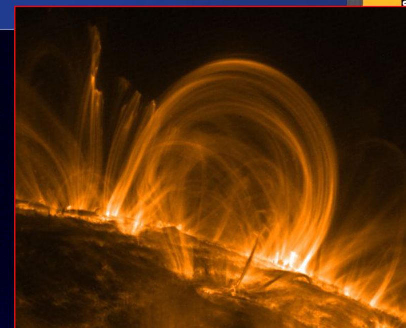
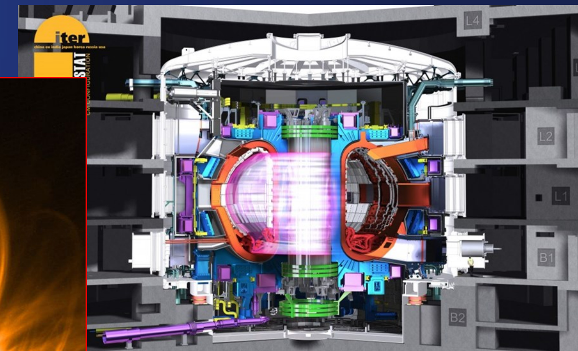
Single Particle Motion





Presented by

Amina Hussein

at the

SULI Introductory Course in Plasma Physics

Princeton Plasma Physics Laboratory

June 15, 2021

Thank you to Dr. Cami Collins (GA) & Will Fox (PPPL) for sharing previous versions of this presentation

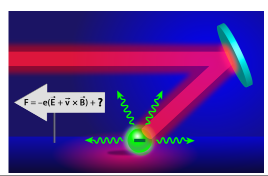
My path in plasma physics

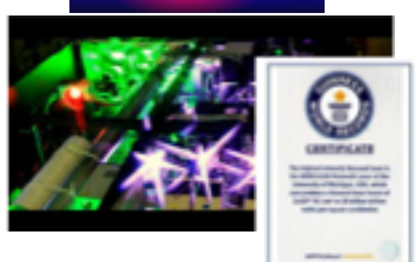
McGill University | Purdue University | University of Michigan | UC Irvine | University of Alberta

Undergrad (2008 – 2013)

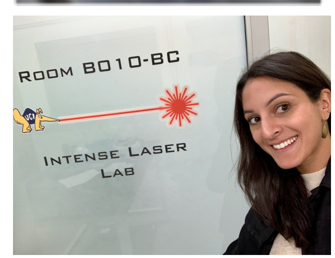
Ugrad Research & Masters (2013-2015)

PhD (2015-2019)

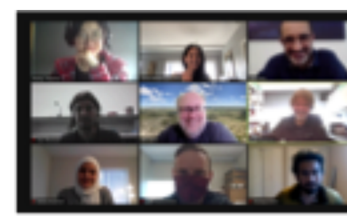




Postdoc (2019-2020)



Professor (since 2020)



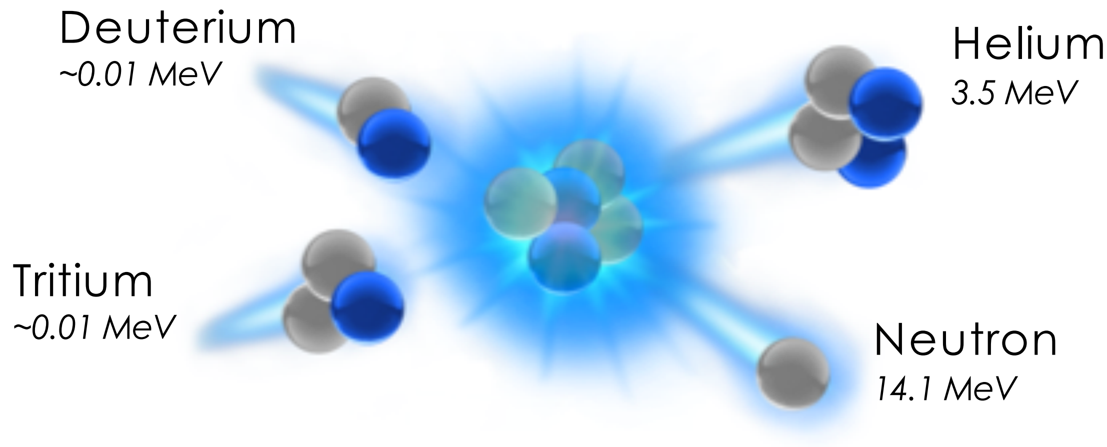
Outline and goals for this lecture

- **Single particle motion is important to understand the behavior of plasmas**
- **We will learn about how magnetic fields modify single particle motion**
  - Gyromotion about a *guiding center*
  - Forces can cause guiding center *drift*
- **Real life consequences and applications**
  - Why do tokamaks have helical B fields?
  - Particle-in-Cell Simulations

References

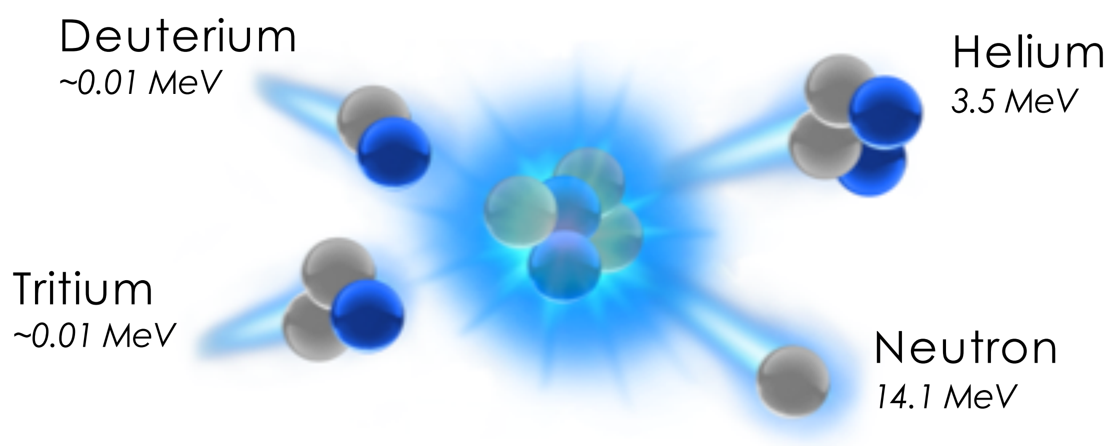
- NRL Plasma Formulary
  www.nrl.navy.mil/ppd/content/nrl-plasma-formulary
- Introduction to Plasma Physics and Controlled Fusion by F. Chen

Plasma physics is the basis of fusion energy research



- **Sustained fusion reactions require enough particles** (density)
- **That are energetic enough** (temperature)
- **And collide often enough** (confinement time)

Plasma physics is the basis of fusion energy research



- **Sustained fusion reactions require enough particles** (density)
- **That are energetic enough** (temperature)
- **And collide often enough** (confinement time)

- **This is the basis of the "triple product":**

$$nT\tau_E \gtrsim 5\times 10^{21}\ \mathrm{keV\ s\ m^{-3}}$$

T~100-200 million K

n~2-3x10^{20} ions/m^3

τ ~1-2 s

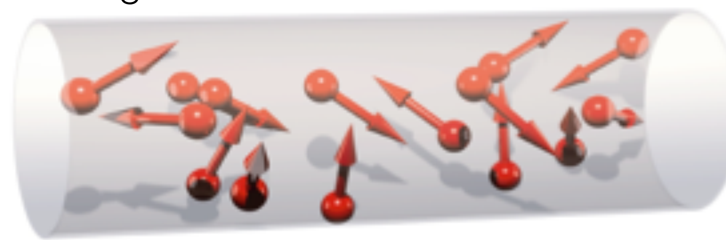
D & T is a plasma at these temperatures

We can understand a lot about how fusion devices confine plasma by studying single particle motion

Typical velocity of a 100 million K ion:

$$kT = \frac{1}{2} m v_{\mathrm{th}}^2$$

$$v_{\mathrm{th}} \sim 6 \times 10^5 \ \mathrm{m/s}$$

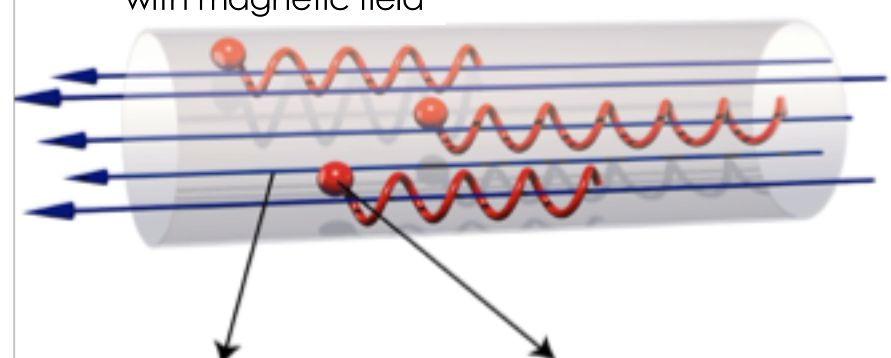


Even with ~10^{20} *ions/m*^{3} , the ion would travel ~10 km before colliding with another

Therefore, these plasmas are effectively collisionless

The ITER tokamak has a minor radius of 2 m, so how are the particles confined?

The trick: use magnetic fields



Charged Particles Feel The (Lorentz) Force

- **A particle with charge (q) moving with velocity ($\mathbf{v}$) in the presence of electric and magnetic fields will experience a force:**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

We know from Newton's second law of motion that force causes acceleration:

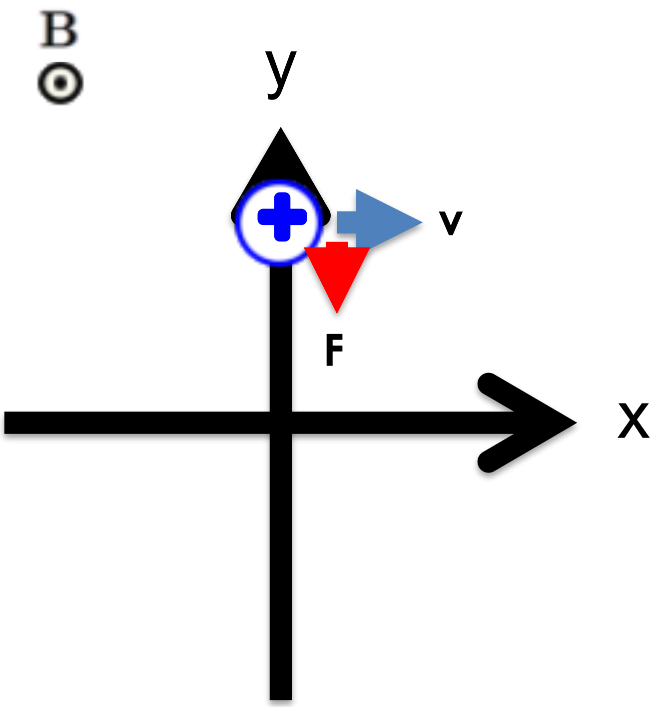
$$\mathbf{F} = m\mathbf{a}$$

A charged particle moving perpendicular to the magnetic field feels a force

How Does a Charged Particle Move in a Magnetic Field?

