Single Particle Motion

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SULI Introductory Course in Plasma Physics
Princeton Plasma Physics Laboratory and on Zoom
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Acknowledgement to Cami Collins (GA) for sharing a previous version of this presentation
My “Single-Researcher Motion” through Plasma Physics

NUF student

High-school Physics and math in Nepal

PhD student

Research Scientist

Undergrad thesis at PPPL


Analysis of current drive using MSE polarimetry without equilibrium reconstruction

VTF experiment (MIT)

Expts at NIF and OMEGA

MRX (PPPL)
Key Points to Take Away

- Motion of single particles important to understand behavior of plasmas
- How magnetic fields modify single particle motion
  - Gyromotion about a guiding center
  - Forces can cause guiding center drift
- Some real life consequences:
  - Why do tokamaks have helical B fields?
  - Why do astrophysical shocks need a magnetic field?

References

- NRL Plasma Formulary
  www.nrl.navy.mil/ppd/content/nrl-plasma-formulary

- Introduction to Plasma Physics and Controlled Fusion by F. F. Chen
Single-particle motion is key intuition to understand many problems in plasma physics.

- Plasma confinement for Magnetic Fusion Energy in stellarators and tokamaks
- Role of magnetic fields in collisionless shocks
- Magnetically-driven Inertial-confinement fusion in Z-pinches
- Structure of solar plasmas
Ideas behind single-particle physics threads all the way through to the most modern simulation techniques

• The Particle-in-Cell simulation technique
  
  Calculate particle motion in $E$ and $B$ fields
  \[ F = ma \]

  Update $E$ and $B$ fields on mesh
  [Maxwell’s Equations]

  Sum over particles to calculate current $J$ and charge density $\rho$

• Idea allows plasma simulations that run on the world’s fastest supercomputers
• Some of you may use “gyrokinetic” simulations during your internship, which take advantage of single-particle results here to speed up calculations
Ideas behind single-particle physics threads all the way through to the most modern simulation techniques

• The Particle-in-Cell simulation technique
  
  Calculate particle motion in $E$ and $B$ fields [$F = ma$]

TODAY

Update $E$ and $B$ fields on mesh [Maxwell’s Equations]

Sum over particles to calculate current $J$ and charge density $\rho$

• Idea allows plasma simulations that run on the world’s fastest supercomputers
• Some of you may use “gyrokinetic” simulations during your internship, which take advantage of single-particle results here to speed up calculations
We can understand a lot about how fusion devices confine plasma by studying single particle motion.

Recall a burning plasma will require a temperature $T \sim 5-10$ keV = 50-100 M Kelvin

$$kT = \frac{1}{2}mv_{th}^2$$

$$v_{th} \sim 6 \times 10^5 \text{ m/s}$$

Even with $\sim 10^{20}$ ions/m$^3$, the ion would travel $\sim 10$ km before colliding with another

Therefore these plasmas are effectively collisionless (a common theme)

The ITER tokamak has minor radius 2 m

How to confine particles?

The trick: use magnetic fields
A particle with charge \((q)\) moving with velocity \((v)\) in the presence of electric and magnetic fields will experience a force:

\[
F = q(E + v \times B)
\]

We know from Newton’s second law of motion that force causes acceleration:

\[
F = ma
\]

A charged particle moving perpendicular to the magnetic field feels a force.
Consider the motion of a particle in a constant, uniform B field.

\[ \mathbf{B} = B_\parallel \mathbf{\hat{z}} \quad \mathbf{E} = 0 \]

Then

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

So we can write

\[ \frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m} \]
How Does a Charged Particle Move in a Magnetic Field?

- Consider the motion of a particle in a constant, uniform B field

\[
B = B_z \hat{z} \\
E = 0
\]

Then

\[
F = q(E + v \times B)
\]

So we can write

\[
\frac{dv}{dt} = \frac{qv \times B}{m}
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\[ \frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m} \]

one “gyration” or “gyro-orbit” completed

Now let’s do the algebra
Goal: Solve the Equations of Motion for a Charged Particle In A Magnetic Field

\[ \frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m} \]

Let's break this into components:

\[ \dot{v}_x = \frac{qv_y B_z}{m} \]
\[ \dot{v}_y = -\frac{qv_x B_z}{m} \]
\[ \dot{v}_z = 0 \]

