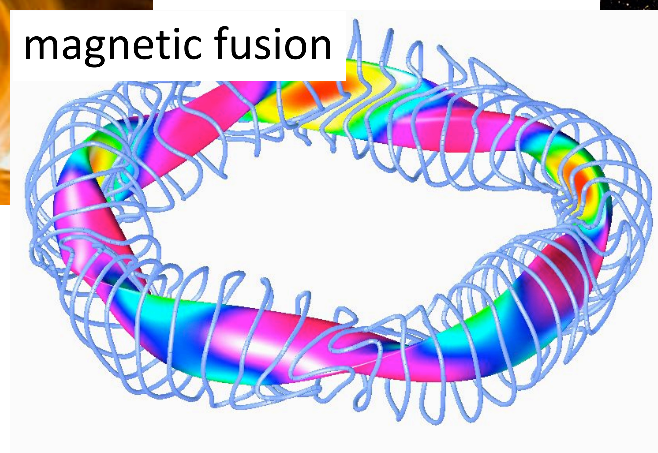
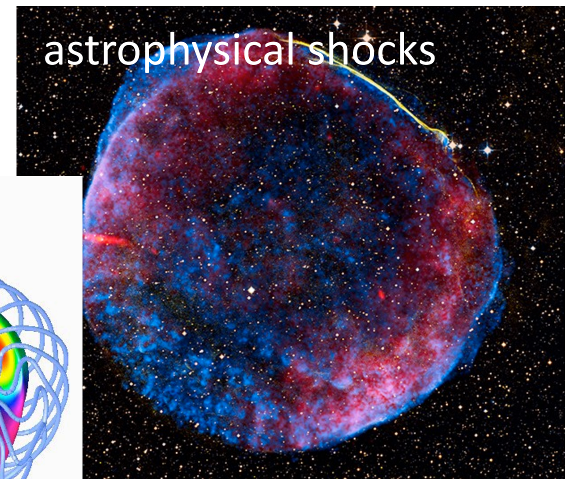
Single Particle Motion

Will Fox

PPPL and Princeton University





SULI Introductory Course in Plasma Physics

Princeton Plasma Physics Laboratory and on Zoom

June 16, 2020

Acknowledgement to Cami Collins (GA) for sharing a previous version of this presentation

My "Single-Researcher Motion" through Plasma Physics

NUF student

GENERAL ATOMICS

High-school Physics and math in Nepal

PhD student

MIT

UNH
Space Science Center

Research Scientist

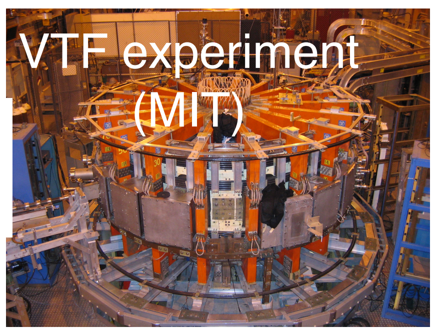
PPPL

Undergrad thesis at PPPL

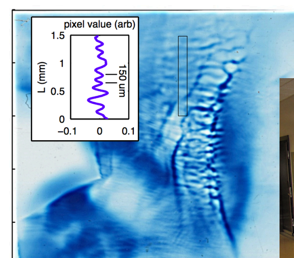
2001 | 2002 | 2002-2009 | 2009-2013 | 2013 - now

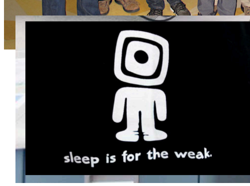
INSTITUTE OF PHYSICS PUBLISHING and INTERNATIONAL ATOMIC ENERGY AGENCY — NUCLEAR FUSION
Nucl. Fusion 42 (2002) 1124–1133 — PII: S0029-5515(02)38137-7

Analysis of current drive using MSE polarimetry without equilibrium reconstruction



Expts at NIF and OMEGA





Key Points to Take Away

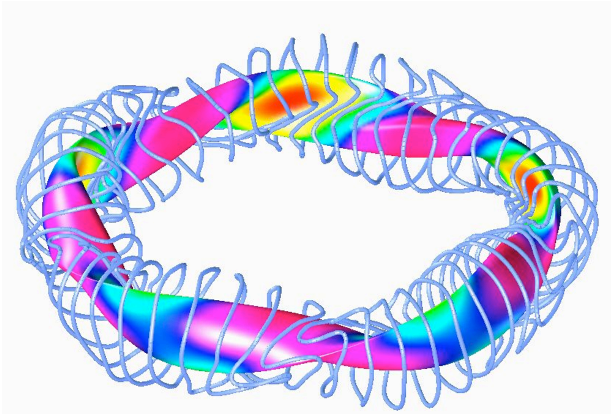
- **Motion of single particles important to understand behavior of plasmas**
- **How magnetic fields modify single particle motion**
    - **Gyromotion about a *guiding center***
    - **Forces can cause guiding center *drift***
- **Some real life consequences:**
    - Why do tokamaks have helical B fields?
    - Why do astrophysical shocks need a magnetic field?

References

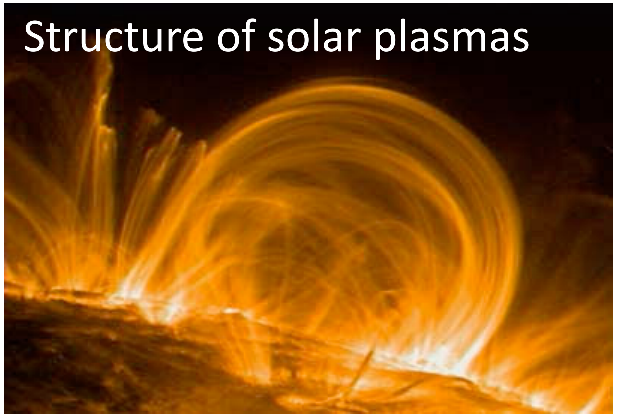
- NRL Plasma Formulary
  www.nrl.navy.mil/ppd/content/nrl-plasma-formulary

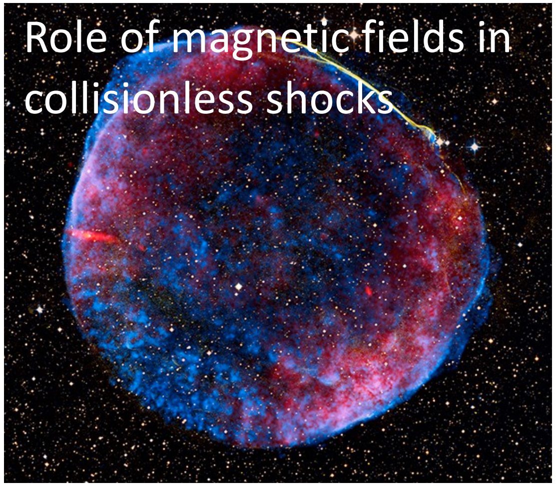
- Introduction to Plasma Physics and Controlled Fusion by F. F. Chen

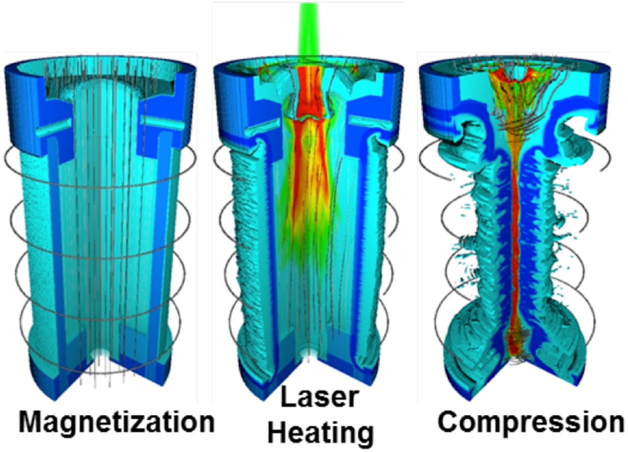
Single-particle motion is key intuition to understand many problems in plasma physics



Plasma confinement for Magnetic Fusion Energy in stellarators and tokamaks

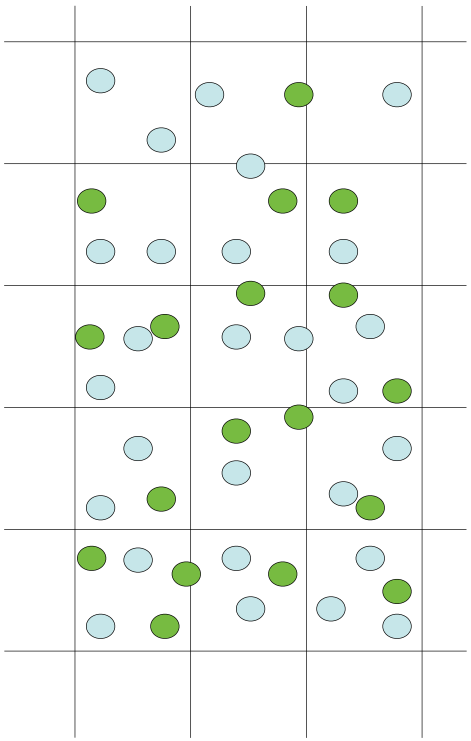






Magnetically-driven Inertial-confinement fusion in Z-pinches

Ideas behind single-particle physics threads all the way through to the most modern simulation techniques



- The Particle-in-Cell simulation technique

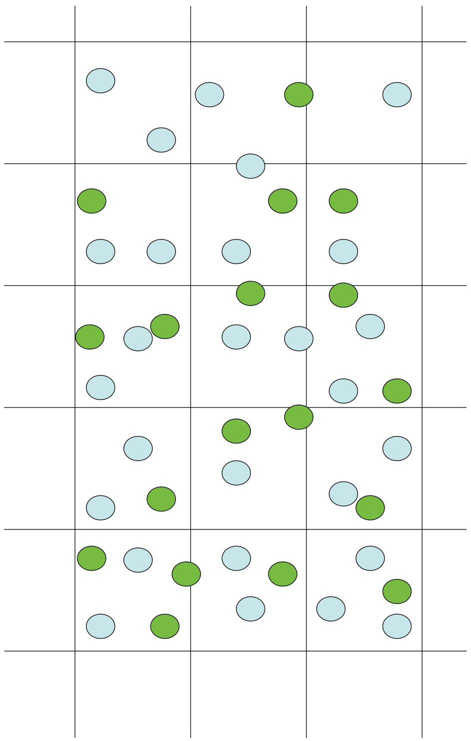
Calculate particle motion in **E** and **B** fields [**F** = m**a**]

Sum over particles to calculate current **J** and charge density **ρ**

Update **E** and **B** fields on mesh [Maxwell's Equations]

- Idea allows plasma simulations that run on the world's fastest supercomputers
- Some of you may use "gyrokinetic" simulations during your internship, which take advantage of single-particle results here to speed up calculations

Ideas behind single-particle physics threads all the way through to the most modern simulation techniques



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TODAY

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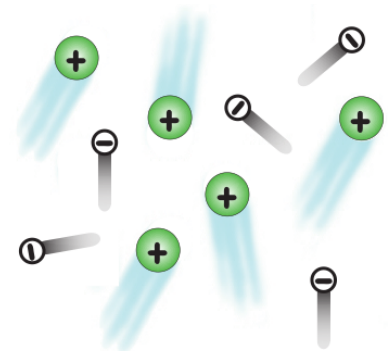
We can understand a lot about how fusion devices confine plasma by studying single particle motion.