- **Consider the motion of a particle in a constant, uniform B field**

$$\mathbf{B} = B_z\hat{\mathbf{z}} \qquad \mathbf{E} = \mathbf{0}$$



Then $\mathbf{F} = q(\cancel{\mathbf{E}} + \mathbf{v} \times \mathbf{B})$

So we can write

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

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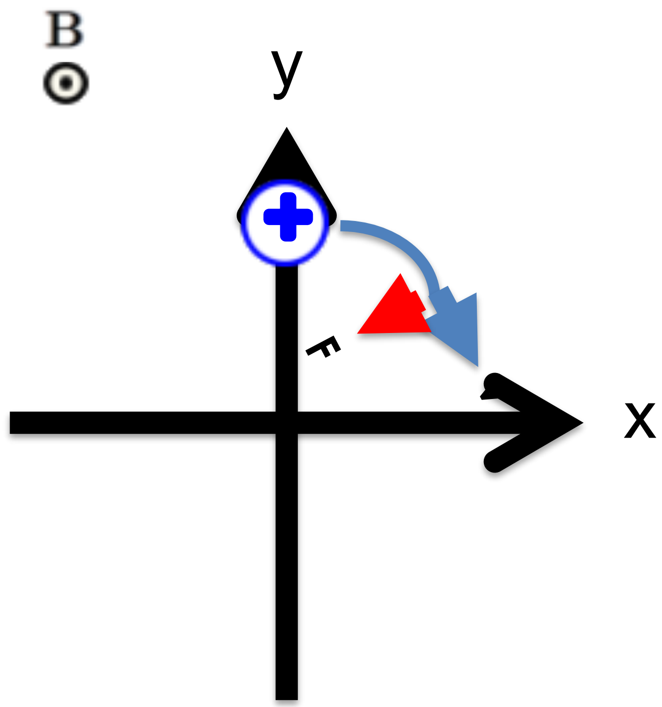

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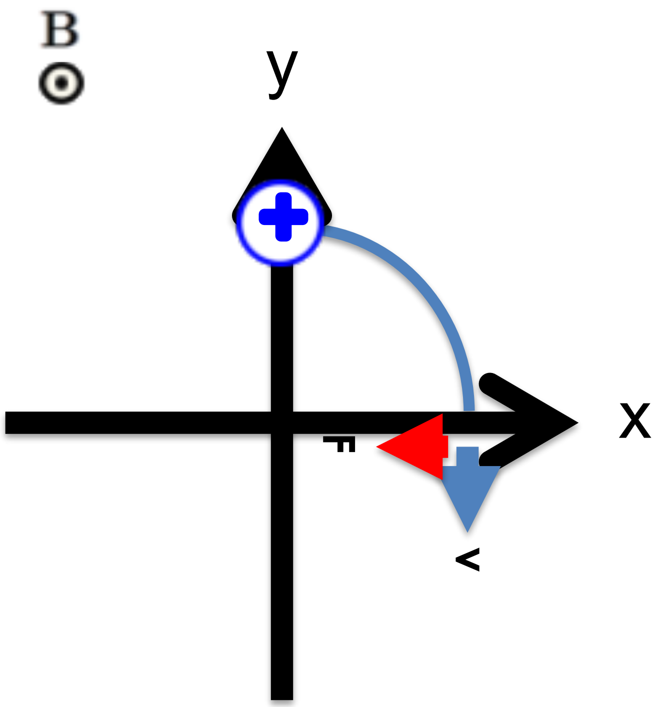

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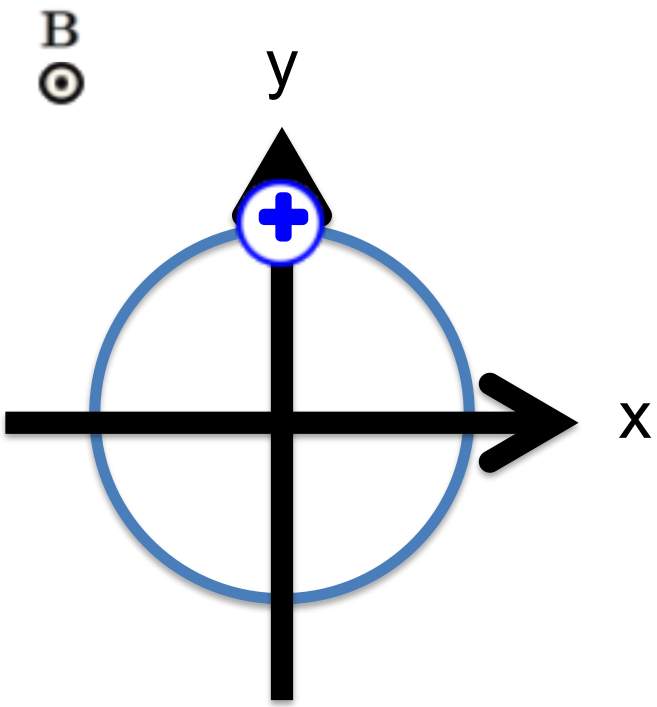

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So we can write

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

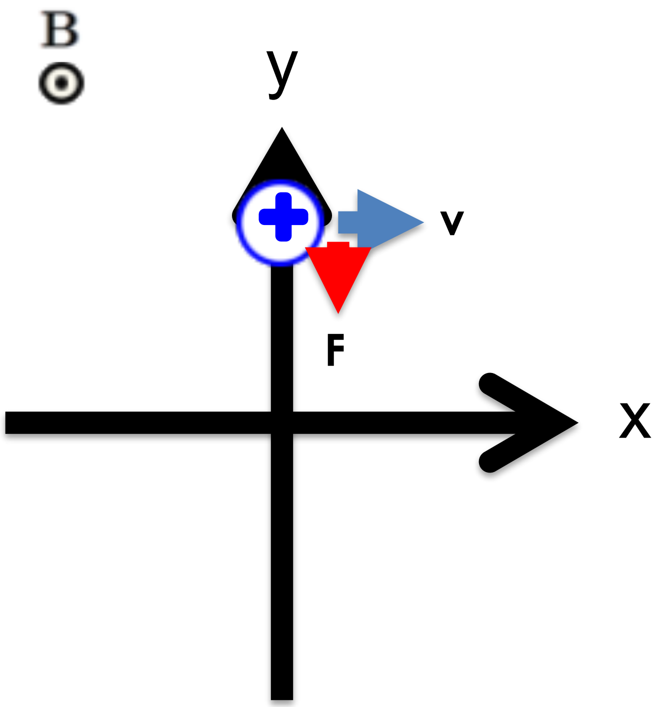
One "gyration" or "gyro-orbit" completed

Goal: Solve the Equations of Motion for a Charged Particle In A Magnetic Field

Let's do the algebra!

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

Let's break this into components:



$$\dot{v}_x\hat{\mathbf{x}} + \dot{v}_y\hat{\mathbf{y}} + \dot{v}_z\hat{\mathbf{z}} = \frac{qv_yB_z\hat{\mathbf{x}} - qv_xB_z\hat{\mathbf{y}}}{m}$$

The 'dot' represents $\frac{d}{dt}$

Matching components:

$$\dot{v}_x = \frac{qv_yB_z}{m} \qquad \dot{v}_y = -\frac{qv_xB_z}{m}$$

$$\dot{v}_z = 0$$

Particles move freely along the field line

Take Another Time Derivative & Substitute to Obtain Differential Equations For Each Spatial Coordinate

These equations are coupled:

$$\dot{v}_x = \frac{q v_y B_z}{m} \qquad \dot{v}_y = -\frac{q v_x B_z}{m}$$

Take another time derivative and substitute:

$$\ddot{v}_x = \frac{q \dot{v}_y B_z}{m} \qquad \ddot{v}_y = -\frac{q \dot{v}_x B_z}{m}$$

Rewriting, we get

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y$$

These may remind you of the equations for a simple harmonic oscillator

Solve the Differential Equations

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y$$

These differential equations can be solved using sines and cosines:

$$v_x = v_\perp \cos\left(\frac{|q|B_z}{m}t + \phi_0\right) \qquad v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right)$$

account for positive or negative q

$$v_\perp = \sqrt{(v_x^2 + v_y^2)}$$

an arbitrary phase to match the initial velocity conditions

the magnitude of the initial velocity perpendicular to $\boldsymbol{B}$

Homework: Work through the details on your own!

The Result: Circular Motion About A Guiding Center

$$v_x = v_\perp \cos\left(\frac{|q|B_z}{m}t + \phi_0\right) \qquad v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right)$$

Integrating, we obtain the positions as a function of time

$$x = \frac{mv_\perp}{|q|B_z}\sin\left(\frac{|q|B_z}{m}t + \phi_0\right) + x_0 \qquad y = \pm\frac{mv_\perp}{|q|B_z}\cos\left(\frac{|q|B_z}{m}t + \phi_0\right) + y_0$$

- **Charged particles undergo circular orbits about a <u>guiding center</u>** (x_0, y_0)

$$r_L \equiv \frac{mv_\perp}{|q|B}$$

"Larmor" radius or "gyro-radius"

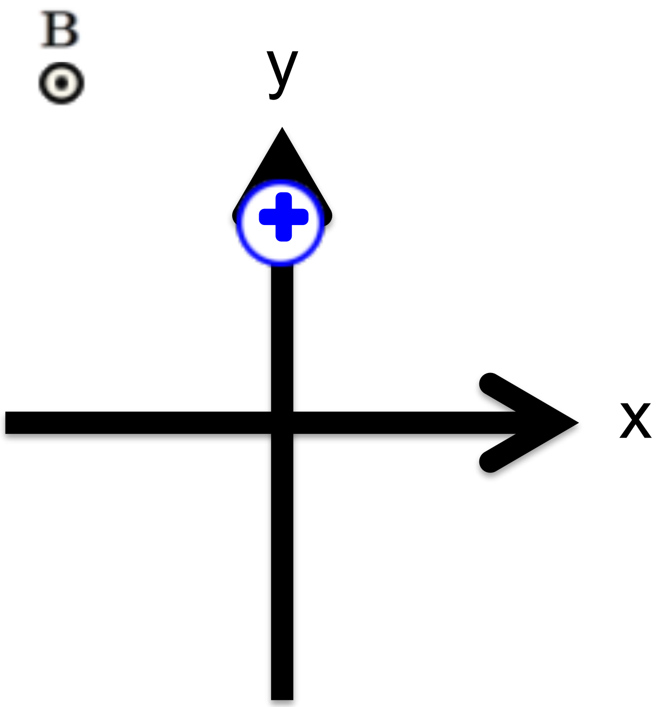
$$\omega_c \equiv \frac{|q|B}{m}$$

"Cyclotron" or "gyro-" frequency

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin\left(\omega_c t + \phi_0\right) + x_0 \qquad y = \pm r_L \cos\left(\omega_c t + \phi_0\right) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a positively charged particle:

