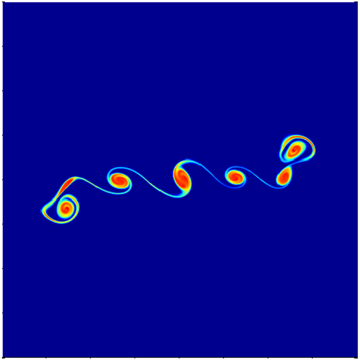
Plasmas as fluids



Antoine Cerfon, Courant Institute, New York University
Email: cerfon@cims.nyu.edu

SULI Introductory Course in Plasma Physics, June 15, 2021

Yours truly in a few words (I)

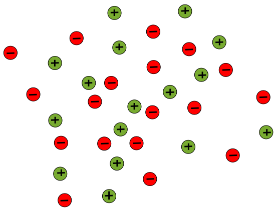
- Undergrad training: Engineering management, in France
- M.Sc in Nuclear Engineering, in France
- Irresistibly attracted to fundamental research + theory
- *Rejected from all PhD programs in US*, except for MIT's Nuclear Engineering
- PhD in Applied Plasma Physics from MIT – Fusion theory

Yours truly in a few words (II)

- Post-doc at MIT – Magnetic confinement fusion and particle accelerator theory
- 2012 until now: Faculty position in mathematics at Courant Institute of Mathematical Sciences, New York University
- Development of new models and high performance numerical methods for plasma physics
- Intermediary between applied mathematicians and plasma physicists
- Enjoys outreach, teaching, mentoring

DESCRIBING PLASMAS

METHOD I: SELF-CONSISTENT PARTICLE PUSHING



Natural idea: Move each particle according to $\mathbf{F}_p = m_p \mathbf{a}_p$

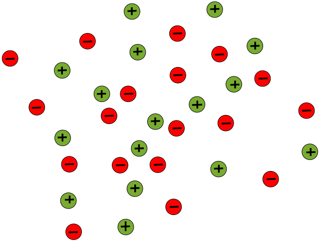
- **Difficulty 1**: There are MANY particles, $N \sim 10^{20} - 10^{22}$ in magnetic fusion grade plasmas
- **Difficulty 2**: $\mathbf{F}_p$ depends on the position and velocity of all the other particles. $\mathbf{F}_p$ is expensive to compute
  e.g.: for electrostatic electric field force

$$\mathbf{F}_p = q_p \sum_{j=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\mathbf{x}_j - \mathbf{x}_p|^2}$$

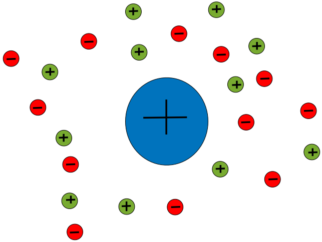
- Problem still not tractable even with the most powerful computers when $N \sim 10^{20} - 10^{22}$ and best algorithms

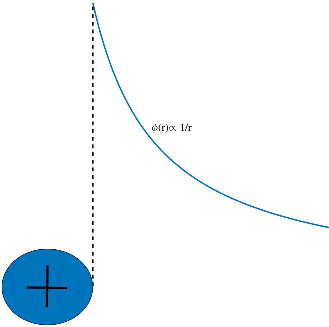
DEBYE SHIELDING

Even if computers were powerful enough, should we ask them to solve this problem?