Particles move freely along the field line

The ‘dot’ notation represents \( \frac{d}{dt} \)
Take Another Time Derivative & Substitute to Obtain Differential Equations For Each Spatial Coordinate

\[ \ddot{v}_x = \frac{qv_y B_z}{m} \quad \ddot{v}_y = -\frac{qv_x B_z}{m} \]

Rewriting, we get

\[ \ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \quad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y \]

These may remind you of the equations for a simple harmonic oscillator

Homework: go through the details on your own
Solve the Differential Equations

\[ \ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \quad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y \]

These differential equations can be solved using sines and cosines:

\[ v_x = v_\perp \cos \left(\frac{|q|B_z}{m} t + \phi_0\right) \quad v_y = \mp v_\perp \sin \left(\frac{|q|B_z}{m} t + \phi_0\right) \]

\[ v_\perp = \sqrt{v_{x}^2 + v_{y}^2} \]

the magnitude of the initial velocity perpendicular to \( B \)

account for positive or negative \( q \)

an arbitrary phase to match the initial velocity conditions

Homework: go through the details on your own
The Result: Circular Motion About A Guiding Center

\[ v_x = v_\perp \cos \left( \frac{|q|B_z}{m} t + \phi_0 \right) \]  
\[ v_y = \mp v_\perp \sin \left( \frac{|q|B_z}{m} t + \phi_0 \right) \]

Integrating, we obtain

\[ x = \frac{mv_\perp}{|q|B_z} \sin \left( \frac{|q|B_z}{m} t + \phi_0 \right) + x_0 \]  
\[ y = \pm \frac{mv_\perp}{|q|B_z} \cos \left( \frac{|q|B_z}{m} t + \phi_0 \right) + y_0 \]

- Charged particles undergo circular orbits about a guiding center \((x_0, y_0)\)

\[ r_L \equiv \frac{mv_\perp}{|q|B} \]  
\[ \omega_c \equiv \frac{|q|B}{m} \]

“Larmor” radius or “gyro-radius”

Cyclotron or gyro-frequency
Gyromotion of a Charged Particle In A Magnetic Field

\[ x = r_L \sin (\omega_c t + \phi_0) + x_0 \]
\[ y = \pm r_L \cos (\omega_c t + \phi_0) + y_0 \]

Let’s take \( \phi_0 = 0 \) and \( x_0 = y_0 = 0 \)

For a positively charged particle:

1. At \( t = 0 \),

\[ x = 0 \]
\[ y = r_L \]
Gyromotion of a Charged Particle In A Magnetic Field

\[ x = r_L \sin (\omega_c t + \phi_0) + x_0 \quad y = \pm r_L \cos (\omega_c t + \phi_0) + y_0 \]

Let's take \( \phi_0 = 0 \) and \( x_0 = y_0 = 0 \)

For a positively charged particle:

1. At \( t = 0 \),
   \[ x = 0 \quad y = r_L \]

2. At \( t = \frac{\pi}{2 \omega_c} \),
   \[ x = r_L \quad y = 0 \]
Gyromotion of a Charged Particle In A Magnetic Field

\[ x = r_L \sin (\omega_c t + \phi_0) + x_0 \quad y = \pm r_L \cos (\omega_c t + \phi_0) + y_0 \]

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Gyromotion of a Charged Particle In A Magnetic Field

\[ x = r_L \sin(\omega_ct + \phi_0) + x_0 \quad y = \pm r_L \cos(\omega_ct + \phi_0) + y_0 \]

Let's take \( \phi_0 = 0 \) and \( x_0 = y_0 = 0 \)

For a negatively charged particle:

1. At \( t = 0 \),

\[ x = 0 \quad y = -r_L \]

\[
\begin{align*}
\text{B} & \quad \text{y} \\
\bigcirc & \quad \text{x}
\end{align*}
\]
Gyromotion of a Charged Particle In A Magnetic Field

\[ x = r_L \sin (\omega_c t + \phi_0) + x_0 \quad y = \pm r_L \cos (\omega_c t + \phi_0) + y_0 \]

Let’s take \( \phi_0 = 0 \) and \( x_0 = y_0 = 0 \)

For a negatively charged particle:

1. At \( t = 0 \),

\[ x = 0 \quad y = -r_L \]