Recall a burning plasma will require a temperature
T ~ 5-10 keV = 50-100 M Kelvin



$$kT = \frac{1}{2} m v_{\mathrm{th}}^2$$

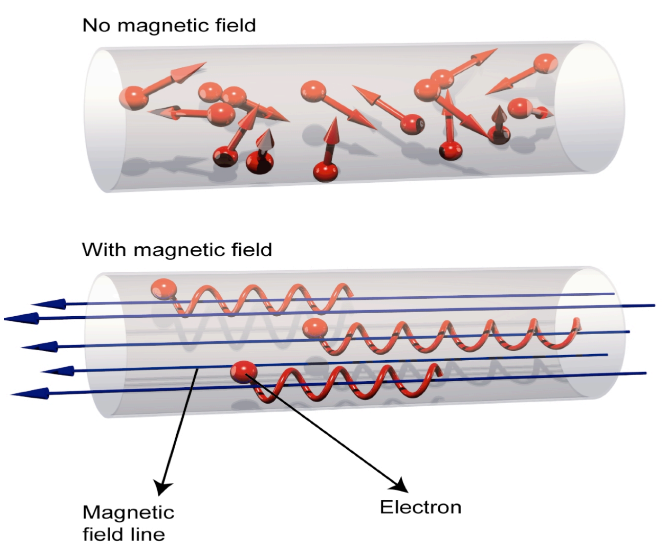
$$v_{\mathrm{th}} \sim 6 \times 10^5 \ \mathrm{m/s}$$

Even with ~10^{20} *ions/m*^{3} , the ion would travel ~10 km before colliding with another

Therefore these plasmas are effectively collisionless (a common theme)

The ITER tokamak has minor radius 2 m
How to confine particles?

The trick: use magnetic fields



Charged Particles Feel The (Lorentz) Force

- **A particle with charge (q) moving with velocity ($\mathbf{v}$) in the presence of electric and magnetic fields will experience a force:**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

We know from Newton's second law of motion that force causes acceleration:

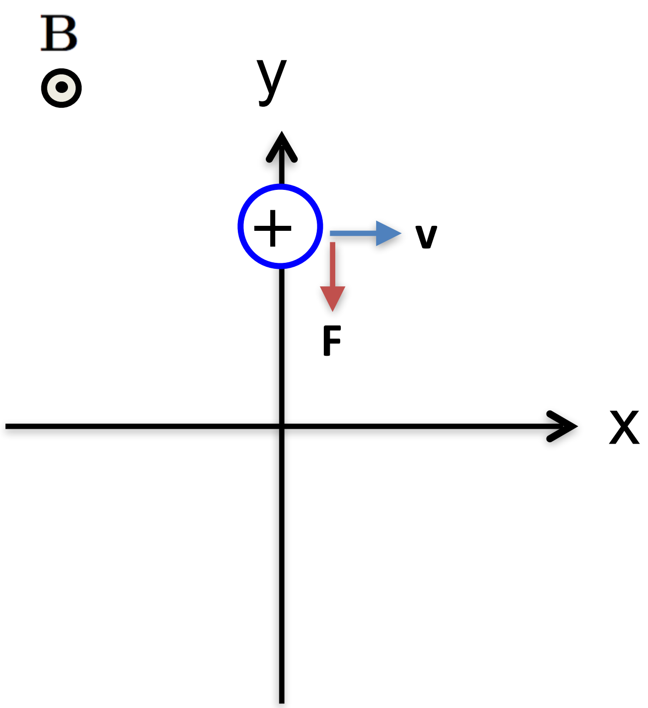
$$\mathbf{F} = m\mathbf{a}$$

A charged particle moving perpendicular to the magnetic field feels a force

How Does a Charged Particle Move in a Magnetic Field?

- **Consider the motion of a particle in a constant, uniform B field**

$$\mathbf{B} = B_z\hat{\mathbf{z}} \qquad \mathbf{E} = \mathbf{0}$$



Then $$\mathbf{F} = q(\cancel{\mathbf{E}} + \mathbf{v} \times \mathbf{B})$$

So we can write

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

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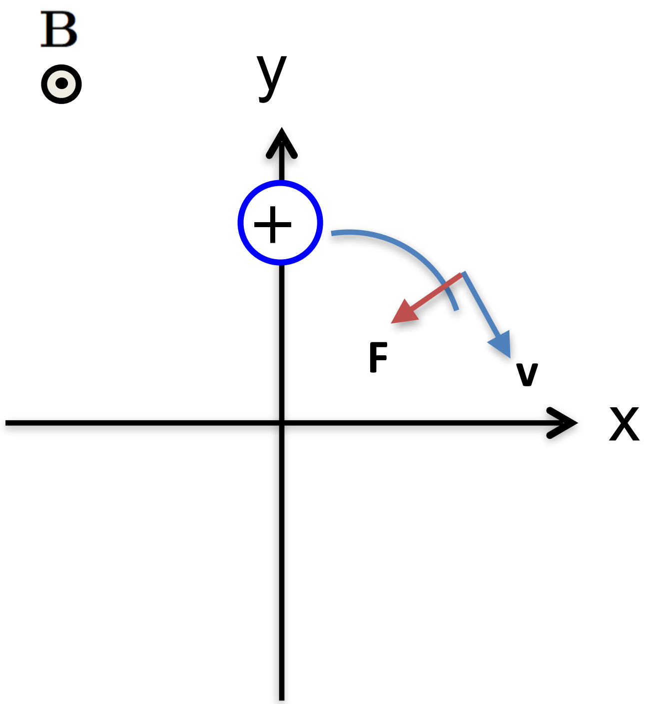

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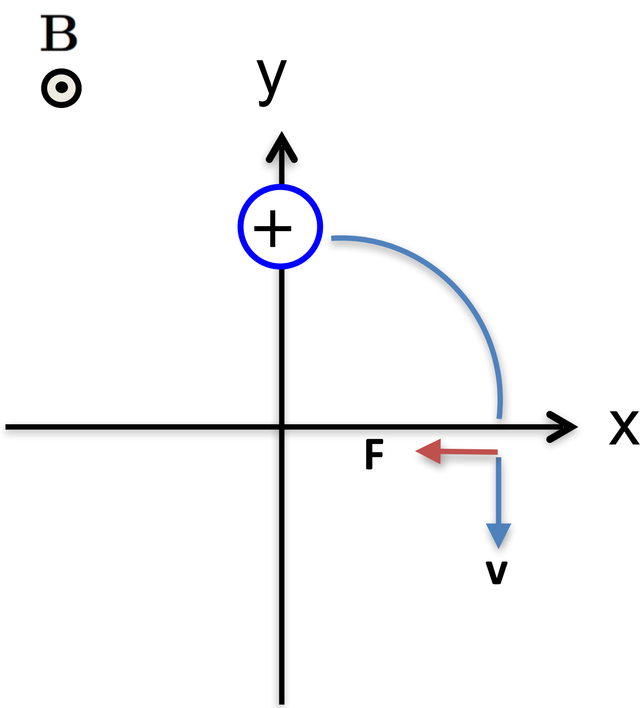

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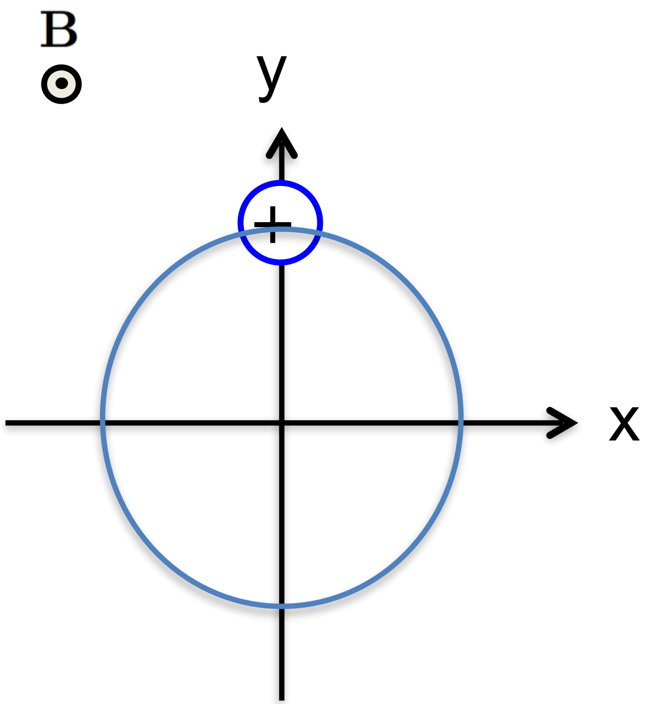