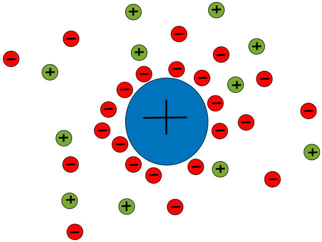


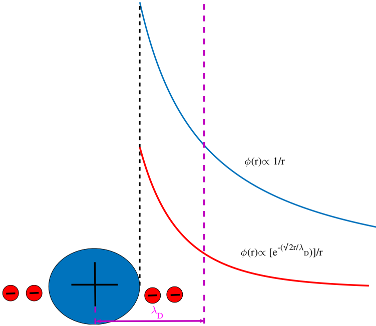
Debye shielding





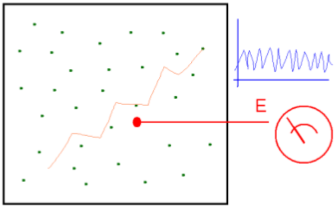
DEBYE SHIELDING

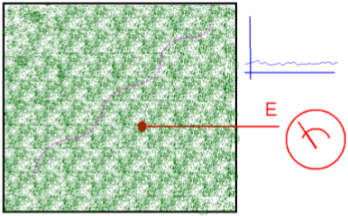




- Local charge imbalance shielded within a few λ_D
- $\lambda_D = \frac{\epsilon_0 T}{e^2 n}$ is called the Debye length

METHOD II: FOR WEAKLY COUPLED PLASMAS, COARSE-GRAIN AVERAGE IN PHASE SPACE





- Weakly coupled plasma: large # of particles in any volume of size λ_D^3
- Replace the discrete particles with smooth distribution function $f(\mathbf{x}, \mathbf{v}, t)$ defined so that

$$f(\mathbf{x}, \mathbf{v}, t)d\mathbf{x}d\mathbf{v} = \#\text{ of particles in 6D phase-space volume } d\mathbf{x}d\mathbf{v}$$

Cartoon figures by G. Lapenta in "Particle In Cell Methods With Application to Simulations in Space Weather"

Distribution function and Vlasov equation

- Macroscopic (fluid) quantities are velocity moments of f

$$n(\mathbf{x},t) = \iiint f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Density}$$

$$n\mathbf{V}(\mathbf{x},t) = \iiint \mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Mean flow}$$

$$\mathbf{P}(\mathbf{x},t) = m\iiint (\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})fd\mathbf{v} \qquad \text{Pressure tensor}$$

- Conservation of f along the phase-space trajectories of the particles determines the time evolution of f:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla f + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}} f = 0$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

$$\Rightarrow \qquad \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = 0$$

This is the Vlasov equation

The Boltzmann equation

- Often separate short range and long range fields, leading to the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

  This equation to be combined with Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

- Nonlinear, integro-differential, 6-dimensional PDE – **Challenging**
- Describes phenomena on widely varying length ($10^{-5} - 10^{3}$ m) and time ($10^{-12} - 10^{2}$ s) scales
- Still not a piece of cake, and never solved as such in computers

Moment approach

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

- Taking the integrals $\iiint d\mathbf{v}$, $\iiint m\mathbf{v} d\mathbf{v}$ and $\iiint mv^2/2 d\mathbf{v}$ of this equation, we obtain the exact **fluid equations**:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0 \qquad \text{(Continuity)}$$

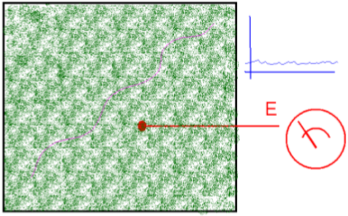
$$mn \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s n_s \left(\mathbf{E} + \mathbf{V}_s \times \mathbf{B} \right) - \nabla \cdot \mathbf{P}_s + \mathbf{R}_s \quad \text{(Moment.)}$$

$$\frac{d}{dt} \left(\frac{3}{2} p_s \right) + \frac{5}{2} p_s \nabla \cdot \mathbf{V}_s + \boldsymbol{\pi}_s : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = W_s \qquad \text{(Energy)}$$

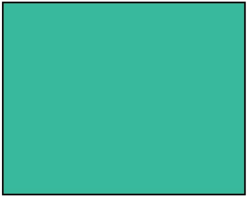
  with $\mathbf{P}_s = p_s \mathbf{I} + \boldsymbol{\pi}_s$.
- Closure problem: for each moment, we introduce a new unknown $\Rightarrow$ End up with too many unknowns
- Need to make approximations to close the moment hierarchy

