frac{L^2}{r_g v_{\mathbf{E}\times\mathbf{B}}}$$

GyroBohm estimate

- To finish our estimate, need the size of the $\mathbf{E}\times\mathbf{B}$ drift

$$v_{\mathbf{E}\times\mathbf{B}} = \frac{1}{B^2}\mathbf{E}\times\mathbf{B} = -\frac{1}{B^2}\nabla\phi\times\mathbf{B} \sim \frac{\phi}{BL}$$

- Energy due to electrostatic potential ~ plasma energy

$$q\phi \sim \underbrace{T = \frac{1}{2}mv_t^2}_{\text{definition of } v_t}$$

- GyroBohm transport

$$v_{\mathbf{E}\times\mathbf{B}} \sim \frac{r_g}{L}v_t \Rightarrow \boxed{\tau_E \sim \frac{L^3}{r_g^2 v_t}}$$

  - Shorter time than collisional because $\tau_c >> \tau_t \sim L/v_t$

Measuring turbulent fluctuations

- We need to detect and model fluctuations of spatial size r_g and characteristic time $\tau_t \sim L/v_t$
- Turbulence in the cold edge measured with probes or with light emitted by the plasma
- Turbulence in the hot core requires more ingenuity
  - Radiation-based methods
    - Beam Emission Spectroscopy (BES)
    - Correlation Electron Cyclotron Emission (CECE)
  - Microwave-based methods
    - Reflectrometry (reflection)
    - Doppler Back-Scattering (DBS) (scattering)
    - Phase Contrast Imaging (PCI) (refraction)

Simulating turbulence

- Fluid models can be used in the very collisional parts of the cold edge
- In the rest of the plasma, collisions are rare, and particles travel long distances without thermalizing with the background ⇒ need kinetic model
  - Turbulence is characterized length scales of the order of r_g and characteristic time $\tau_t \sim L/v_t$ << gyrofrequency ⇒ gyrokinetics ≈ use drifts averaged over finite gyro-orbit
- Simulations require a large range of scales to include both the injection and the dissipation scale

Take-home message

- Turbulence theory usually builds on Kolmogorov's ideas on how to dissipate the injected energy
  - Kolmogorov's ideas are constantly challenged and modified because turbulence encompasses many different phenomena that do not fit his original work
- In magnetic confinement fusion, we are interested in the transport produced by turbulent fluctuations
  - The gyroBohm estimate is a starting point: there are many effects that modify it significantly: macroscopic flows, magnetic field fluctuations, energetic particles, impurities…
  - There is still plenty to learn from experiments and theory: many observations still baffle us!