one “gyration” or “gyro-orbit” completed

Then $$\mathbf{F} = q(\cancel{\mathbf{E}} + \mathbf{v} \times \mathbf{B})$$

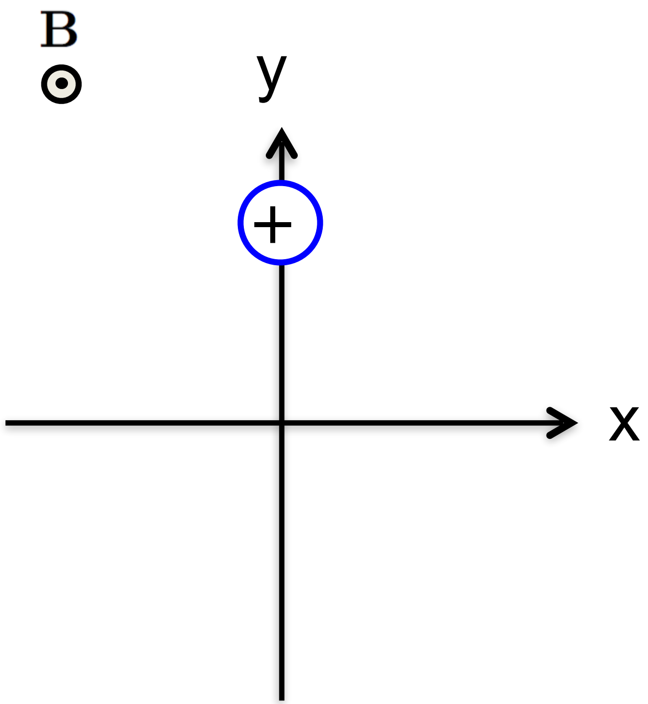
So we can write

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

Now let’s do the algebra

Goal: Solve the Equations of Motion for a Charged Particle In A Magnetic Field

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$



Let's break this into components:

$$\dot{v}_x = \frac{qv_yB_z}{m} \qquad \dot{v}_y = -\frac{qv_xB_z}{m}$$

$$\dot{v}_z = 0$$

Particles move freely along the field line

The 'dot' notation represents $\frac{d}{dt}$

Take Another Time Derivative & Substitute to Obtain Differential Equations For Each Spatial Coordinate

$$\dot{v}_x = \frac{q v_y B_z}{m} \qquad \dot{v}_y = -\frac{q v_x B_z}{m}$$

$$\ddot{v}_x = \frac{q \dot{v}_y B_z}{m} \qquad \ddot{v}_y = -\frac{q \dot{v}_x B_z}{m}$$

Rewriting, we get

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y$$

These may remind you of the equations for a simple harmonic oscillator

Homework: go through the details on your own

Solve the Differential Equations

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y$$

These differential equations can be solved using sines and cosines:

$$v_x = v_\perp \cos\left(\frac{|q|B_z}{m}t + \phi_0\right) \qquad v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right)$$

account for positive or negative q

$$v_\perp = \sqrt{(v_x^2 + v_y^2)}$$

the magnitude of the initial velocity perpendicular to $\boldsymbol{B}$

an arbitrary phase to match the initial velocity conditions

Homework: go through the details on your own

The Result: Circular Motion About A Guiding Center

$$v_x = v_\perp \cos\left(\frac{|q|B_z}{m}t + \phi_0\right) \qquad v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right)$$

Integrating, we obtain

$$x = \frac{mv_\perp}{|q|B_z}\sin\left(\frac{|q|B_z}{m}t + \phi_0\right) + x_0 \qquad y = \pm\frac{mv_\perp}{|q|B_z}\cos\left(\frac{|q|B_z}{m}t + \phi_0\right) + y_0$$

- **Charged particles undergo circular orbits about a guiding center** (x_0, y_0)

$$r_L \equiv \frac{mv_\perp}{|q|B}$$

“Larmor” radius or “gyro-radius”

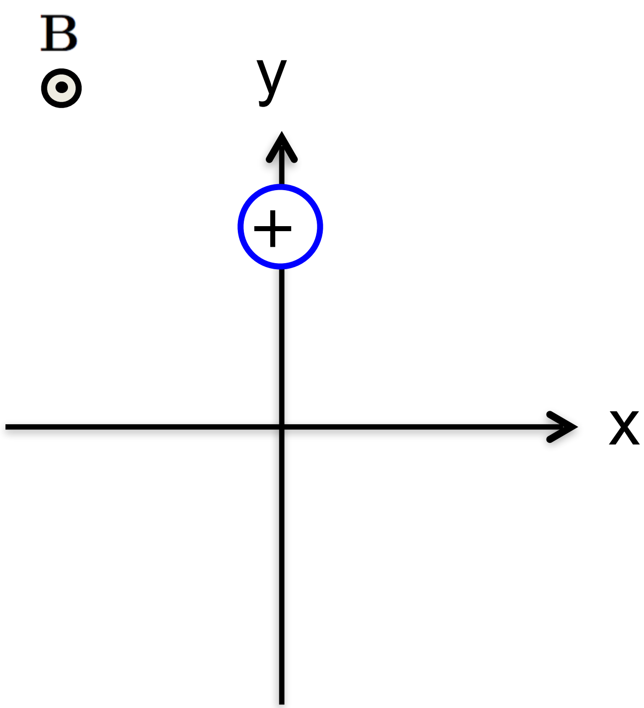
$$\omega_c \equiv \frac{|q|B}{m}$$

Cyclotron or gyro-frequency

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin\left(\omega_c t + \phi_0\right) + x_0 \qquad y = \pm r_L \cos\left(\omega_c t + \phi_0\right) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a positively charged particle:

1. At $t = 0$,

$$x = 0 \qquad y = r_L$$

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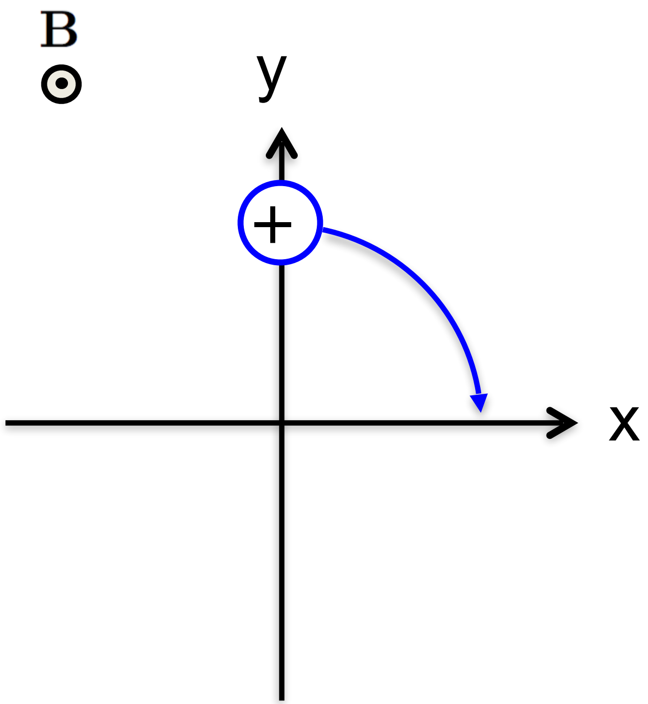

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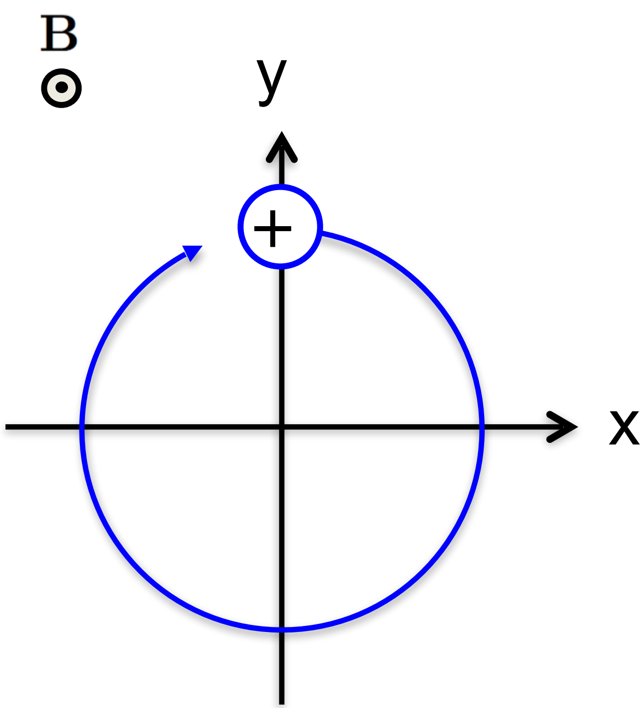

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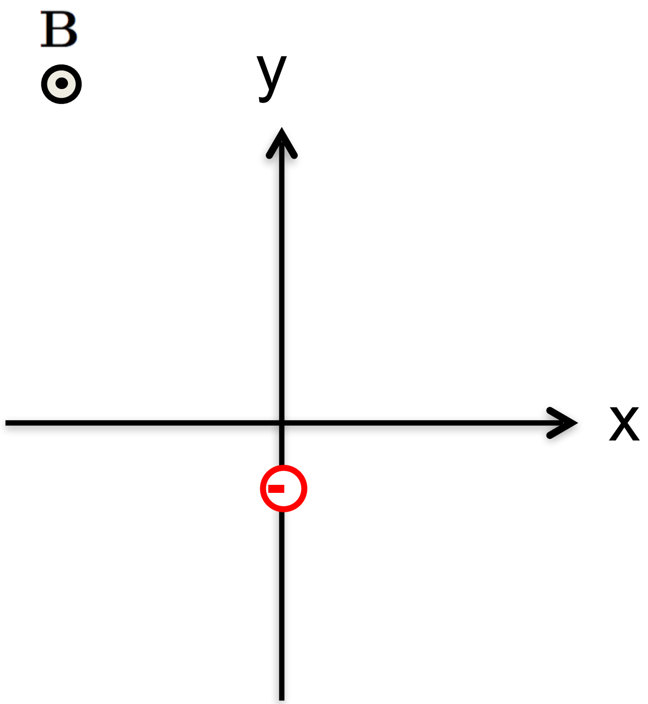

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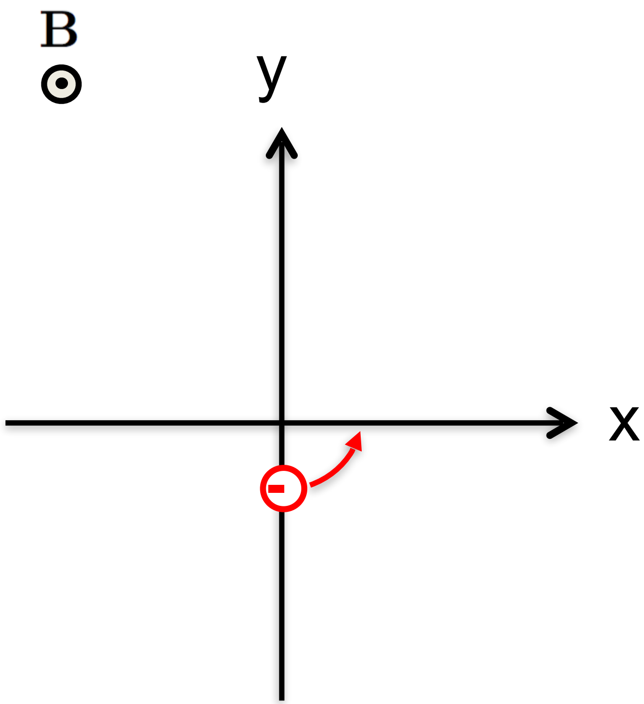