FLUID DESCRIPTIONS OF PLASMAS

High collisionality



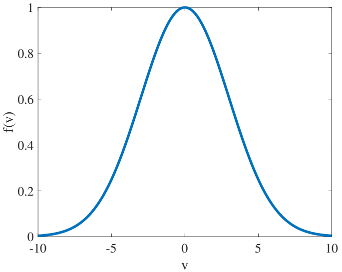
Low density – low collisionality

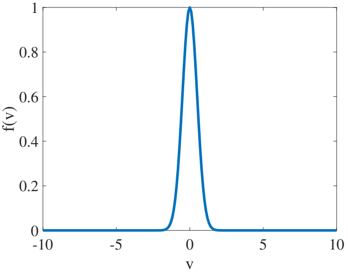


High density – high collisionality

- When there are many collisions, nearby particles remain nearby during entire evolution.
- Thermal equilibrium established locally
- **Fluid-like behavior**

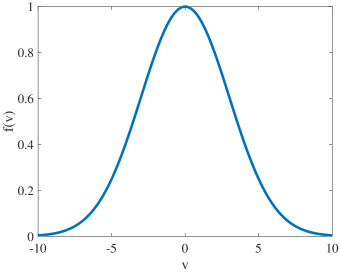
Low temperature



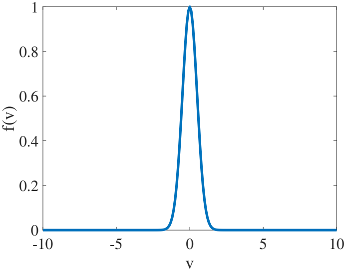


- At low temperature, nearby particles have similar total velocity
- Nearby particles remain nearby during the entire evolution
- **Fluid-like behavior**

Low temperature



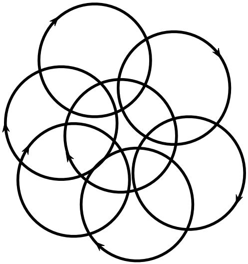
High temperature



Low temperature

- At low temperature, nearby particles have similar total velocity
- Nearby particles remain nearby during the entire evolution
- **Fluid-like behavior**

Strong magnetic field



Weak magnetic field

Strong magnetic field

- At high magnetic field, short mean free path perpendicular to the magnetic field
- Projections of nearby particles in plane ⊥ to magnetic field remain nearby during the entire evolution
- **Fluid-like behavior perpendicular to the magnetic field**

Fluid models

- For some fusion applications/plasma regimes (heating and current drive, transport), kinetic treatment cannot be avoided
- Fluid models are based on approximate expressions for higher order moments (off-diagonal entries in pressure tensor, heat flux) in terms of lower order quantities(density, velocity, diagonal entries in pressure tensor)
- Approximations depend on the physics regime of interest
- Benefits of fluid models:
  - Much less computationally expensive
  - Intuitive interpretation for the terms in the equations

Closure example

Quasi-neutral plasma of electrons and ions, weakly magnetized

$$\mathbf{q}_e = -\kappa_e \nabla T_e - 0.71 \frac{T_e \mathbf{J}}{e}$$

$$\mathbf{q}_i = -\kappa_i \nabla T_i$$

$$\boldsymbol{\pi}^s_{jk} = -\eta_s \left(\frac{\partial V^s_j}{\partial x_k} + \frac{\partial V^s_k}{\partial x_j} - \frac{2}{3} \nabla \cdot \mathbf{V} \delta_{jk} \right)$$

κ_e,κ_i: electron and ion thermal conductivities

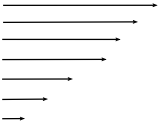
η_s: electron and ion viscosity

We recognize:

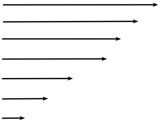
- Fourier's law of thermal conduction for both electrons and ions – Intuitive term. Heat flows from hot places to cold places.
- Perhaps unexpected extra term: convective heat transport due to fast electrons
- Viscous diffusion of momentum associated with derivatives of velocities