1. At $t = 0$,

$$x = 0 \qquad y = r_L$$

Gyromotion of a Charged Particle In A Magnetic Field

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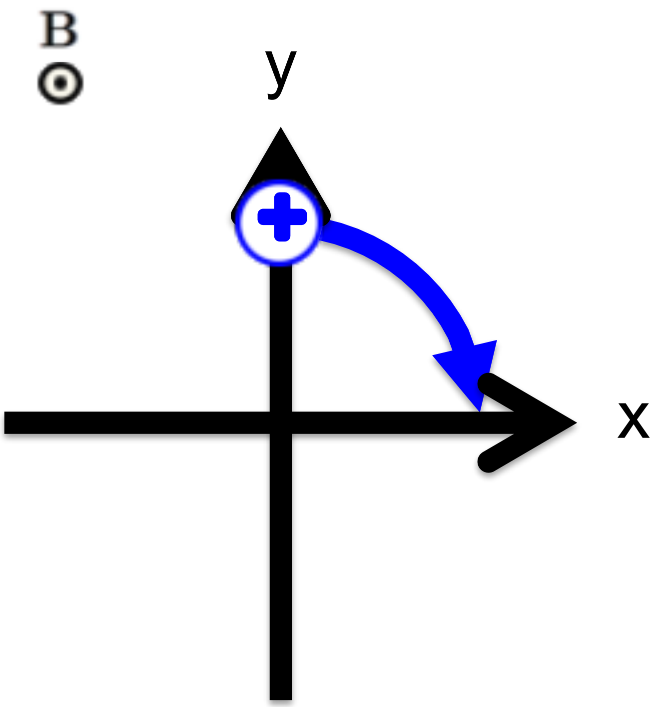

For a positively charged particle:

1. At $t = 0$,

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2. At $t = \frac{\pi}{2}\frac{1}{\omega_c}$,

$$x = r_L \qquad y = 0$$

Gyromotion of a Charged Particle In A Magnetic Field

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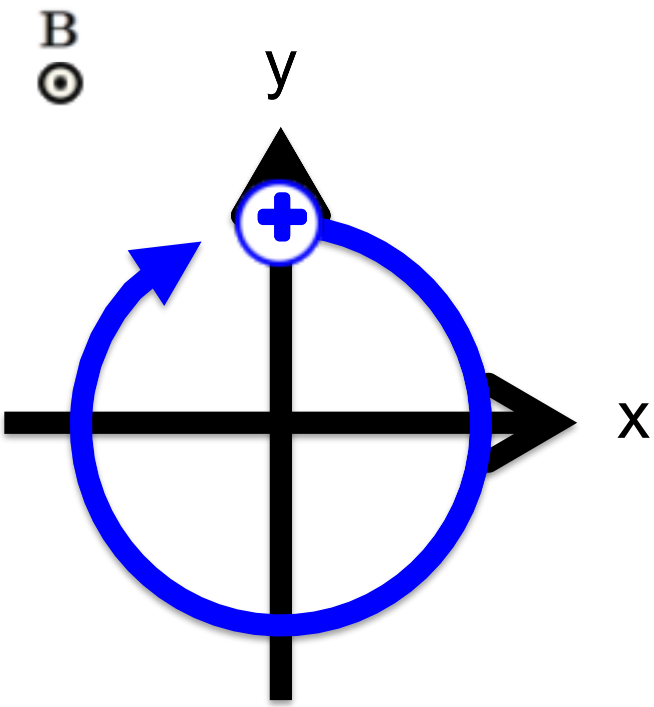

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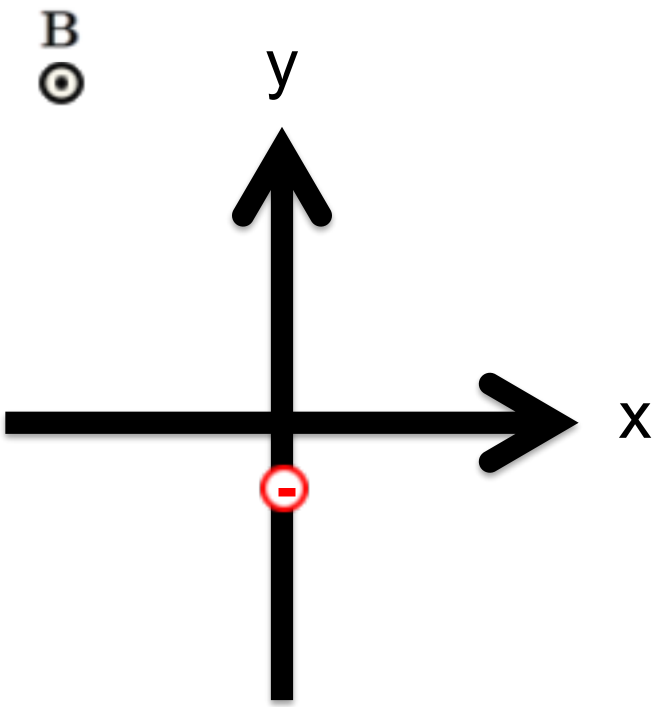

For a negatively charged particle:

1. At $t = 0$,

$$x = 0 \qquad y = -r_L$$

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin(\omega_c t + \phi_0) + x_0 \qquad y = \pm r_L \cos(\omega_c t + \phi_0) + y_0$$

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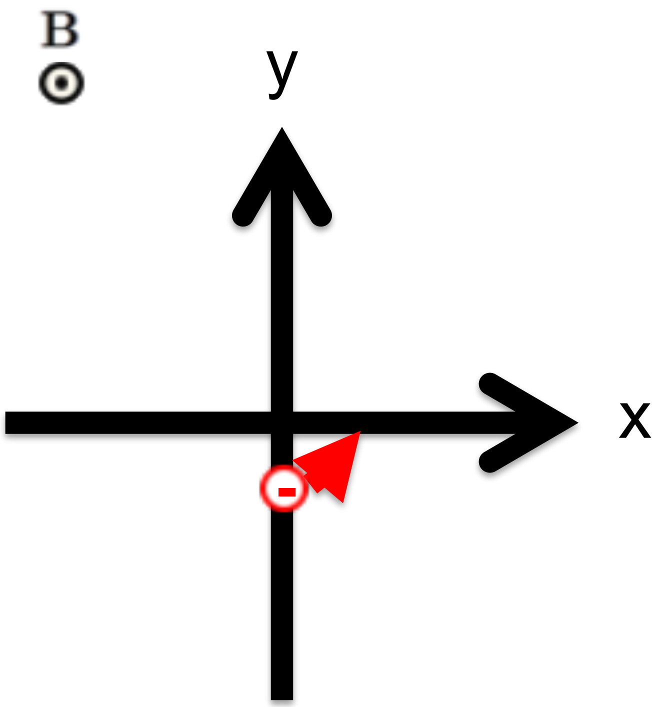

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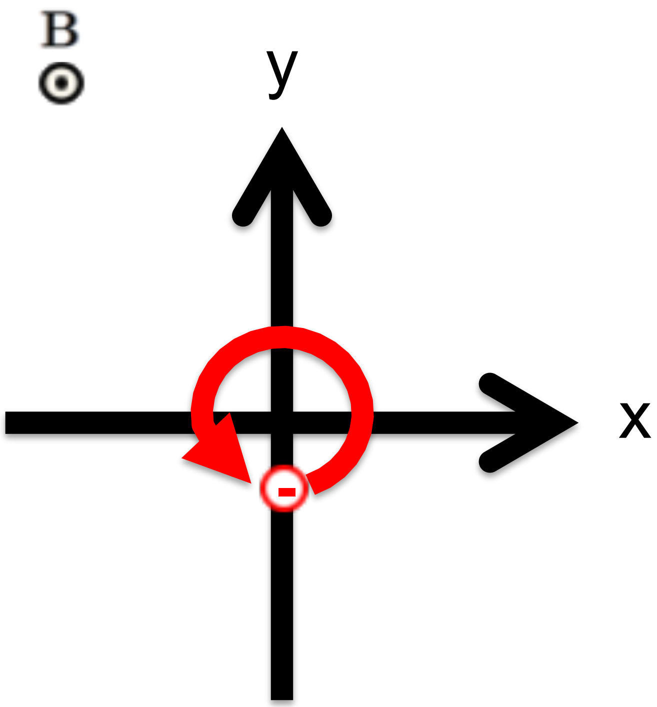

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1. At $t = 0$,

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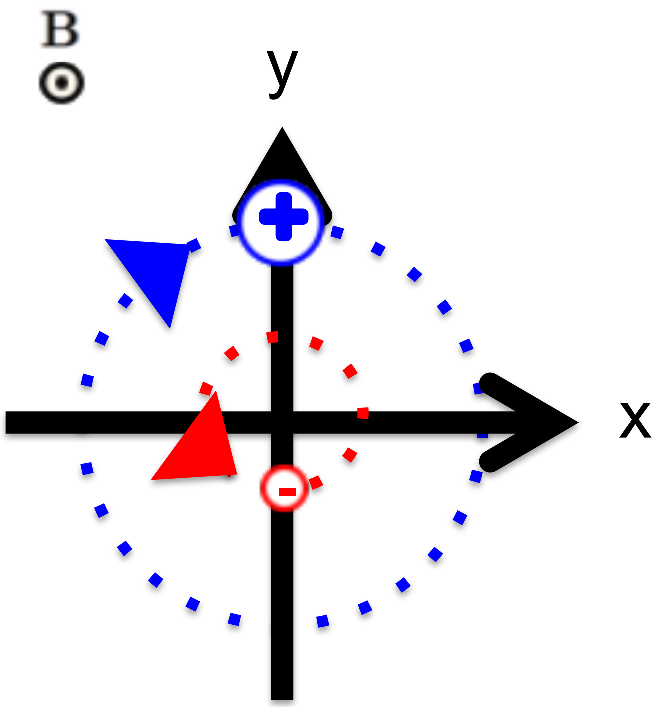
Gyromotion of Ions vs. Electrons

- **The direction of gyromotion depends on the sign of the charge**
- **Ions generally have a much larger Larmor radius than electrons**



In ITER, a typical deuterium ion with T_i=10 keV and B=5 Tesla would have

$$v_{Ti} = \sqrt{\frac{kT_i}{m_i}} \approx 700 \text{ km/s}$$

$$r_L \equiv \frac{mv_\perp}{|q|B} \approx 3 \text{ mm}$$

An electron with T_e=10 keV and B=5 Tesla has

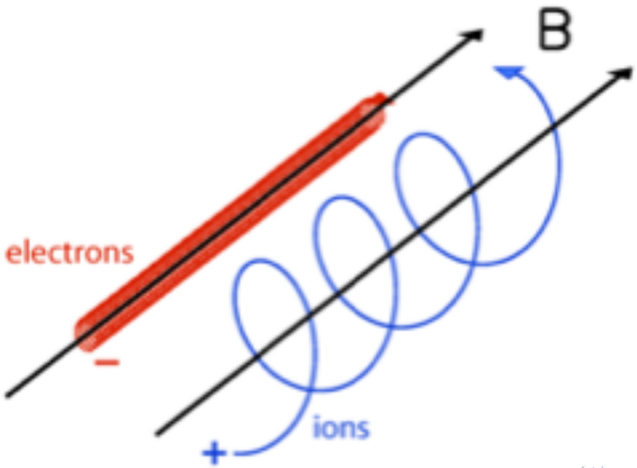
$$r_L \approx 0.05 \text{ mm}$$ (60 times smaller)