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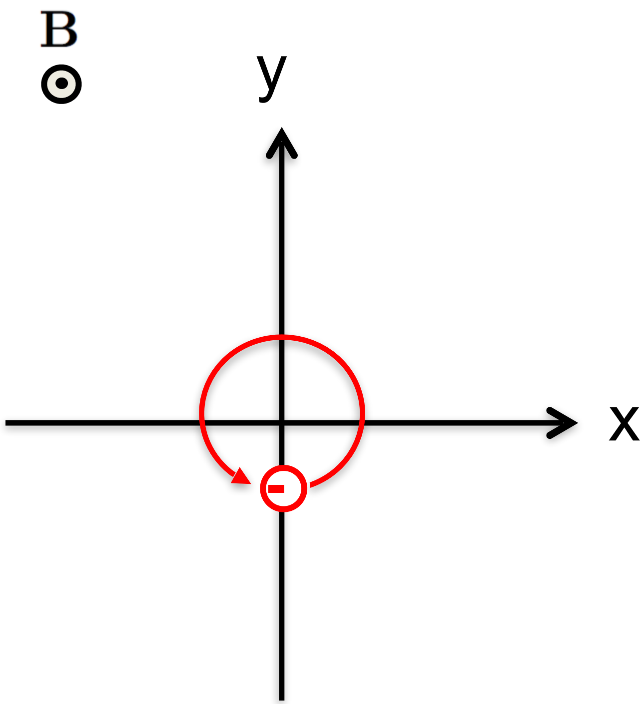

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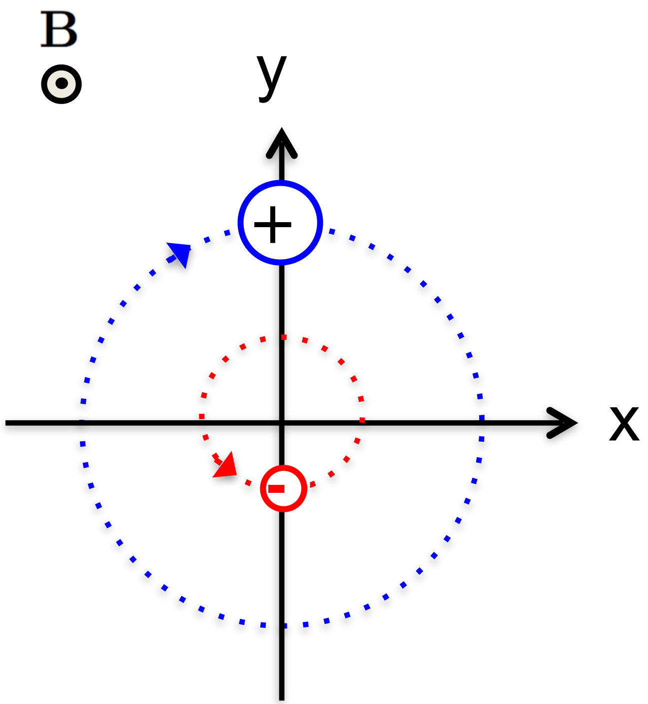
Gyromotion of Ions vs. Electrons

- **The direction of gyromotion depends on the sign of the charge**
- **Ions generally have a much larger Larmor radius than electrons**



In ITER, for a typical deuterium ion with T_i=10 keV and B=5 Tesla would have

$$v_{Ti} = \sqrt{\frac{kT_i}{m_i}} \approx 700 \text{ km/s}$$

$$r_L \equiv \frac{mv_\perp}{|q|B} \approx 3 \text{ mm}$$

An electron with T_e=10 keV and B=5 Tesla has

$$r_L \approx 0.05 \text{ mm}$$ (60 times smaller)

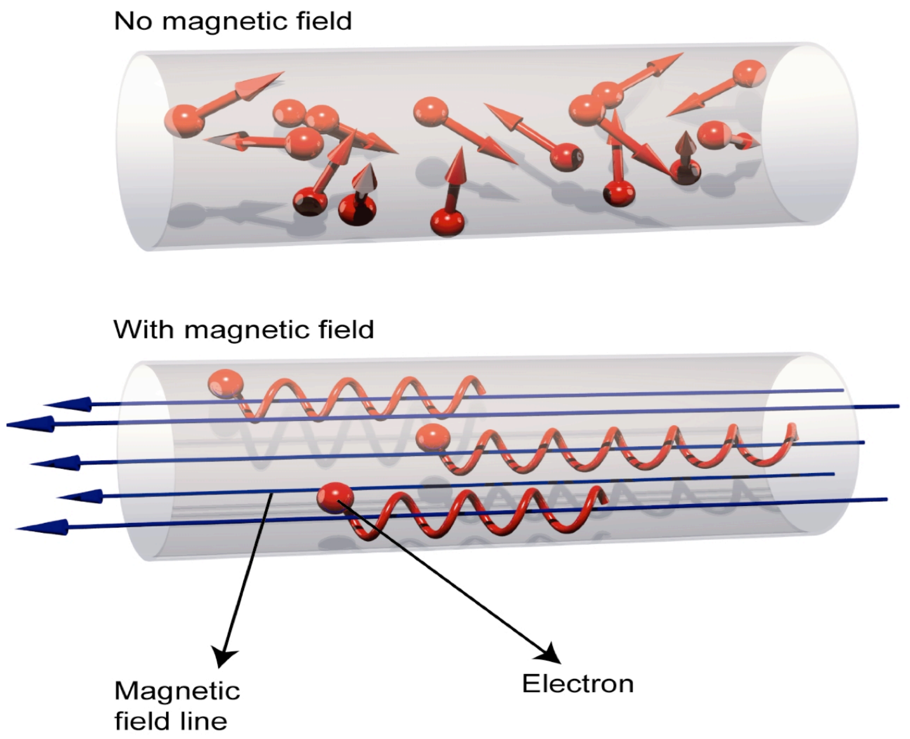
Particle motion and the "guiding center"

The gyration strongly constrains the motion of the particle perpendicular to the B field.

So the particle motion is intuitively decomposed to a sum of (1) gyration plus (2) "drift" of a "guiding center"

The guiding center is what is left when you average over the gyration

The guiding center can have:
(1) fairly unconstrained parallel motion
(2) slower "drifts" in the perpendicular direction



http://iter.rma.ac.be/en/img/MagneticConfinement.jpg

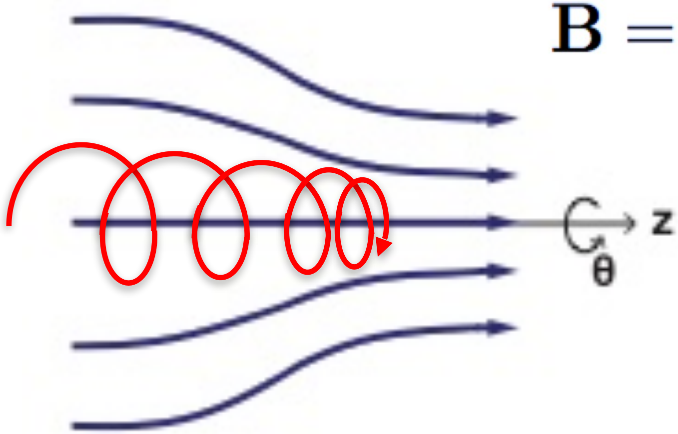
Fusion devices should be much larger than the gyro-radii.
Homework: Is it true for ITER?

Magnetic structure of the solar corona

Solar coronal plasma
Te = 10 MK (100 eV)
B = 100 G

Which way is the magnetic field running?
Is the gradient stronger along or across the field?

Some guiding center dynamics: Magnetic Mirrors



$$\mathbf{B} = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}} \qquad \mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

The B_r ends up causing additional acceleration in the z direction:

$$m\frac{dv_z}{dt} = -qv_\theta B_r$$

Result:
gyromotion + mirror force in the $-\hat{\mathbf{z}}$ direction

$$F_z = -\frac{mv_\perp^2}{2B}\frac{\partial B_z}{\partial z}$$

The magnetic moment is $\mu \equiv \dfrac{mv_\perp^2}{2B}$

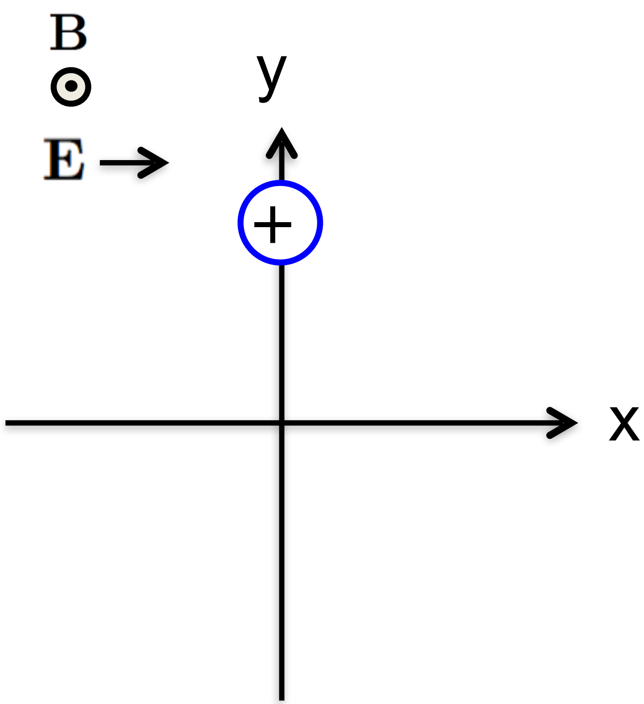
mirror force $\mathbf{F}_{||} = -\mu\nabla_{||}B$

μ is an "adiabatic invariant" - a deeply important concept for magnetized plasmas

Next Simplest Case to Analyze: Drift under Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{E} = E_x \hat{\mathbf{x}} \qquad \mathbf{B} = B_z \hat{\mathbf{z}}$$