Viscous diffusion

No viscosity

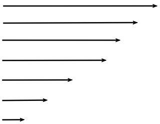


Initial flow profile

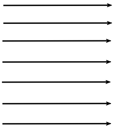


Some time later

With viscosity



Initial flow profile



Some time later

Viscous momentum exchange/momentum diffusion

A GREAT, SIMPLE FLUID MODEL: IDEAL MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Valid under the conditions

$$\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{v_i \tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2 \left(\frac{m_e}{m_i}\right)^{1/2} \frac{a}{v_{Ti}\tau_{ii}} \ll 1$$

Word of caution

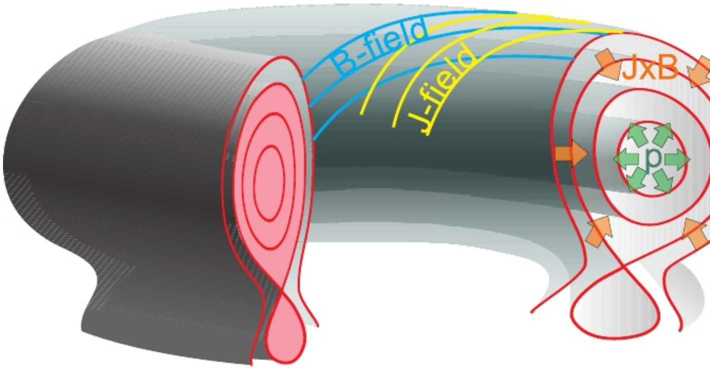
The validity of a fluid approximation depends both on the plasma of interest and on the characteristics of the phenomena of interest in that plasma

For a given plasma, fluid models can be excellent for some phenomena (e.g. macroscopic phenomena), and unreliable for other phenomena (e.g. phenomena at the DeBye scale)

SUCCESSFUL FLUID MODELS

Magnetic confinement fusion equilibrium

Ideal MHD is used to accurately predict the equilibrium magnetic configuration of fusion devices.



- The regions of constant pressure are nested toroidal surfaces
- Magnetic fields and currents lie on these nested surfaces

Vertical instability

MHD models and codes accurately calculate the loss of vertical stability in tokamaks

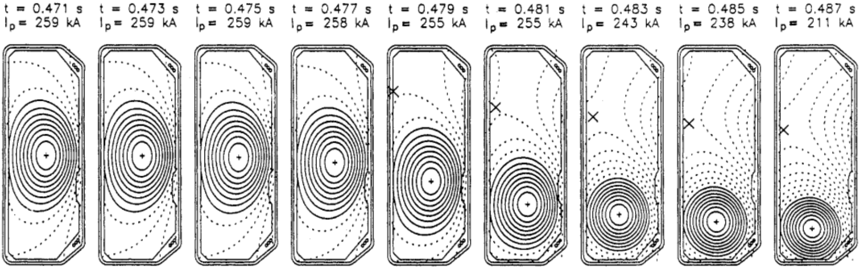
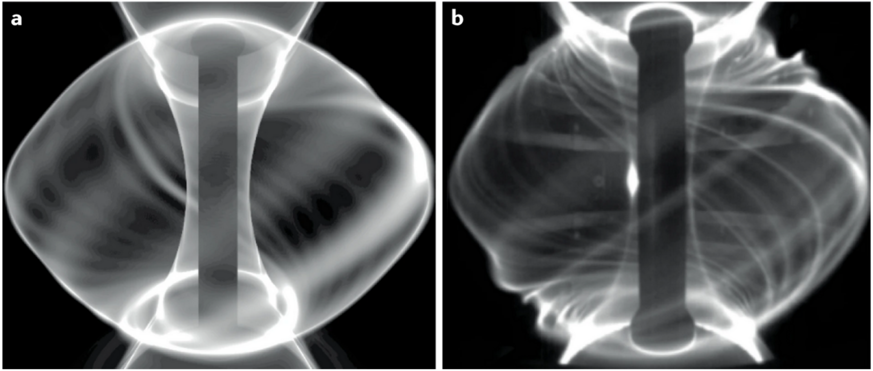


FIG. 2. Loss of vertical position control following a feedback cut at 0.470 s.