Summary: Particle motion and the guiding center

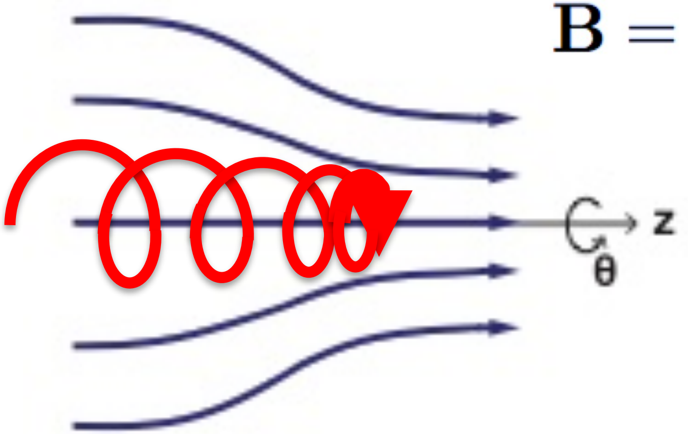
- Gyration strongly confines the motion of particles **perpendicular** to the applied magnetic field
- The *guiding center* is what is left over when you average of the gyration
- The *radius* of the particle motion is the Larmor or Gyro- radius
- The cyclotron or gyro frequency is the *frequency* of a charged particle moving perpendicular to a uniform B field
- The particle motion is intuitively composed of
  (1) Gyration and
  (2) "Drift" of the guiding center

$$r_L \equiv \frac{mv_\perp}{|q|B}$$

$$\omega_c \equiv \frac{|q|B}{m}$$



Some guiding center dynamics: Magnetic Mirrors



$$\mathbf{B} = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}} \qquad \mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

The B_r ends up causing additional acceleration in the z direction:

$$m\frac{dv_z}{dt} = -qv_\theta B_r$$

Result:
gyromotion + *mirror force* in the $-\hat{\mathbf{z}}$ direction

$$F_z = -\frac{mv_\perp^2}{2B}\frac{\partial B_z}{\partial z}$$

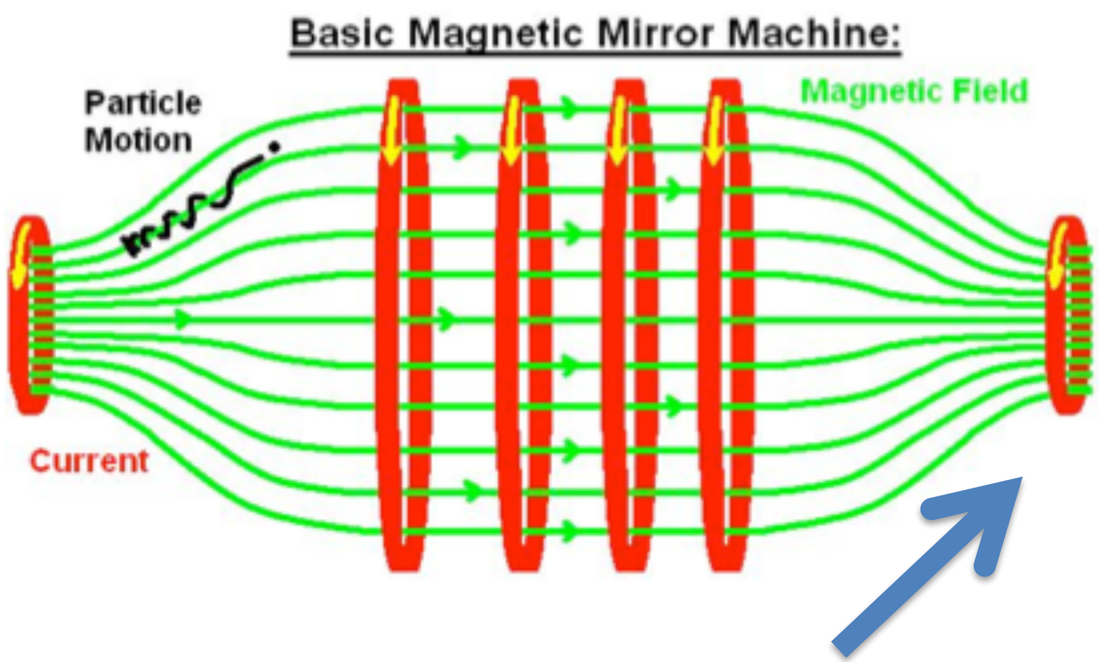
The magnetic moment is $\mu \equiv \frac{mv_\perp^2}{2B}$ mirror force is: $\mathbf{F}_\parallel = -\mu\nabla_\parallel B$

$$\frac{d\mu}{dt} = 0$$ **The magnetic moment is a constant of motion**

Magnetic Mirror Confinement In Action

Early Fusion Experiments

Ex: Tandem Mirror Experiment (LLNL,1980's) and other variants (Polywell devices)

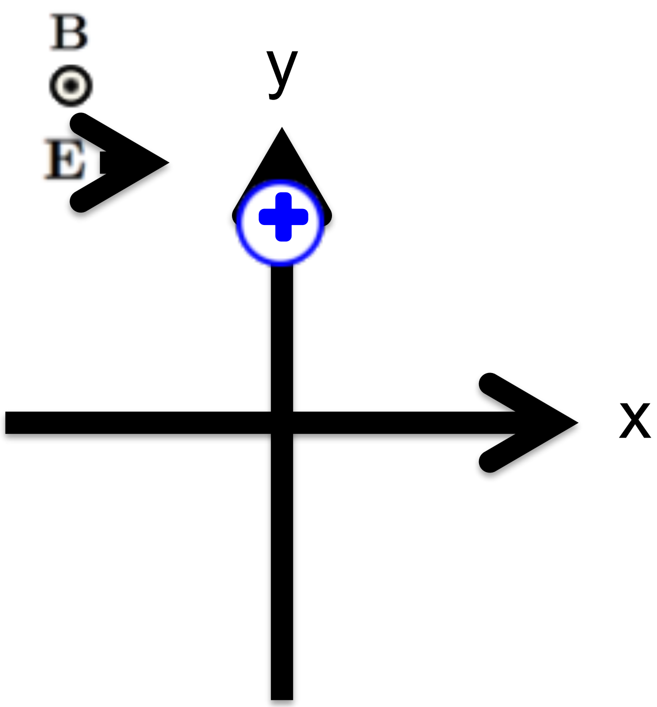


Particles with enough $v_{||}$ can still escape

Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{E} = E_x\hat{\mathbf{x}} \qquad \mathbf{B} = B_z\hat{\mathbf{z}}$$



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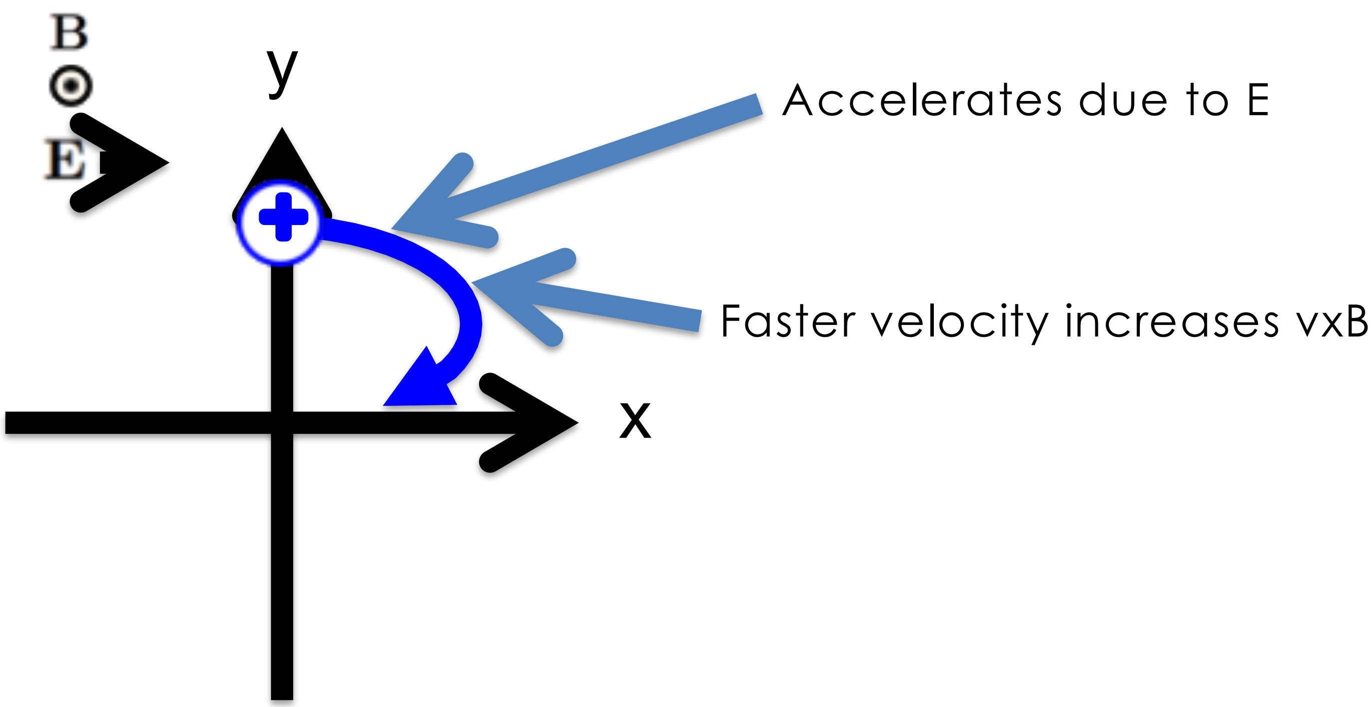

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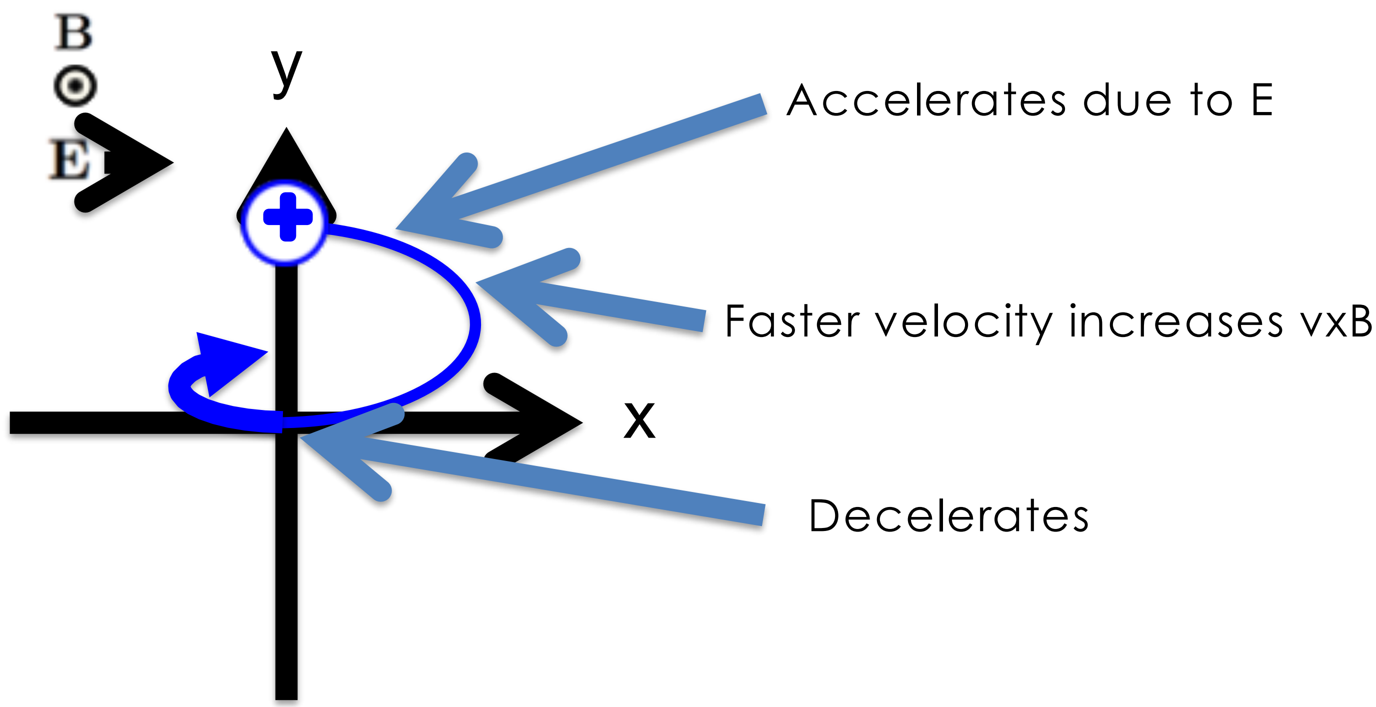