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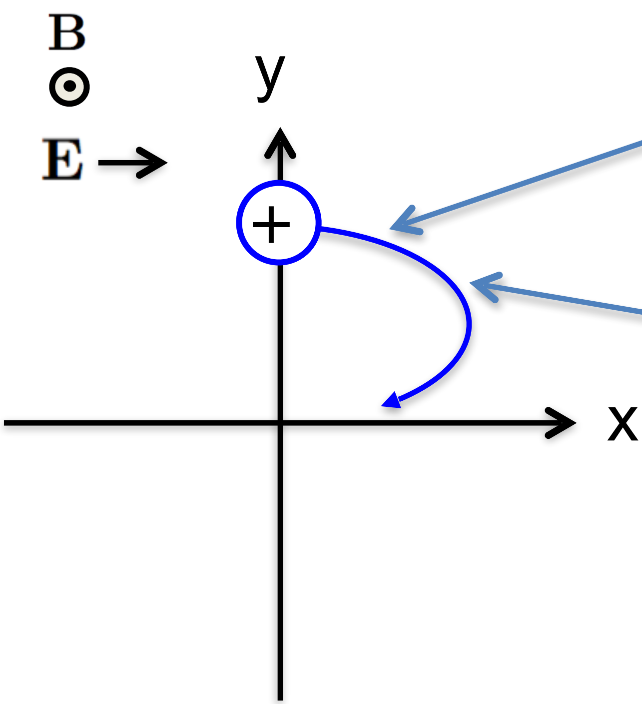

Accelerates due to E

Faster velocity increases vxB

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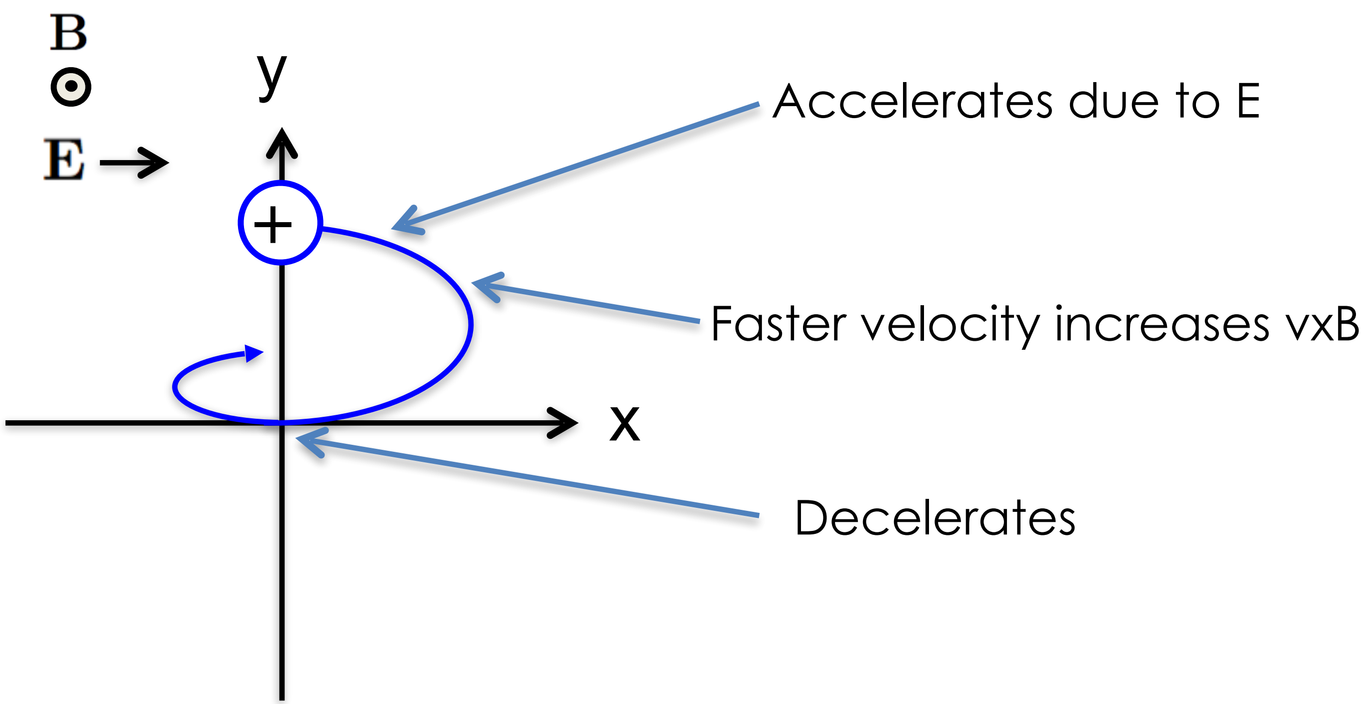

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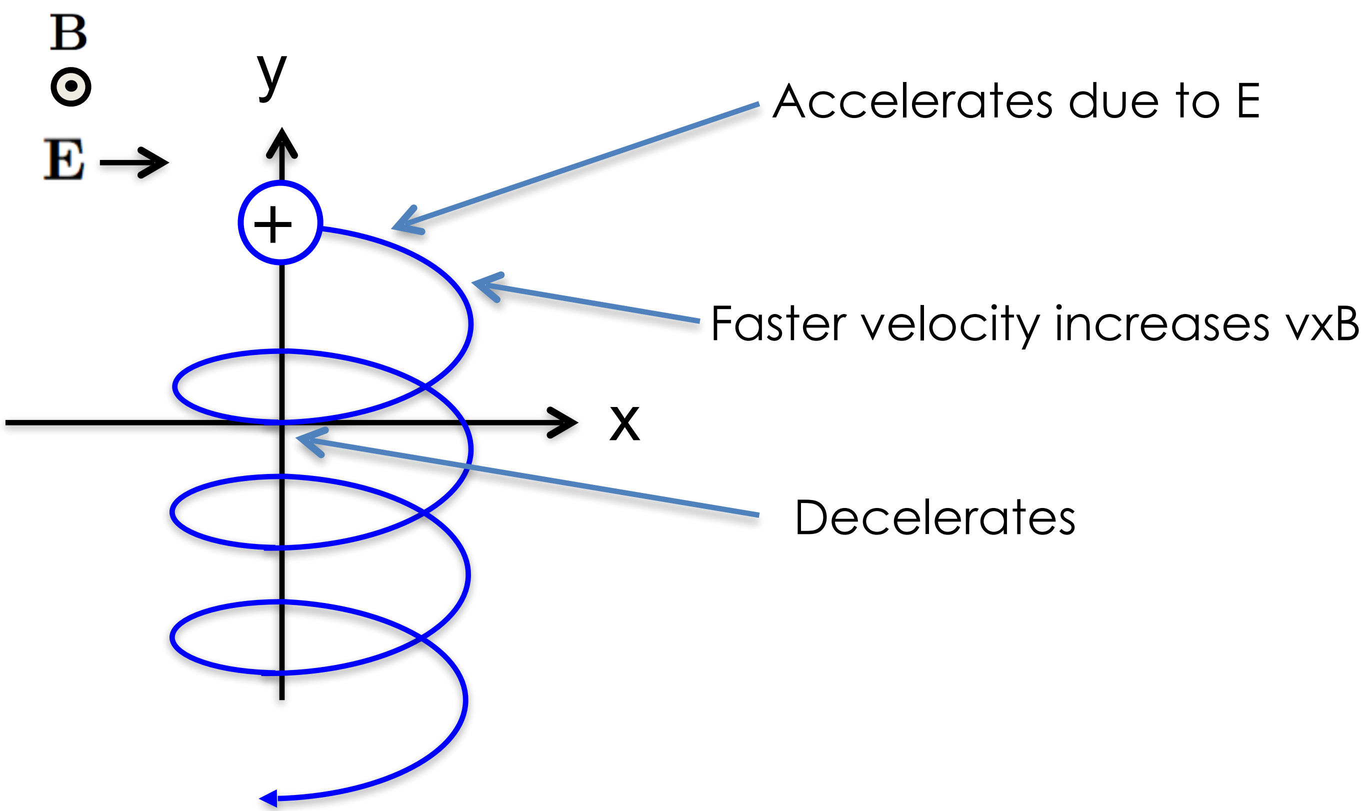

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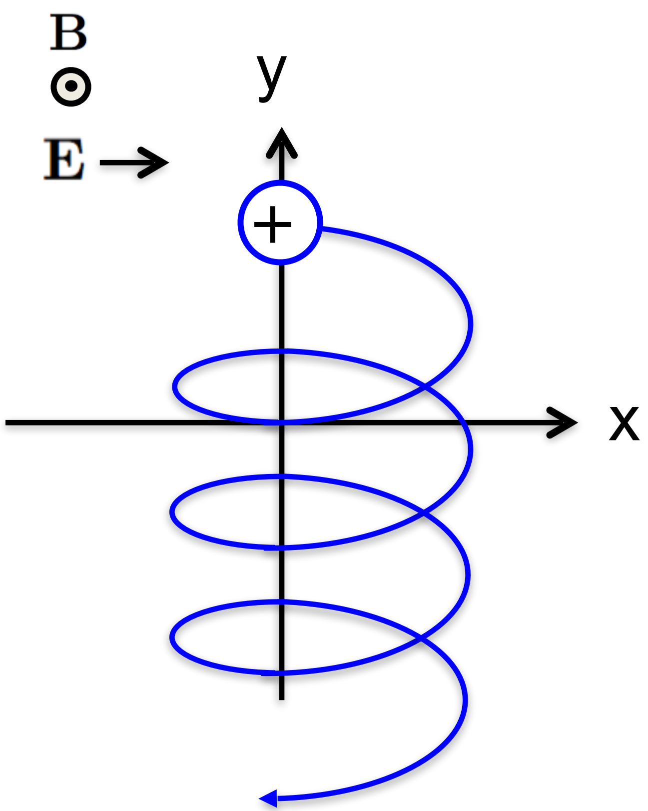


$$\dot{v}_x = \frac{q}{m}(v_y B_z + E_x)$$

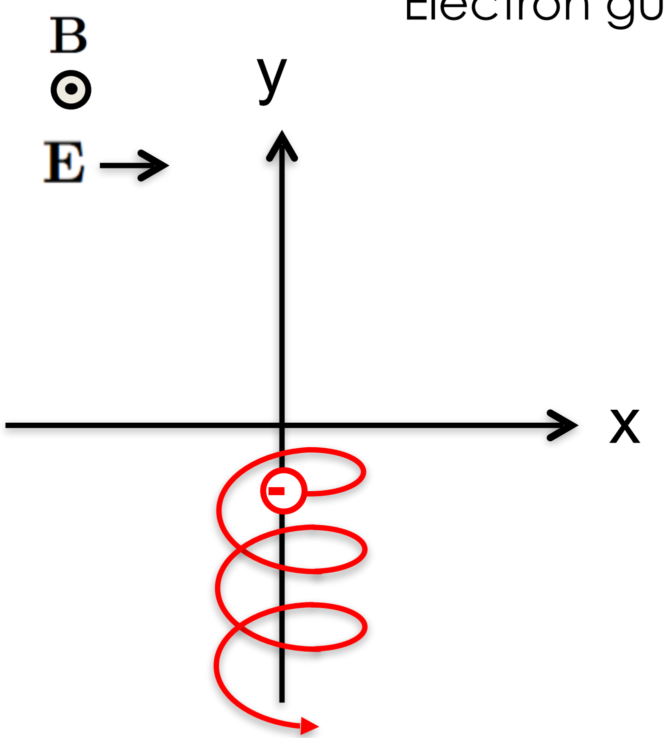
$$v_y = \mp v_\perp \sin\left(\frac{|q| B_z}{m} t + \phi_0\right) - \frac{E_x}{B_z}$$