Figure from F. Hofmann *et al.*, Nuclear Fusion **37** 681 (1997)

Edge localized modes in tokamaks

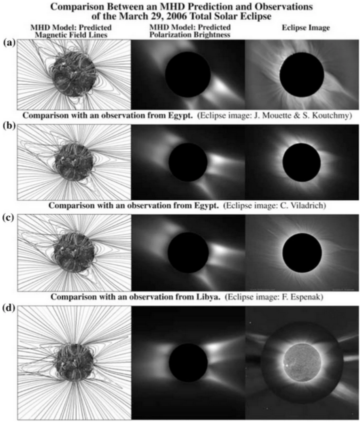
MHD models and codes can capture the filamentary nature of ″ballooning″ modes in the edge of tokamaks



S.J.P. Pamela *et al.* 2013 *Plasma Phys. Control. Fusion* **55** 095001

Solar Corona

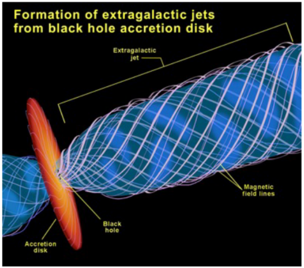
MHD models and codes are used to explain the dynamics of the solar corona.

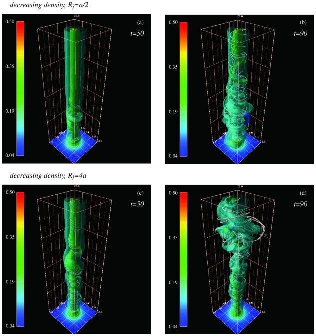


Miki Z, Linker J, Lionello R, Riley P, Titov V (2007) Predicting the structure of the solar corona for the total solar eclipse of March 29, 2006. In: Demircan O, Selam SO, Albayrak B (eds) Solar and stellar physics through eclipses. ASP conference series, vol 370. Astronomical Society of the Pacific, San Francisco, pp 299307

Dynamics of astrophysical plasmas

Relativistic MHD models and codes are used to understand the structure and dynamics of plasmas around accreting black holes and neutron star systems





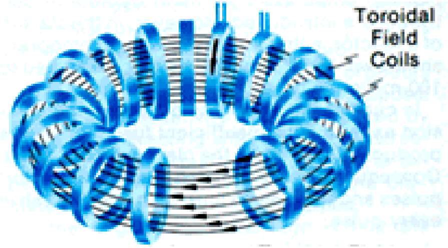
Cartoon figure from Christopher Carey
Simulation from Yosuke Mizuno *et al.* 2014 *ApJ* **784** 167

COMPLEMENTARITY OF PARTICLE AND FLUID PICTURES

The value of qualitative understanding

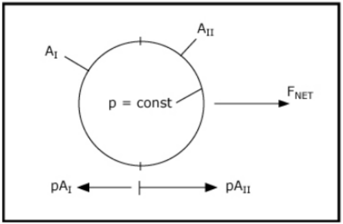
- In general, both a single-particle model of the plasma and a fluid model have mathematical flaws
- They are nevertheless useful to interpret results and build intuition
- Insights are amplified by combining both view points
- Let us see this with two simple examples

Can one confine a hot plasma with a purely toroidal field?



- Bend a solenoid into a torus to confine a hot fusion plasma
- Rely solely on toroidal fields
- Relatively simple, thus great! Wait, does it work?
- What does the fluid picture have to say?