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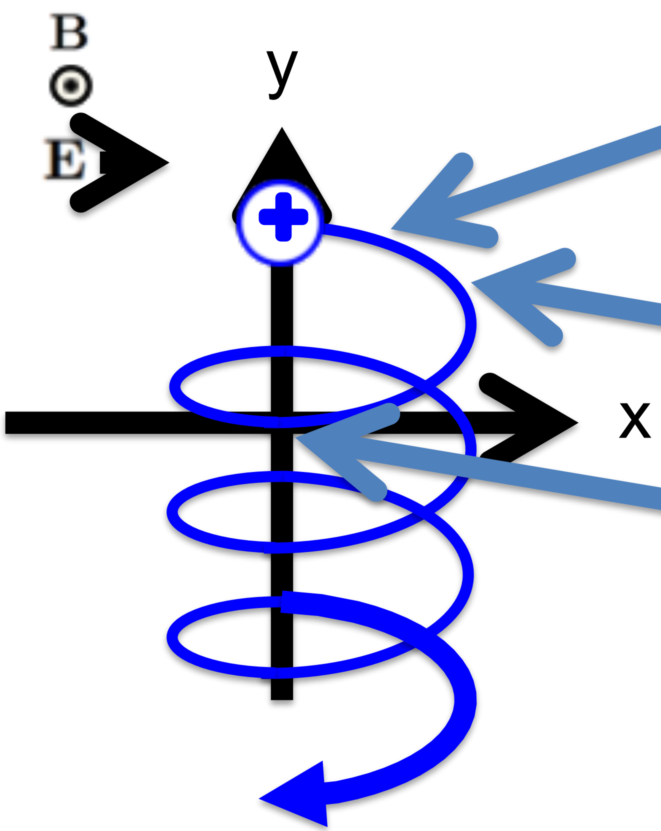


Accelerates due to E

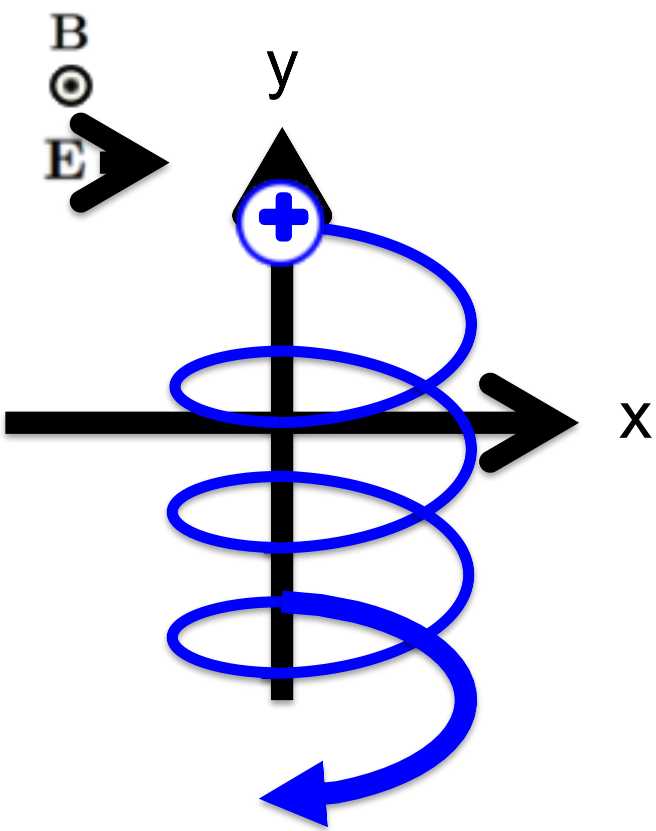
Faster velocity increases vxB

Decelerates

This is called the E x B drift!

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{E} = E_x \hat{\mathbf{x}} \qquad \mathbf{B} = B_z \hat{\mathbf{z}}$$



$$\dot{v}_x = \frac{q}{m}(v_y B_z + E_x)$$

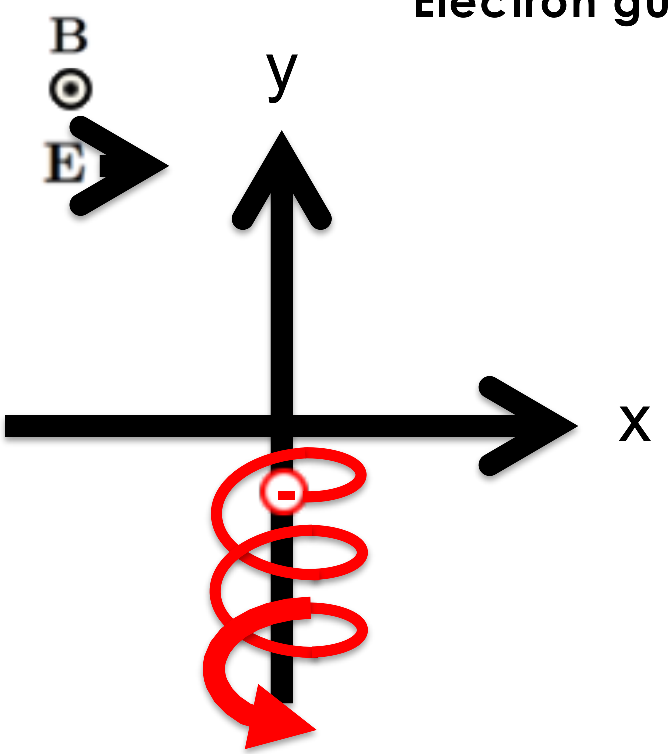
$$v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right) - \frac{E_x}{B_z}$$

Ion guiding center drifts in the direction $-\hat{\mathbf{y}}$

Guiding Center Drift Due to E x B

$$v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right) - \frac{E_x}{B_z}$$

Electron guiding center also drifts in the direction $-\hat{\mathbf{y}}$



The ExB drift can be written more generally as

$$\mathbf{v_E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- **ExB drift is independent of charge and mass**
- **Both electrons and ions move together**

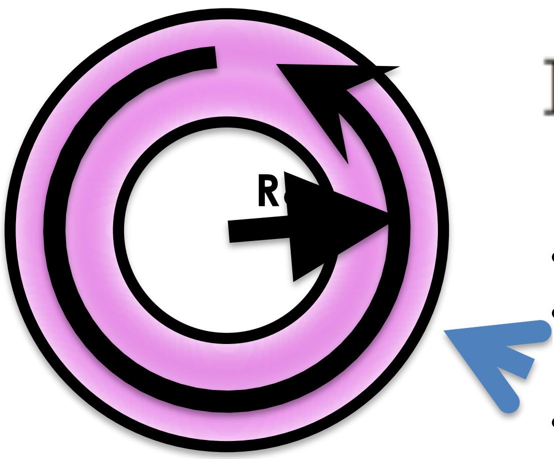
Other Forces Can Cause Guiding Center Drift

- **Any force perpendicular to B can cause particles to drift**

Drift due to force: $$\mathbf{v_d} = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2}$$

Examples of forces: $$\mathbf{F_g} = m\mathbf{g}$$ gravity

$$\mathbf{F_{cf}} = \frac{mv_\parallel^2}{R_c}\hat{\mathbf{r}}$$ centrifugal

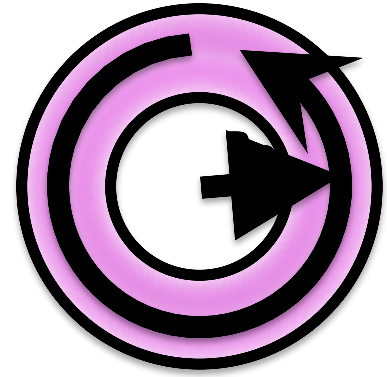


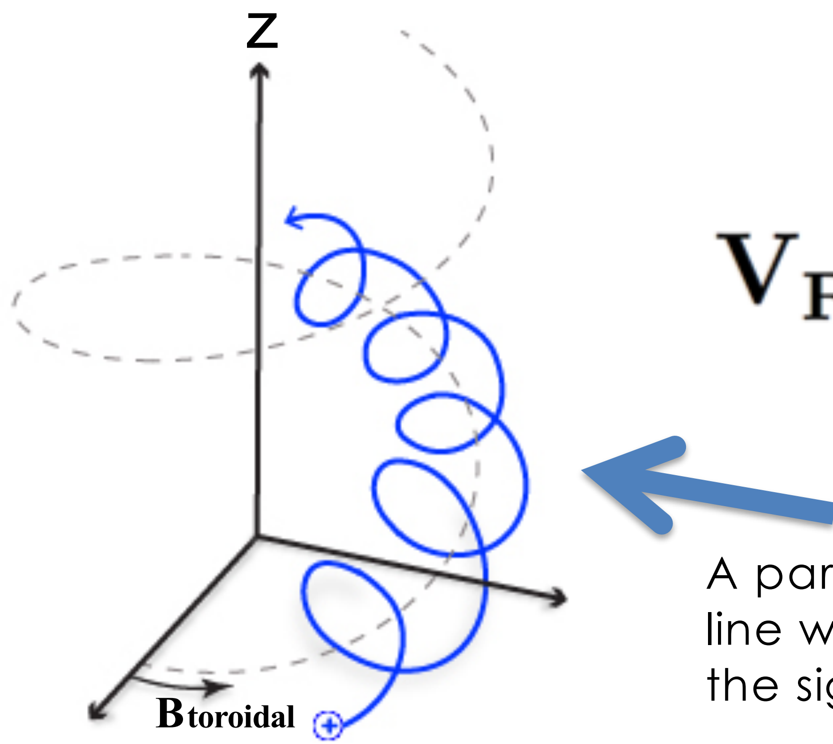
- Bend the magnetic field into a donut shape
- No end losses because the field lines go around and close on themselves
- BUT a particle following a toroidal magnetic field would experience $\mathbf{F_{cf}}$