Ion guiding center drifts in the direction $-\hat{\mathbf{y}}$

Guiding Center Drift Due to E x B

$$v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right) - \frac{E_x}{B_z}$$

Electron guiding center also drifts in the direction $-\hat{\mathbf{y}}$



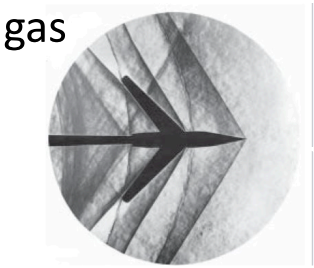
The ExB drift can be written more generally as

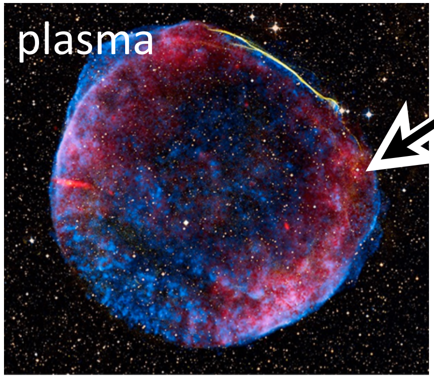
$$\mathbf{v_E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- **ExB drift is independent of charge and mass**
- **Both electrons and ions move together**

Preview for MHD lecture: Magnetic field also moves with plasma, via Faraday's law. Plasma and field can be thought of being "frozen together"

Digression: How magnetic fields mediate collisionless shocks in astrophysical plasmas





SNR1006

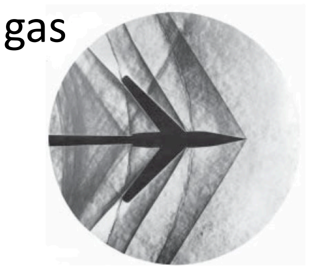
Collisionless SNR shocks shown to be the sites of cosmic ray acceleration. [Ackerman Science 2013]

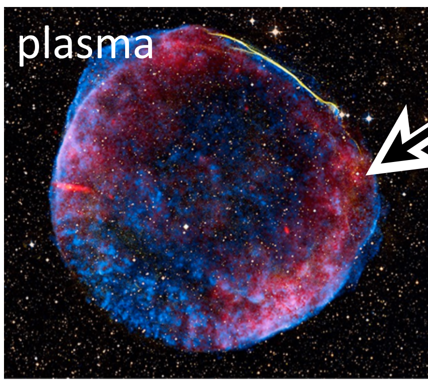
Shocks occur where supersonic flows interact.
Shocks convert kinetic energy to heat

In gas, inter-particle collisions mediate the shock

In plasma, *collisionless* shocks have shock width << mean-free-path.

Magnetic fields mediate collisionless shocks in astrophysical plasmas





SNR1006

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Shocks convert of kinetic energy to heat

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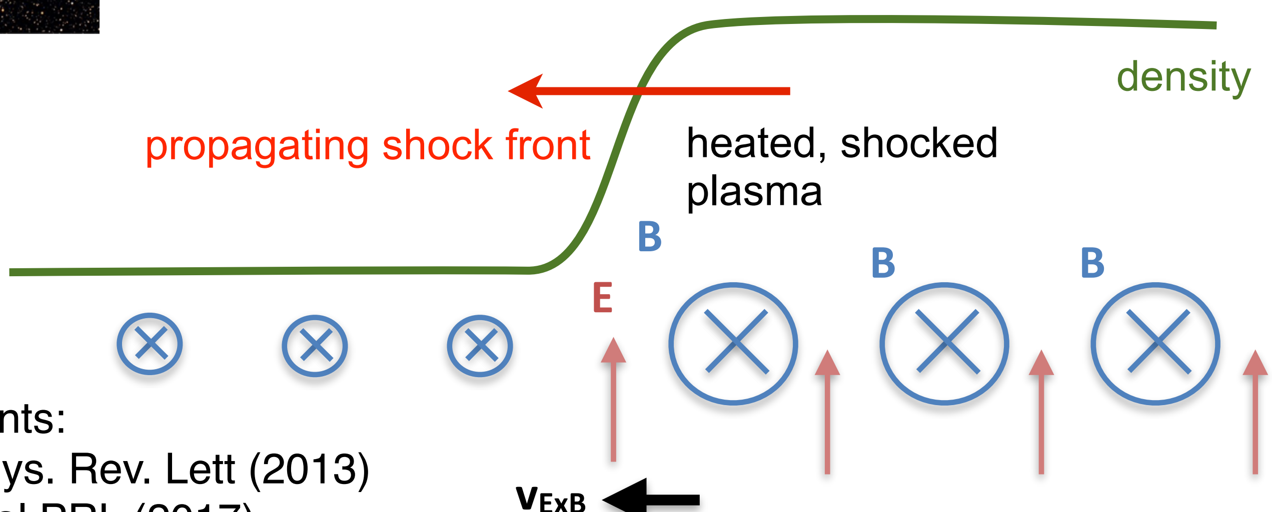
In plasma, *collisionless* shocks have shock width << mean-free-path.

Collective electromagnetic fields mediate shock in collisionless plasmas

Possible mechanisms:

- Pile-up of pre-existing field ("magnetized shocks")
- **Self-generation** of a turbulent magnetic field near shock by Weibel instability *



See recent experiments:
WF, G.Fiksel et al Phys. Rev. Lett (2013)
D. Schaeffer, WF, et al PRL (2017)

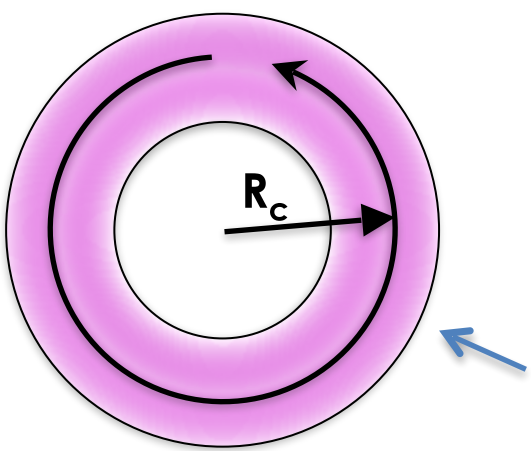
Other Forces Can Also Cause Guiding Center Drift

- **Any force perpendicular to B can cause particles to drift**

Drift due to force: $$\mathbf{v_d} = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}$$

Examples of forces: $$\mathbf{F_g} = m\mathbf{g}$$ gravity

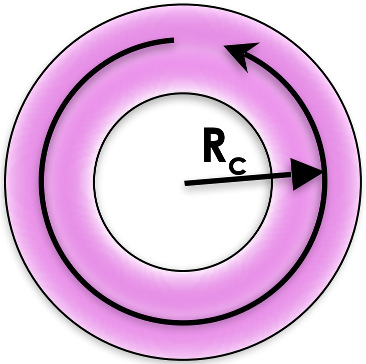
$$\mathbf{F_{cf}} = \frac{mv_{\parallel}^2}{R_c}\hat{\mathbf{r}}$$ centrifugal



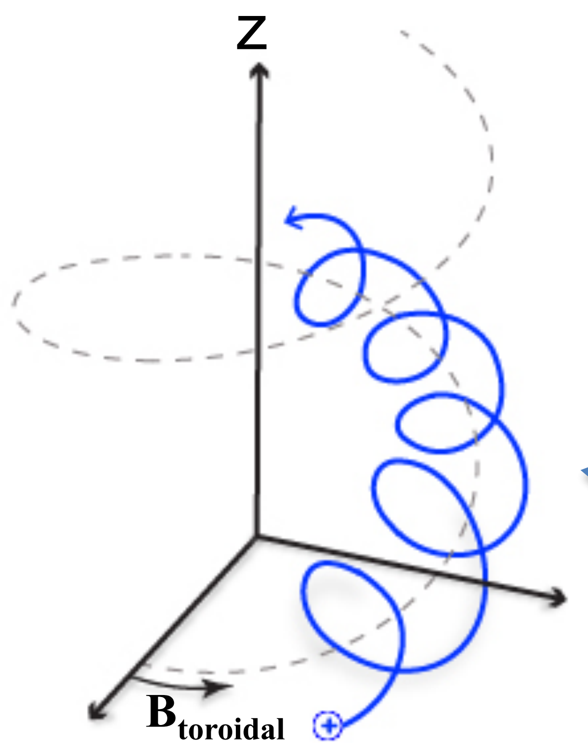
- Bend the magnetic field into a donut shape
- No end losses because the field lines go around and close on themselves
- BUT a particle following a toroidal magnetic field would experience $\mathbf{F_{cf}}$