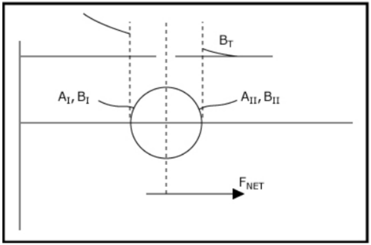
PURELY TOROIDAL FIELDS DO NOT WORK – FLUID PICTURE



"Tire-tube" force

$pA_{II} > pA_I$

$\Rightarrow$ Net outward force



"1/R" force

$B_T \propto 1/R$

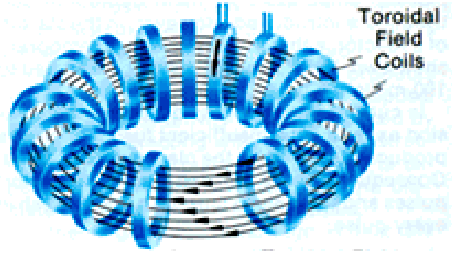
$B_{T1} > B_{T2}, A_1 < A_2$

$B_{T1}^2 A_1 > B_{T2}^2 A_2$

$\Rightarrow$ Net outward force

Cartoon figures from J.P. Freidberg, MHD Theory of Fusion Systems, MIT Open Courseware 2007

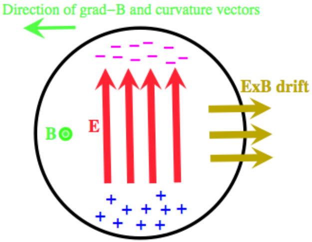
Purely toroidal fields do not work – single particle picture



- Magnetic field in the toroidal direction
- Grad-B and curvature vectors in the radial direction

$\Rightarrow$ **Grad-B and curvature drifts in the vertical direction**

PURELY TOROIDAL FIELDS DO NOT WORK – SINGLE PARTICLE PICTURE

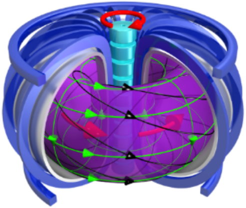


- The electrons always drift upwards
- The ions always drift downwards
  ⇒**This leads to charge separation!**
- Electric field forms because of charge separation

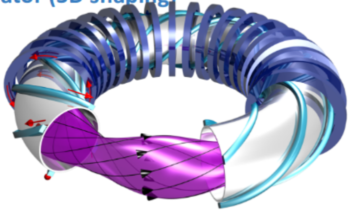
⇒ **All the particles $\mathbf{E} \times \mathbf{B}$ drift out of the device!**

SOLUTIONS

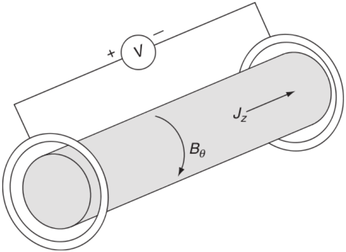
Tokamak (Axisymmetric)

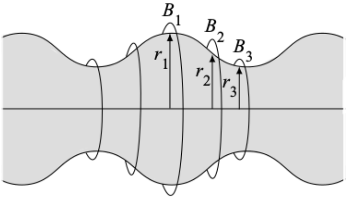


Stellarator (3D shaping)

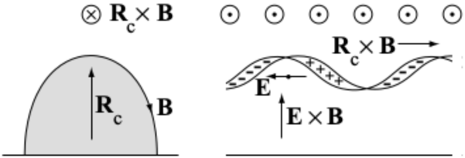


Interchange instability





Single-particle picture of the same instability



J.P. Freidberg, *Plasma Physics and Fusion Energy*, Cambridge University Press (2007)

PLASMAS EXPERIMENTS TO VALIDATE THEORETICAL RESULTS IN FLUID DYNAMICS

Isomorphism between perpendicular plasma dynamics and 2D Euler equations

Non-neutral plasma in strong B field, $\perp$ dynamics

$$\frac{\partial n}{\partial t} + \nabla\phi \times \mathbf{e}_z \cdot \nabla n = 0$$
$$\nabla^2 \phi = -n$$

n: plasma density; ϕ: electrostatic potential

2D incompressible Euler

$$\frac{\partial \omega}{\partial t} + \nabla\psi \times \mathbf{e}_z \cdot \nabla \omega = 0$$
$$\nabla^2 \psi = -\omega$$

ω: z-directed vorticity; ψ: stream function for the flow

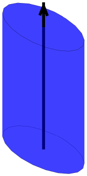
- Isomorphism recognized a long time ago[1]
- We can use non-neutral plasma physics experiments to study inviscid (ideal) fluids!