Curvature Drift Due to Bending Field Lines

The outward centrifugal force causes **curvature drift**

$$\mathbf{F_{cf}} = \frac{mv_{\parallel}^2}{R_c}\hat{\mathbf{r}}$$

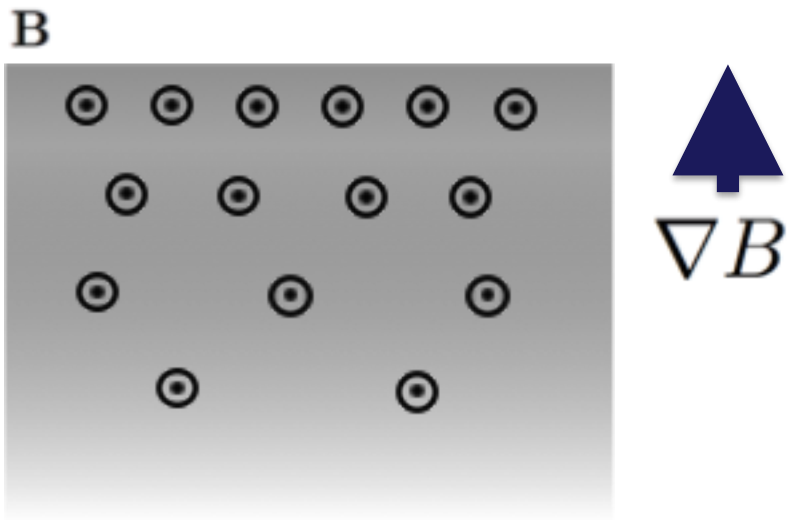


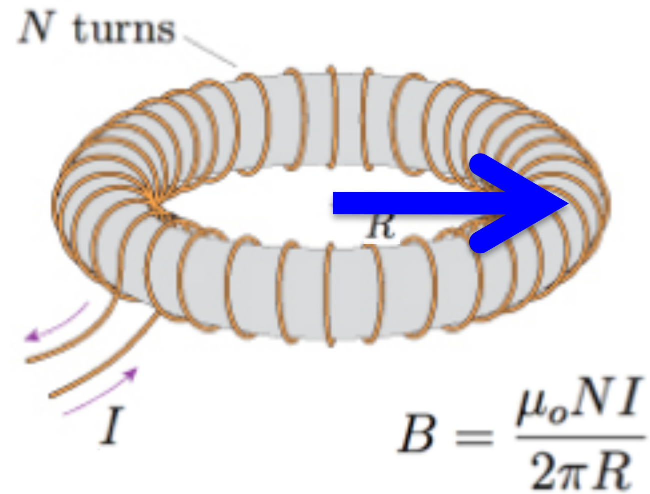


$$\mathbf{V_R} = \frac{mv_{\parallel}^2}{qB^2}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2}$$

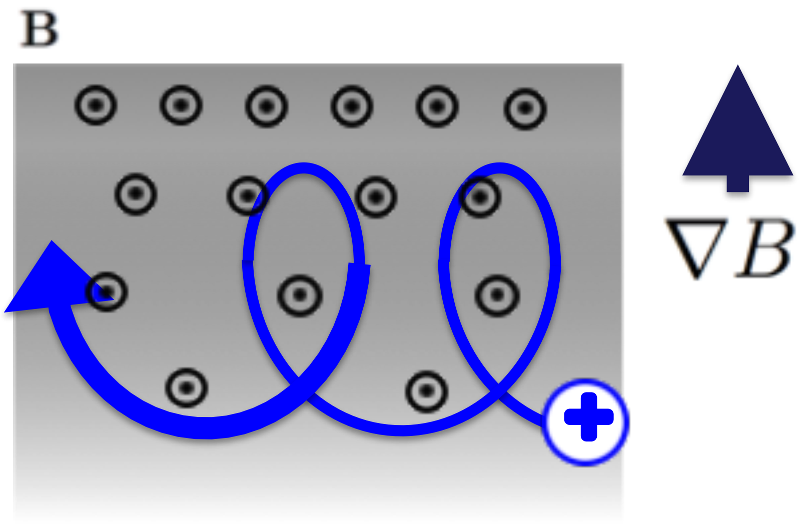
A particle moving along a curved field line will drift up or down, depending on the sign of the charge

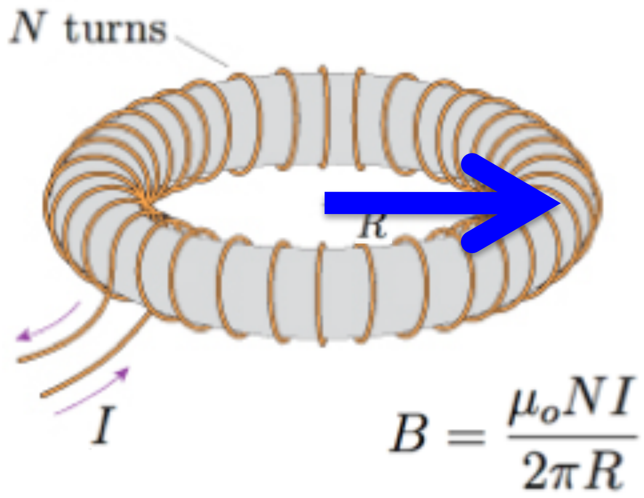
Spatially Varying Magnetic Field Strength Also Causes Drift: The Grad B drift





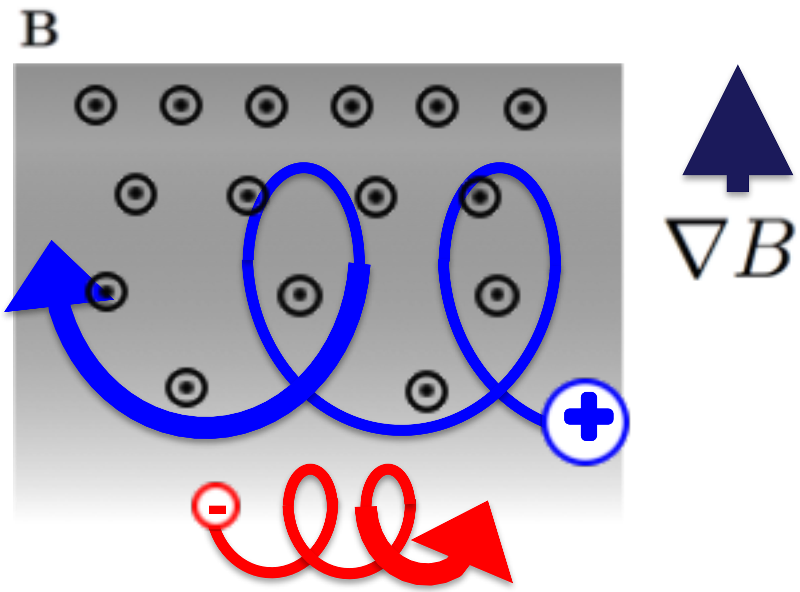
Spatially Varying Magnetic Field Strength Also Causes Drift

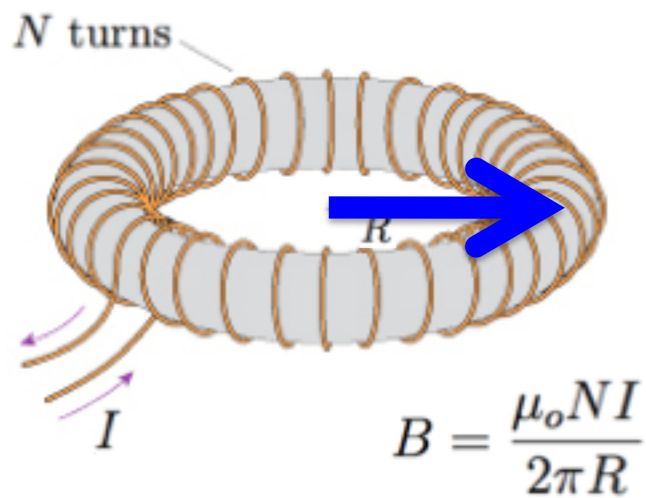




- **The gyro-radius will be larger where the field is weaker and smaller where the field is stronger**

Spatially Varying Magnetic Field Strength Also Causes Drift





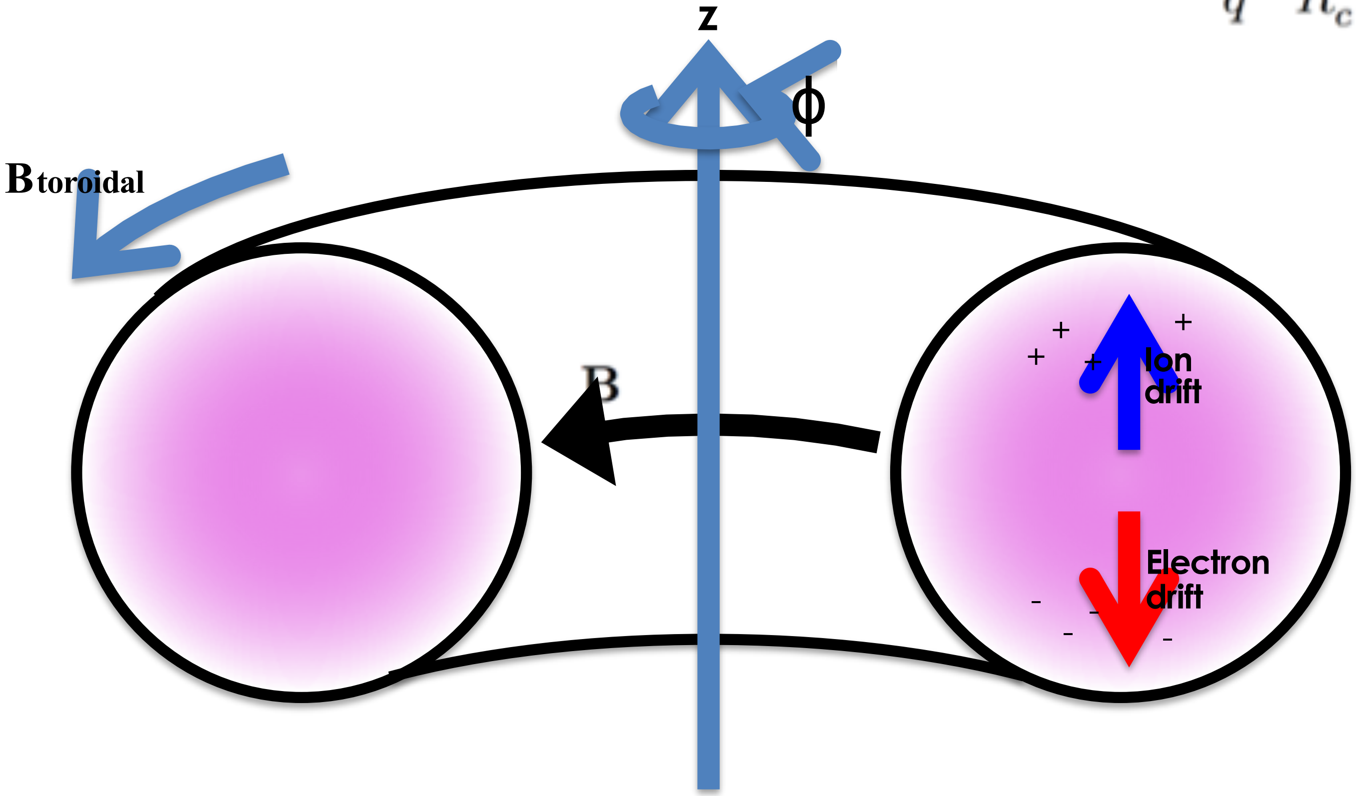
- The gyro-radius will be larger where the field is weaker and smaller where the field is stronger
- The resulting drift velocity is described by:

$$\mathbf{V}_{\nabla \mathbf{B}} = \frac{m v_\perp^2}{2qB} \frac{\mathbf{B} \times \nabla B}{B^2}$$

What Happens To Charged Particles In A Purely Toroidal Magnetic Field?