2. At \( t = \frac{\pi}{2} \frac{1}{\omega_c} \),

\[ x = r_L \quad y = 0 \]
Let's take \( \phi_0 = 0 \) and \( x_0 = y_0 = 0 \)

For a negatively charged particle:

1. At \( t = 0 \),
   \[
   x = 0 \quad \text{and} \quad y = -r_L
   \]

2. At \( t = \frac{\pi}{2 \omega_c} \),
   \[
   x = r_L \quad \text{and} \quad y = 0
   \]
Gyromotion of Ions vs. Electrons

- The direction of gyromotion depends on the sign of the charge
- Ions generally have a much larger Larmor radius than electrons

In ITER, for a typical deuterium ion with $T_i=10$ keV and $B=5$ Tesla would have

$$v_{Ti} = \sqrt{\frac{kT_i}{m_i}} \approx 700 \text{ km/s}$$

$$r_L \equiv \frac{mv_{\perp}}{|q|B} \approx 3 \text{ mm}$$

An electron with $T_e=10$ keV and $B=5$ Tesla has

$$r_L \approx 0.05 \text{ mm} \quad (60 \text{ times smaller})$$
Particle motion and the “guiding center”

The gyration strongly constrains the motion of the particle perpendicular to the B field.

So the particle motion is intuitively decomposed to a sum of (1) gyration plus (2) “drift” of a "guiding center”

The guiding center is what is left when you average over the gyration

The guiding center can have:
(1) fairly unconstrained parallel motion
(2) slower “drifts” in the perpendicular direction

Fusion devices should be much larger than the gyro-radii.
Homework: Is it true for ITER?
Solar coronal plasma
Te = 10 MK (100 eV)
B = 100 G

Which way is the magnetic field running?
Is the gradient stronger along or across the field?
Some guiding center dynamics: Magnetic Mirrors

\[ B = B_r \hat{r} + B_z \hat{z} \quad F = q(\mathbf{v} \times \mathbf{B}) \]

The \( B_r \) ends up causing additional acceleration in the \( z \) direction:

\[ m \frac{dv_z}{dt} = -qv_\theta B_r \]

Result:

gyromotion + mirror force in the \(-\hat{z}\) direction

\[ F_z = -\frac{mv_\perp^2}{2B} \frac{\partial B_z}{\partial z} \]

The magnetic moment is

\[ \mu \equiv \frac{mv_\perp^2}{2B} \]

mirror force

\[ F_\parallel = -\mu \nabla_\parallel B \]

\( \mu \) is an “adiabatic invariant” - a deeply important concept for magnetized plasmas
Next Simplest Case to Analyze: Drift under Constant, Uniform Electric Field Perpendicular to Magnetic Field

\[ F = q(E + v \times B) \]

\[ E = E_x \hat{x} \quad B = B_z \hat{z} \]
Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

\[ F = q(E + v \times B) \]

\[ E = E_x \hat{x} \quad B = B_z \hat{z} \]

Accelerates due to E

Faster velocity increases vxB
Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

\[ \mathbf{E} = E_x \hat{x} \quad \mathbf{B} = B_z \hat{z} \]

Accelerates due to \( E \)

Faster velocity increases \( v \times B \)

Decelerates
Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

\[ F = q(E + v \times B) \]

\[ E = E_x \hat{x} \quad B = B_z \hat{z} \]

- Accelerates due to E
- Faster velocity increases \( v \times B \)
- Decelerates
Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

\[ \mathbf{E} = E_x \hat{x} \quad \mathbf{B} = B_z \hat{z} \]

\[ \dot{v}_x = \frac{q}{m} \left( v_y B_z + E_x \right) \]

\[ v_y = \mp v_\perp \sin \left( \frac{|q| B_z}{m} t + \phi_0 \right) - \frac{E_x}{B_z} \]

Ion guiding center drifts in the direction \(-\hat{y}\)
Guiding Center Drift Due to $E \times B$

The $E \times B$ drift can be written more generally as

$$v_y = \mp v_\perp \sin \left( \frac{|q|B_z}{m} t + \phi_0 \right) - \frac{E_x}{B_z}$$

Electron guiding center also drifts in the direction $-\hat{y}$

The $E \times B$ drift can be written more generally as

$$v_E = \frac{E \times B}{B^2}$$

- $E \times B$ drift is independent of charge and mass
- Both electrons and ions move together

Preview for MHD lecture: Magnetic field also moves with plasma, via Faraday’s law. Plasma and field can be thought of being “frozen together”
Digression: How magnetic fields mediate collisionless shocks in astrophysical plasmas

Shocks occur where supersonic flows interact. Shocks convert kinetic energy to heat

In gas, inter-particle collisions mediate the shock

In plasma, collisionless shocks have shock width $\ll$ mean-free-path.