Curvature Drift Due to Bending Field Lines

The outward centrifugal force causes curvature drift



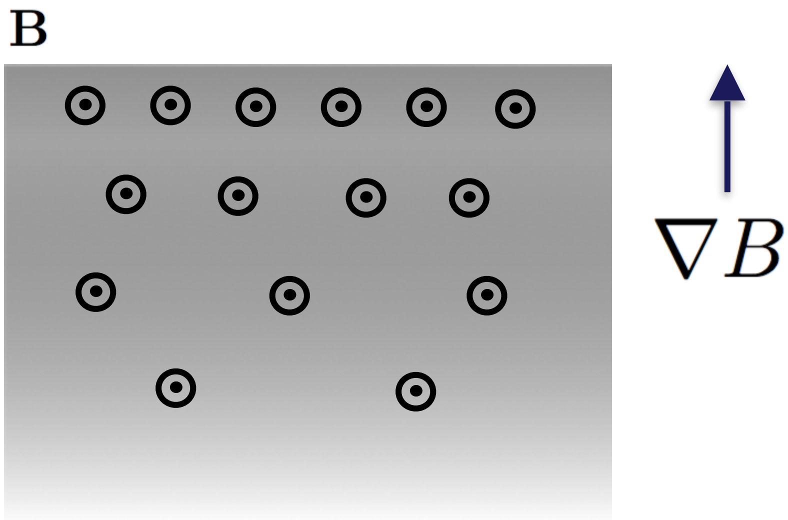
$$\mathbf{F_{cf}} = \frac{mv_\parallel^2}{R_c}\hat{\mathbf{r}}$$

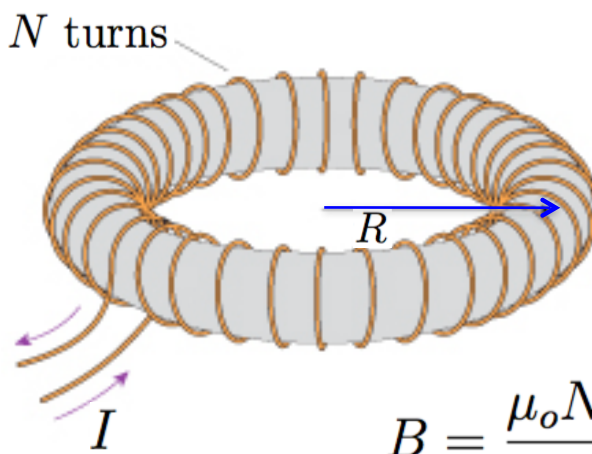


$$\mathbf{V_R} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2}$$

A particle moving along a curved field line will drift up or down, depending on the sign of the charge

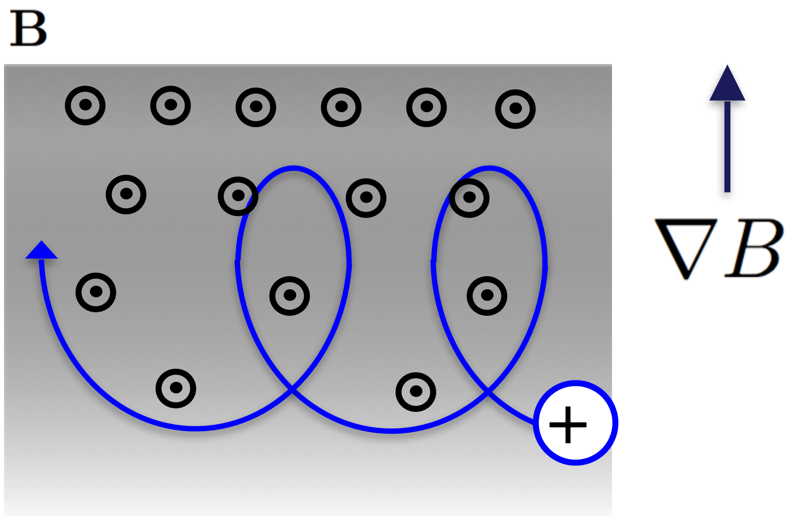
Spatially Varying Magnetic Field Strength Also Causes Drift

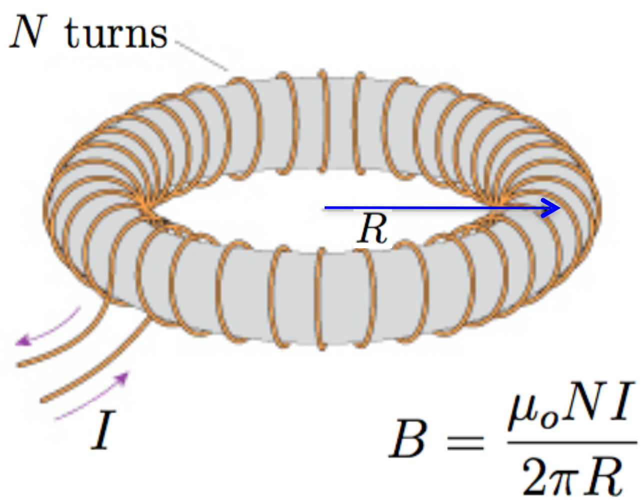




$$B = \frac{\mu_o N I}{2\pi R}$$

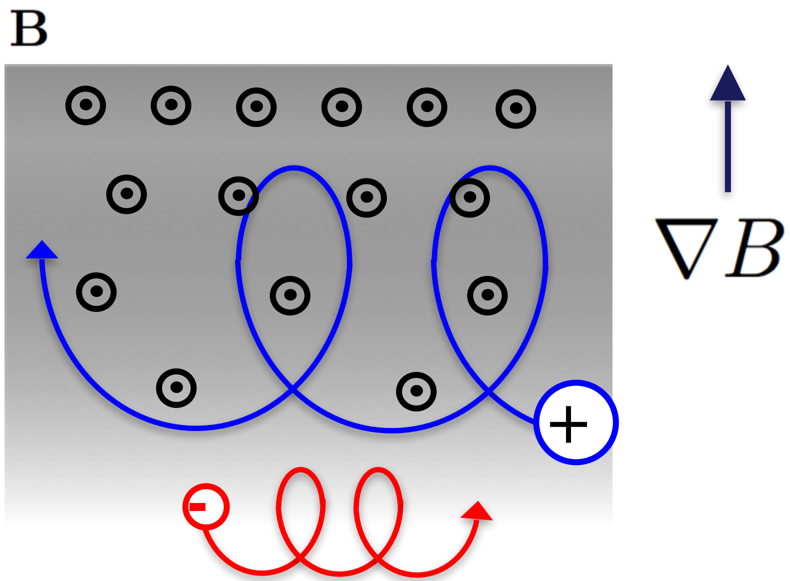
Spatially Varying Magnetic Field Strength Also Causes Drift

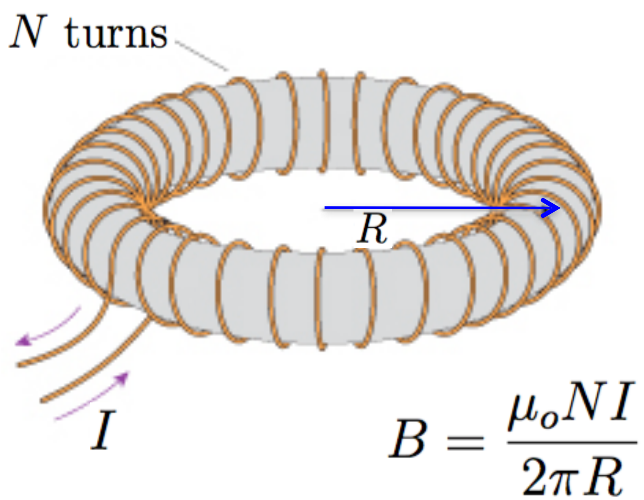




- **The gyro-radius will be larger where the field is weaker and smaller where the field is stronger**

Spatially Varying Magnetic Field Strength Also Causes Drift





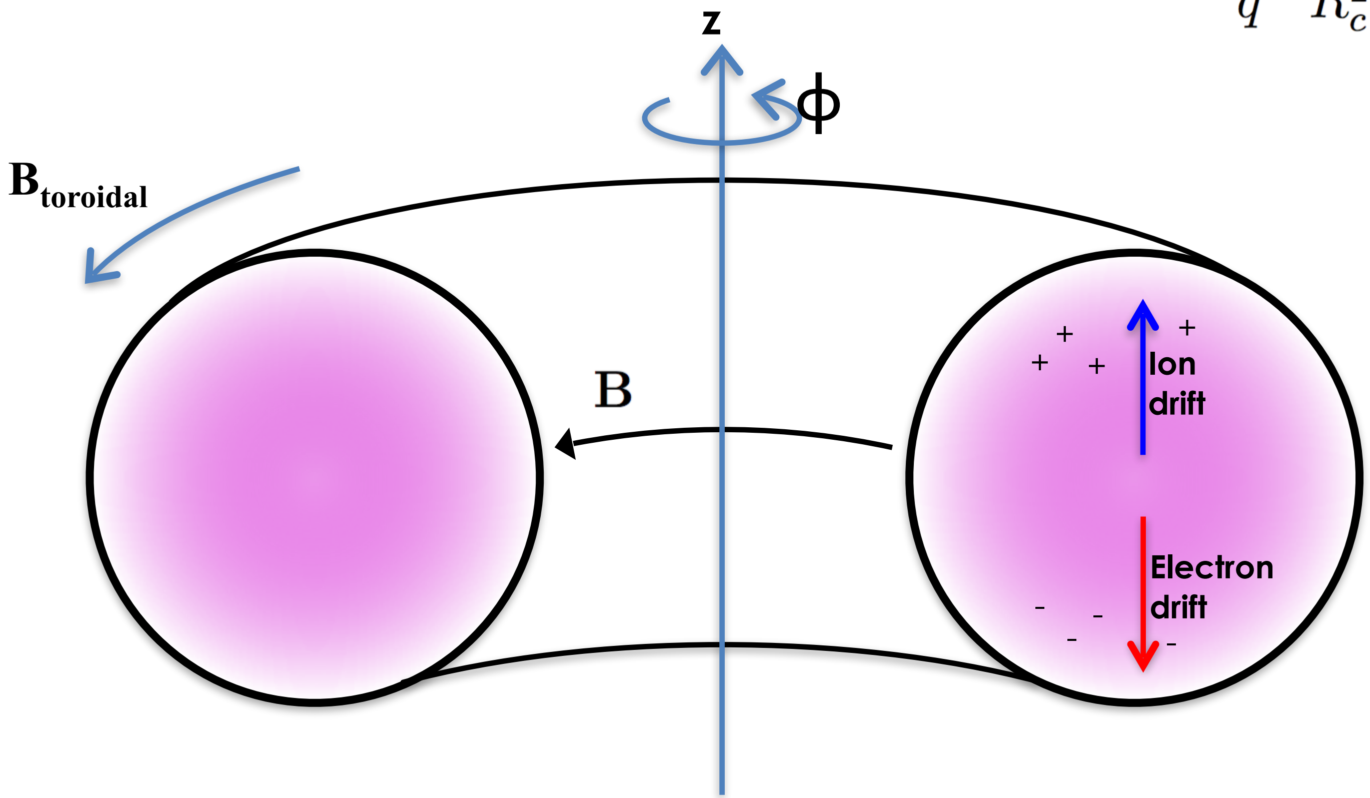
- The gyro-radius will be larger where the field is weaker and smaller where the field is stronger
- The resulting drift velocity is described by:

$$\mathbf{V}_{\nabla \mathbf{B}} = \frac{m v_\perp^2}{2qB} \frac{\mathbf{B} \times \nabla B}{B^2}$$

Application: What Happens To Charged Particles In A Purely Toroidal Magnetic Field?