- Charged particles in a curved magnetic field will experience both ∇B and curvature drift: these effects add

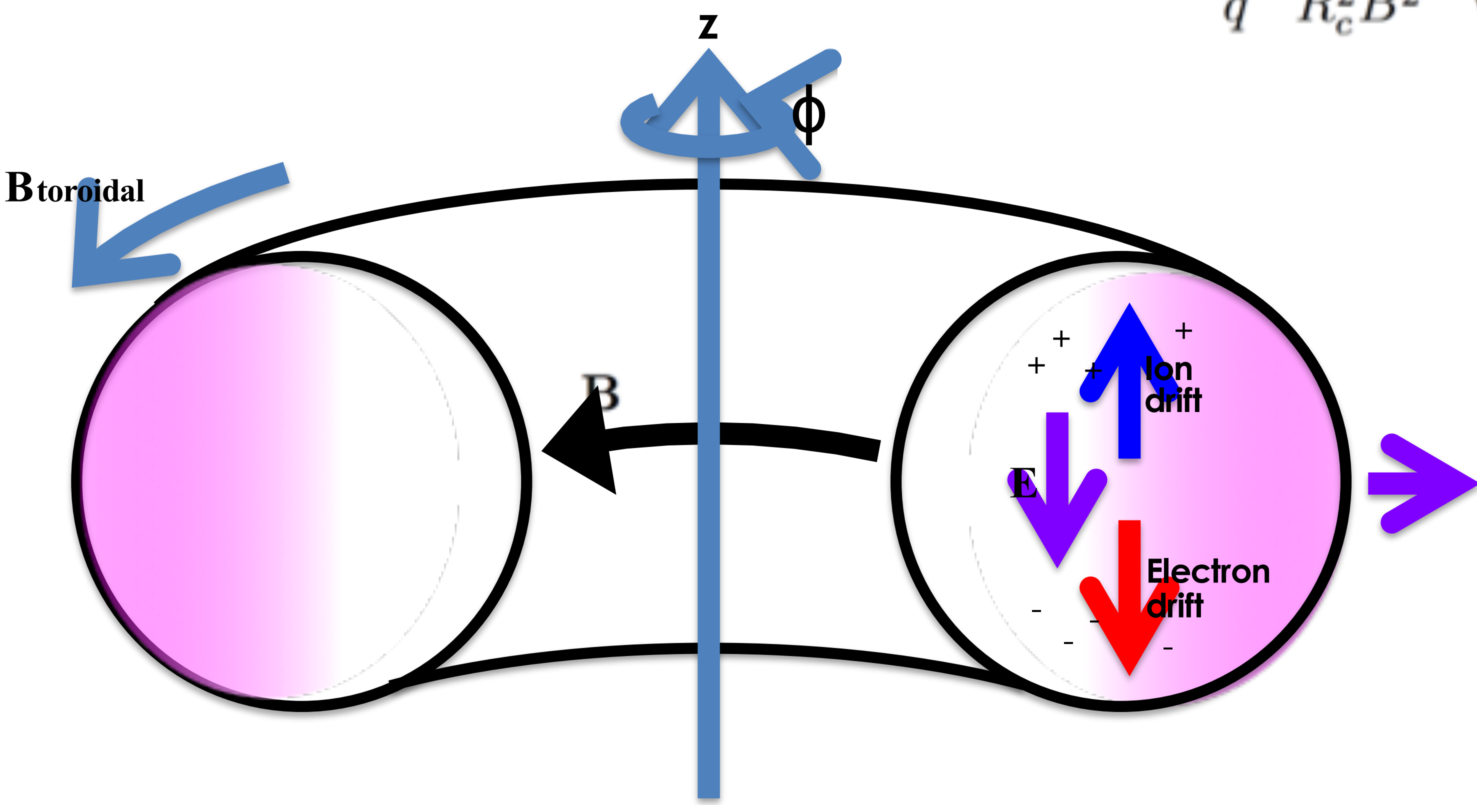
$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c}\times\mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$



Charged Particles Will Drift Outward

- Charged particles in a curved magnetic field will experience both ∇B and curvature drift

$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$



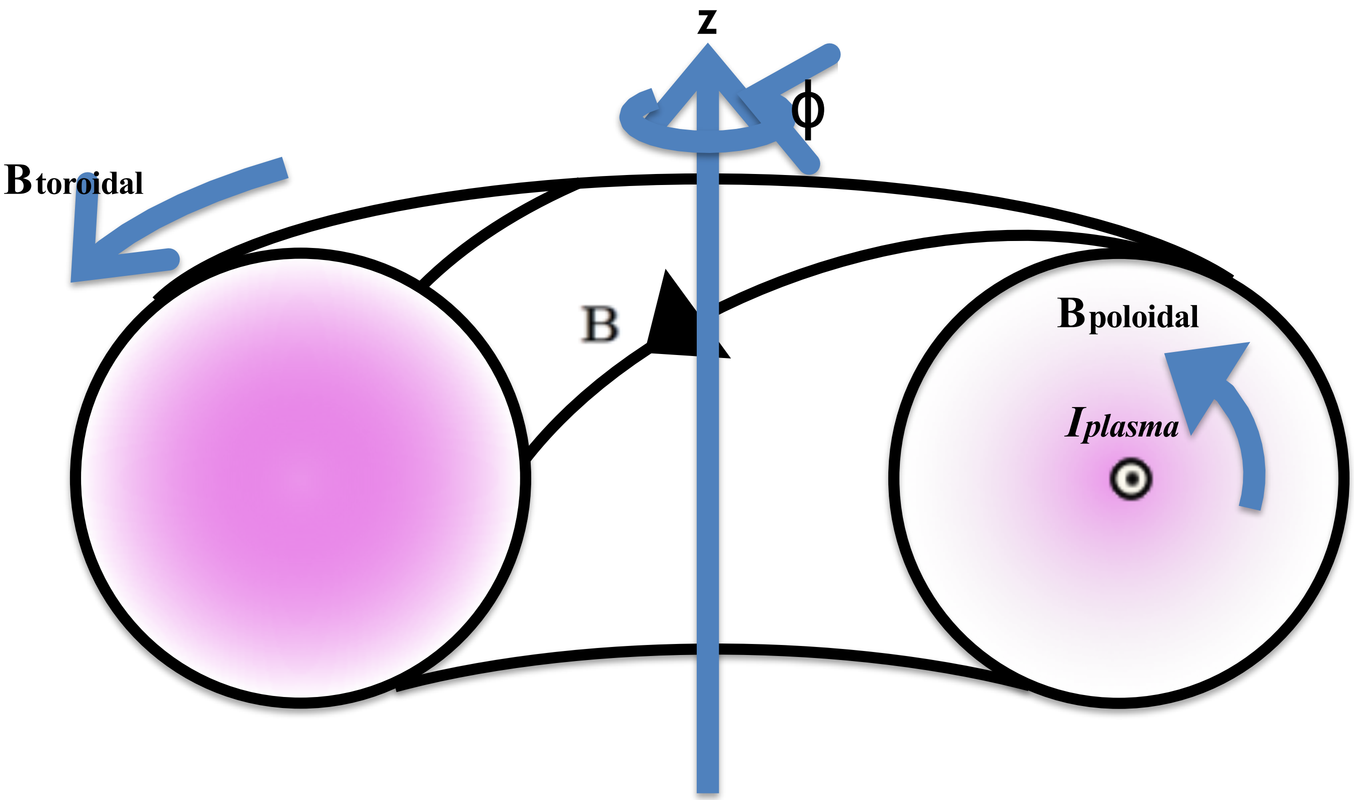
$$\mathbf{v_E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- This means that no matter what, particles in a torus with a purely toroidal field will drift radially out and hit the walls.

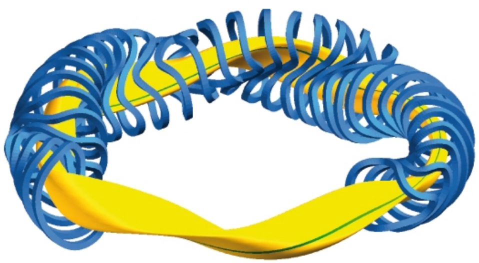
Tokamak Solution: Add Poloidal Magnetic Field

Toroidal: long way around
Poloidal: short way around

More on this on Day 5!



Stellerator:



Twist is applied by external magnets

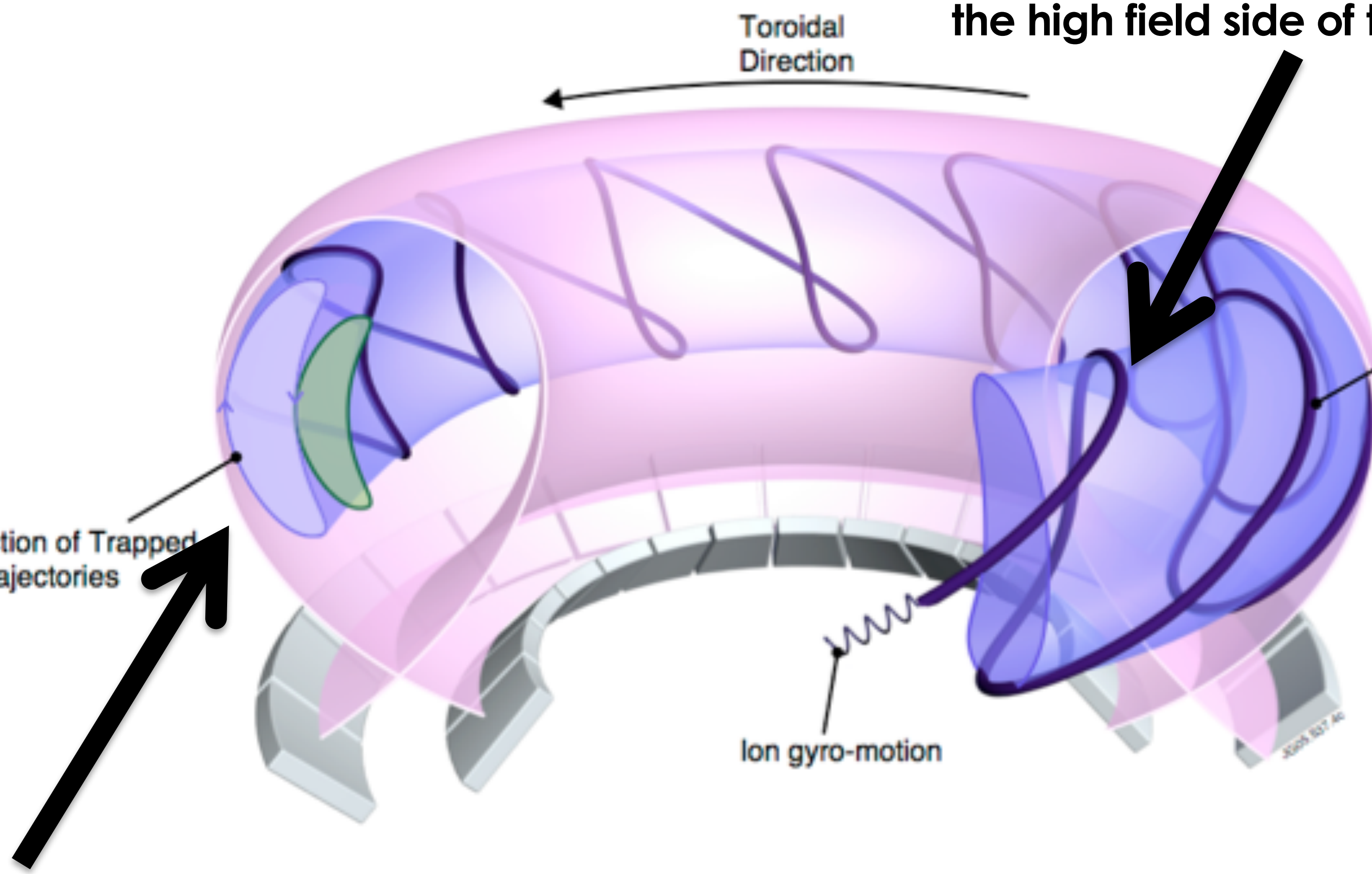
Tokamak:

1. Use external coils to apply a toroidal magnetic field
2. Drive **current** in the plasma to generate a **poloidal magnetic field**

The resulting helical magnetic field is much better at confining charged particles.