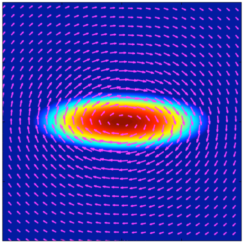
---

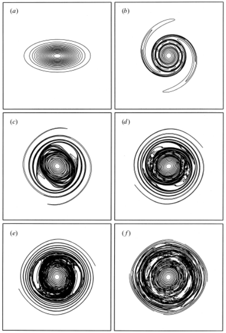
[1] C.F. Driscoll and K.S. Fine, *Phys.Fluids B* **2** 1359 (1990)

INVISCID DAMPING

$$\frac{\partial n}{\partial t} + \nabla\phi \times \mathbf{e}_z \cdot \nabla n = 0 \qquad \nabla^2\phi = -n$$



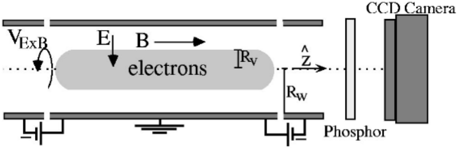


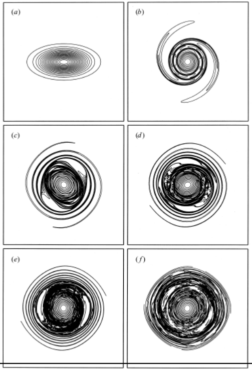


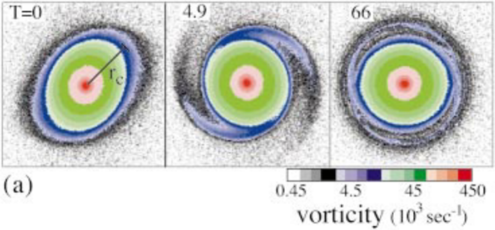
Movie!

M.V. Melander, J.C. McWilliams and N.J. Zabusky, *J. Fluid Mech.* **178** 137 (1987)
A.J. Cerfon, *Phys. Rev. Lett.* **116**, 174801 (2016)

INVISCID DAMPING – EXPERIMENTS

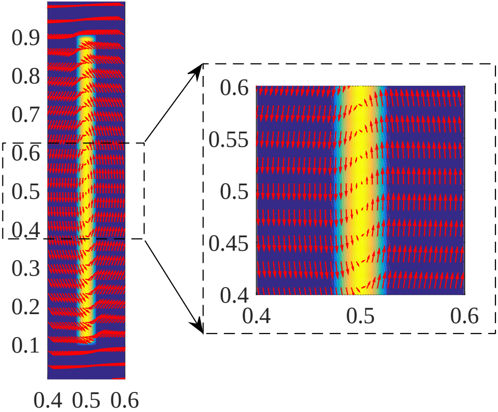






D.A. Schecter *et al.*, *Physics of Fluids* **12**, 2397 (2000)

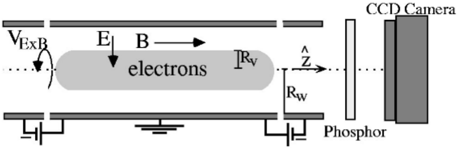
Shear layer instability

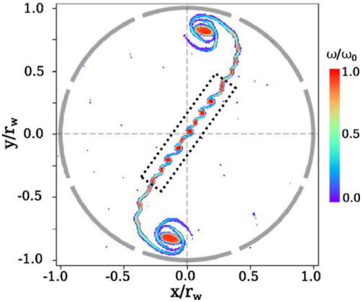


Movie!

A.J. Cerfon, *Phys. Rev. Lett.* **116**, 174801 (2016)

Shear layer instability – Experiment!





N.C. Hurst *et al.*, *Physics of Plasmas* **27**, 042101 (2020)

Thanks for listening!

Comments? Questions?