Collisionless SNR shocks shown to be the sites of cosmic ray acceleration. [Ackerman Science 2013]
Magnetic fields mediate collisionless shocks in astrophysical plasmas

Shocks occur where supersonic flows interact. Shocks convert kinetic energy to heat

In gas, inter-particle collisions mediate the shock

In plasma, collisionless shocks have shock width $<<$ mean-free-path.

**Collective electromagnetic fields mediate shock in collisionless plasmas**

Possible mechanisms:
- Pile-up of pre-existing field (“magnetized shocks”)
- **Self-generation** of a turbulent magnetic field near shock by Weibel instability *

See recent experiments:
Other Forces Can Also Cause Guiding Center Drift

• Any force perpendicular to $B$ can cause particles to drift

Drift due to force:

$$v_d = \frac{1}{q} \frac{F \times B}{B^2}$$

Examples of forces:

$$F_g = mg \quad \text{gravity}$$

$$F_{cf} = \frac{mv^2}{R_c} \hat{r} \quad \text{centrifugal}$$

• Bend the magnetic field into a donut shape
• No end losses because the field lines go around and close on themselves
• BUT a particle following a toroidal magnetic field would experience $F_{cf}$
Curvature Drift Due to Bending Field Lines

The outward centrifugal force causes curvature drift

\[ F_{cf} = \frac{mv^2}{R_c} \hat{r} \]

\[ V_R = \frac{mv^2}{qB^2} \frac{R_c \times B}{R_c^2} \]

A particle moving along a curved field line will drift up or down, depending on the sign of the charge.
Spatially Varying Magnetic Field Strength Also Causes Drift

\[ B = \frac{\mu_0 NI}{2\pi R} \]
The gyro-radius will be larger where the field is weaker and smaller where the field is stronger.
The gyro-radius will be larger where the field is weaker and smaller where the field is stronger.

The resulting drift velocity is described by:

\[ V_{\nabla B} = \frac{mv_{\perp}^2}{2qB} \frac{B \times \nabla B}{B^2} \]
Charged particles in a curved magnetic field will experience both $\nabla B$ and curvature drift: these effects add.

\[
V_R + V_{\nabla B} = \frac{m}{q} \frac{R_c \times B}{R_c^2 B^2} \left( v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right)
\]
Charged Particles Will Drift Outward

- Charged particles in a curved magnetic field will experience both $\nabla B$ and curvature drift.

\[
V_R + V_{\nabla B} = \frac{m}{q} \frac{R_c \times B}{R_c^2 B^2} \left( v_{\parallel}^2 + \frac{1}{2} v_{\perp}^2 \right)
\]

- This means that no matter what, particles in a torus with a purely toroidal field will drift radially out and hit the walls.
Solutions: Add twist to the field with *poloidal* fields

Toroidal: long way around  
Poloidal: short way around

Tokamak:

Stellarator:

1. Use external coils to apply a toroidal magnetic field
2. Drive **current** in the plasma to generate **poloidal** magnetic field

Twist is applied by **3-D shaping fields**.

Will be returned to in lectures by D. Battaglia, F. Laggner, and A. Bader!
There Are Two Main Classes of Particle Orbits In Tokamaks

\[
V_R + V_{\nabla B} = \frac{m \mathbf{R}_c \times \mathbf{B}}{q R_c^2 B^2} \left( v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)
\]

Particles with sufficient \( v_\parallel \) will follow the helical magnetic field around the torus

Trapped

Particles with lower \( v_\parallel \) are reflected as they encounter stronger \( B \) and therefore execute “banana” orbits as they precess around the torus \( B \)
Conclusions - single particle and gyro motion in plasmas

- Magnetic fields needed to *confine* hot collisionless fusion plasmas.
- Guiding center undergoes $E \times B$, $\nabla \times B$ and curvature drifts.
- Cross-field confinement structures space, solar, and astrophysical plasmas, too.
- Magnetic fields mediate collective behavior of plasmas, via drifts.

- Work hard and soak it in. Have a good summer!