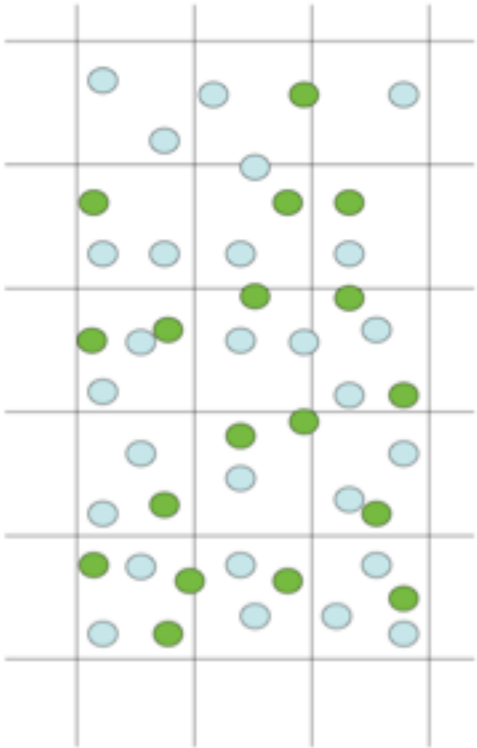
Particle orbits in Tokamaks (Banana Orbits)

Particles that don't have enough $v_{||}$ are reflected by the mirror force at the high field side of the tokamak

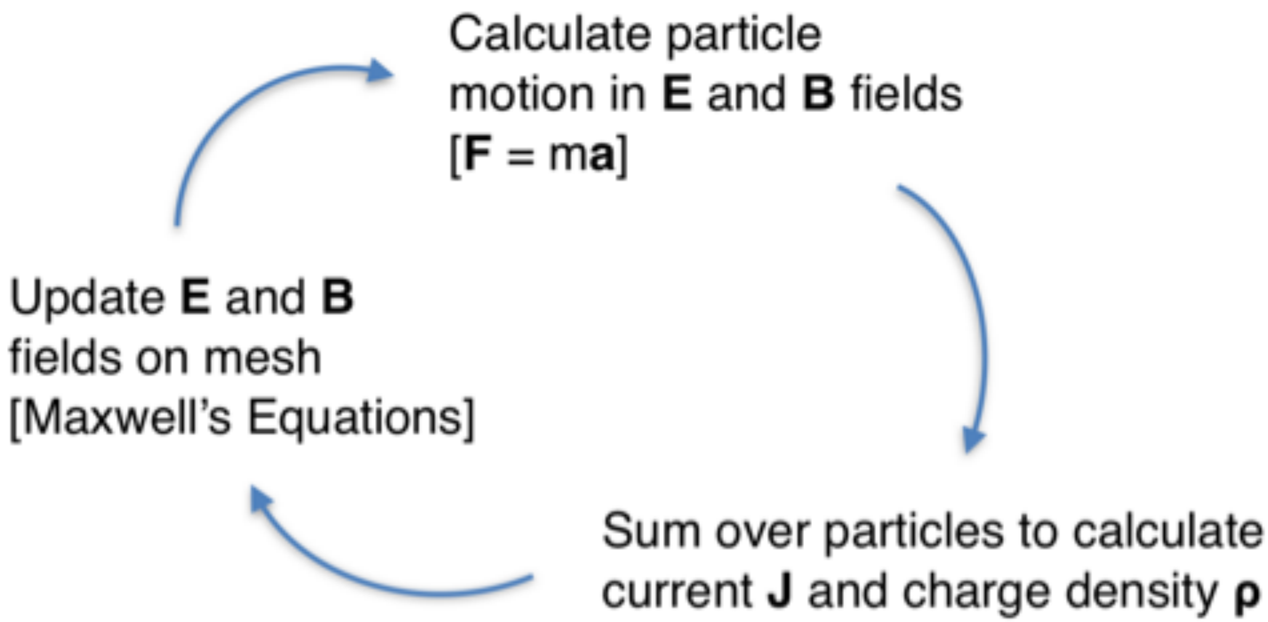


Trapped particles won't hit the wall if the banana orbit width is small enough

The ideas behind single-particle physics scale all the way to advanced simulation techniques

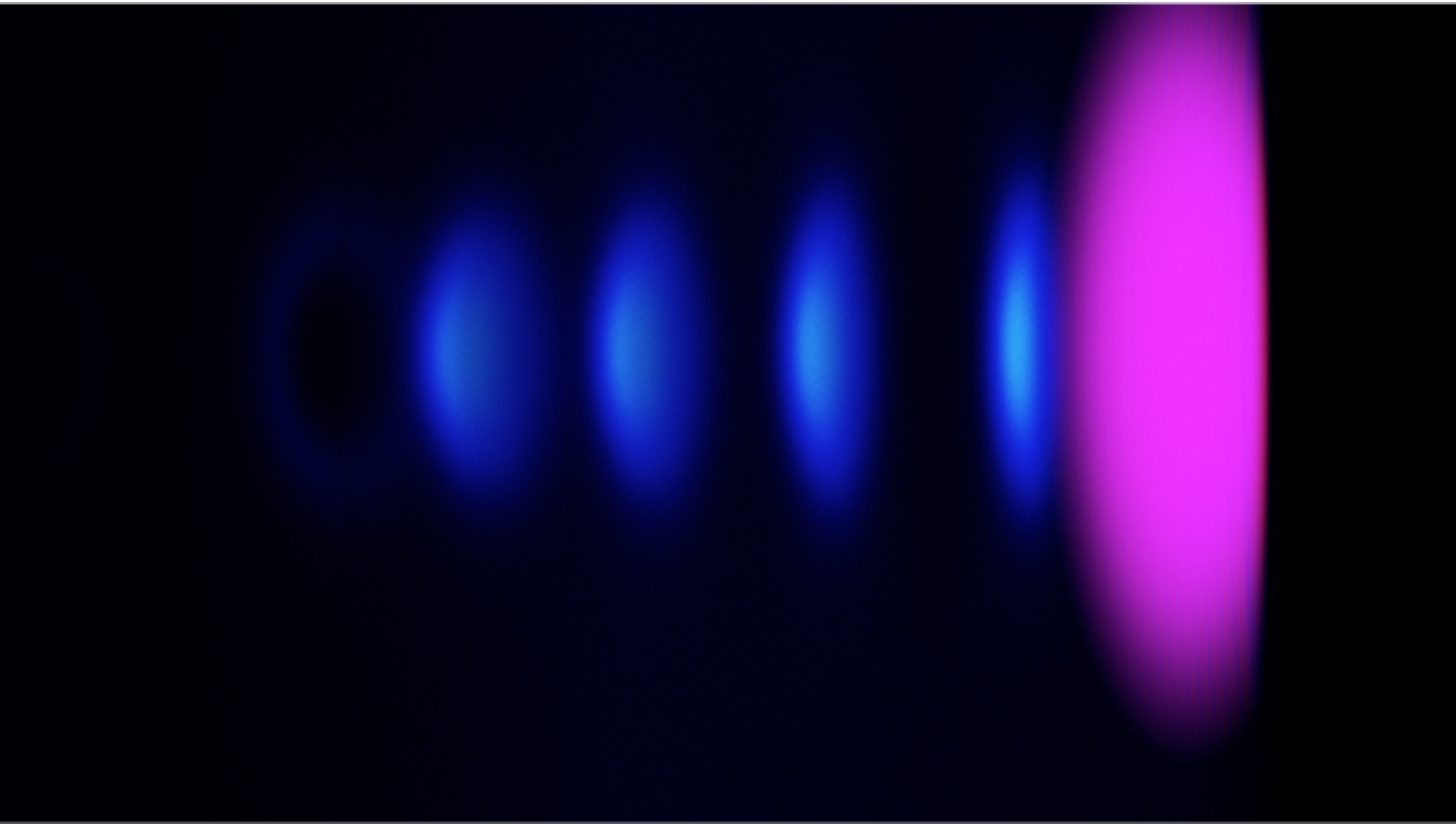


- The Particle-in-Cell simulation technique



- **Plasma simulations run on the world's fastest supercomputers to study the trajectories of charged particles in electromagnetic fields**
- **Used to study electron acceleration and ion heating**

We use PIC codes to study compact Laser Wakefield Accelerators to generate energetic electron beams

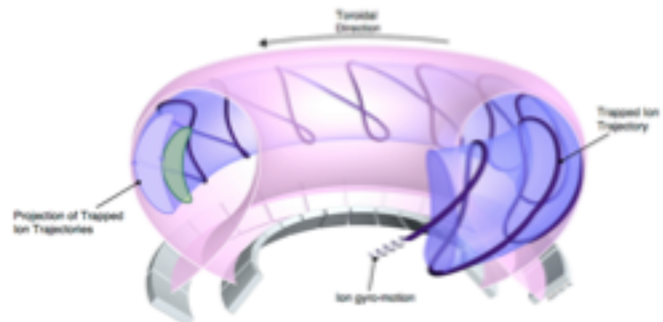


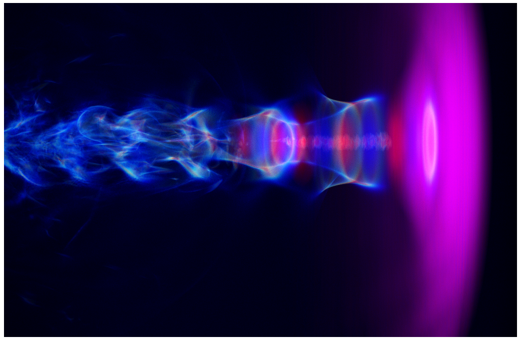
Conclusions

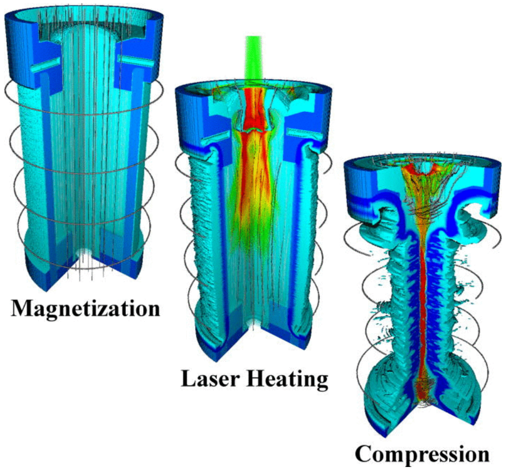
- Magnetic fields are required to confine hot, collisionless fusion plasmas

- Depending on magnetic field geometry or the presence of other forces like electric fields, the guiding center can undergo drifts (e.g. ExB, gradB and curvature drifts)

- Single particle motion enables us to understand complex plasma behavior – even towards achieving fusion!







Have a great summer and feel free to get in touch!

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