- **Charged particles in a curved magnetic field will experience both ∇B and curvature drift: these effects add**

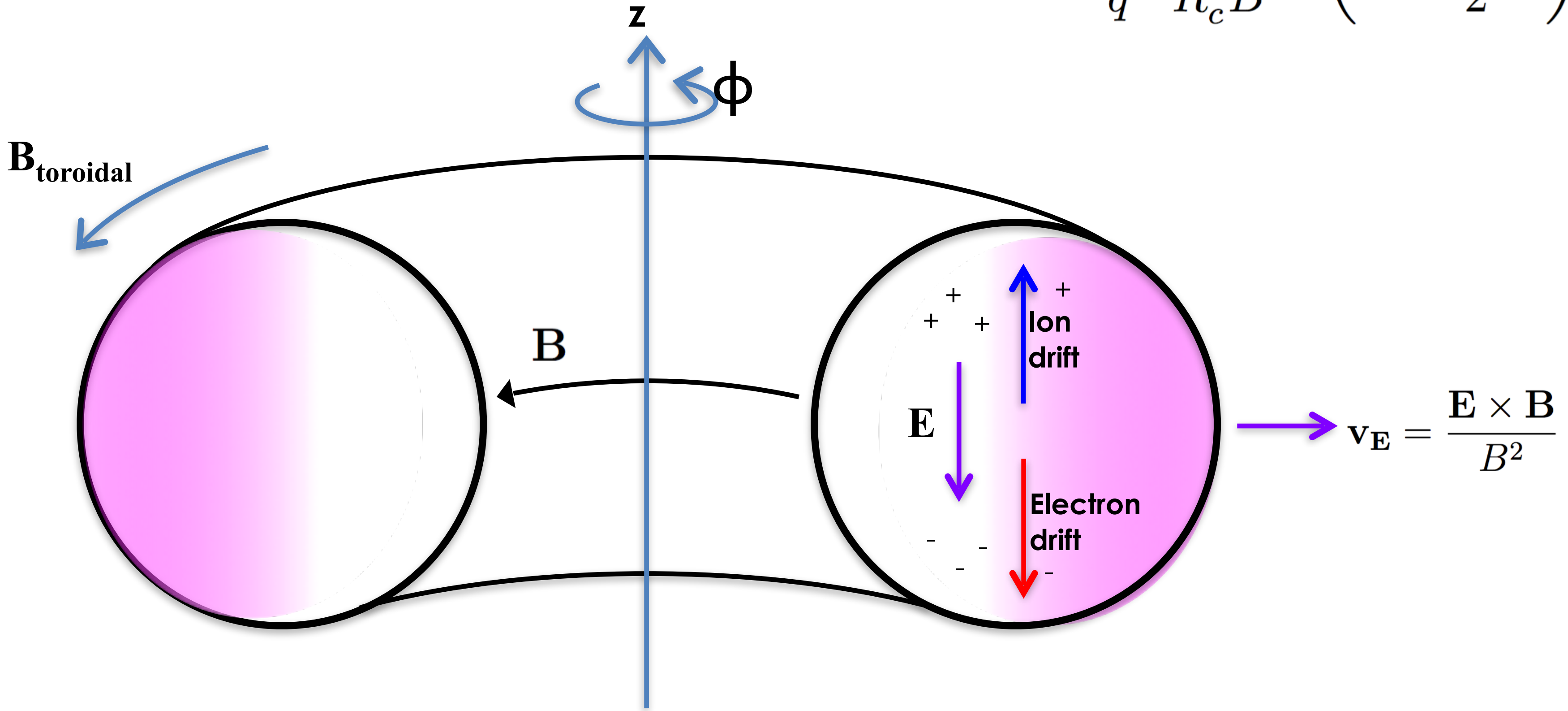
$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$



Charged Particles Will Drift Outward

- Charged particles in a curved magnetic field will experience both ∇B and curvature drift

$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$

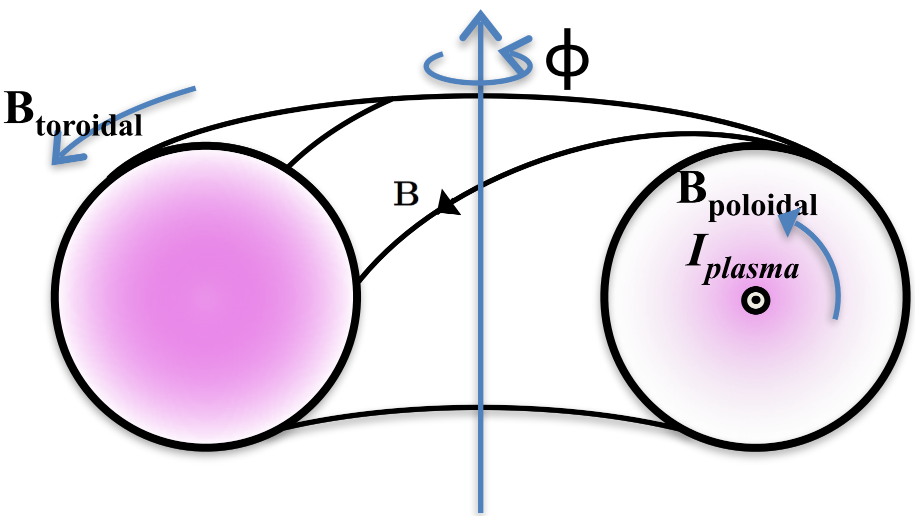


- This means that no matter what, particles in a torus with a purely toroidal field will drift radially out and hit the walls.

Solutions: Add twist to the field with *poloidal* fields

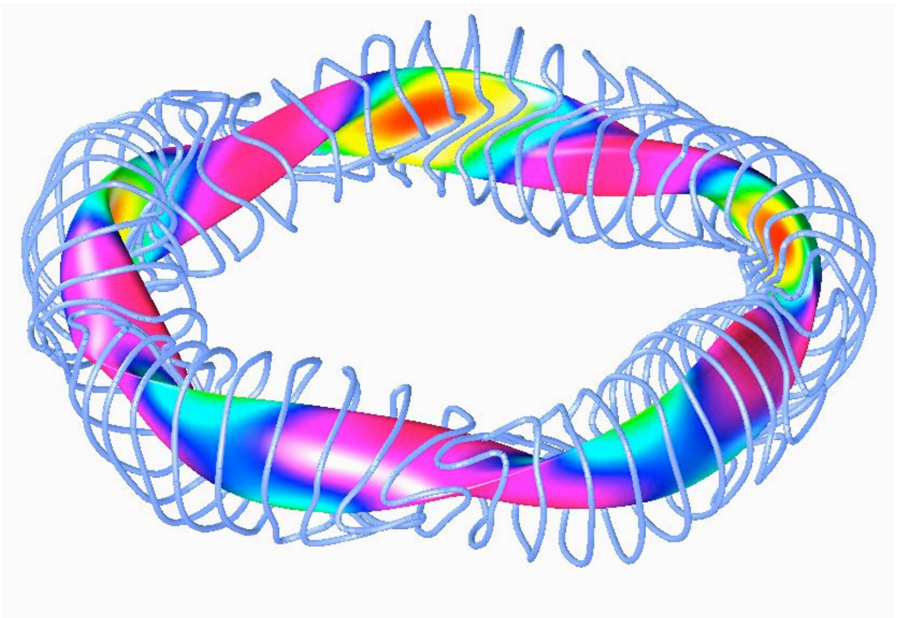
Toroidal: long way around
Poloidal: short way around

Tokamak:



1. Use external coils to apply a toroidal magnetic field
2. Drive **current** in the plasma to generate **poloidal magnetic field**

Stellarator:

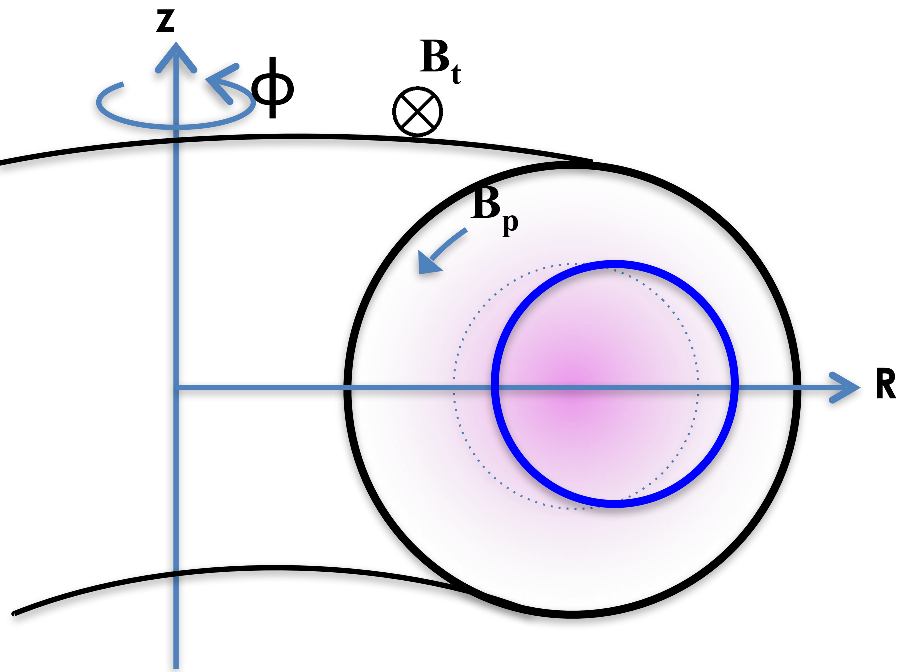


Twist is applied by **3-D shaping fields.**

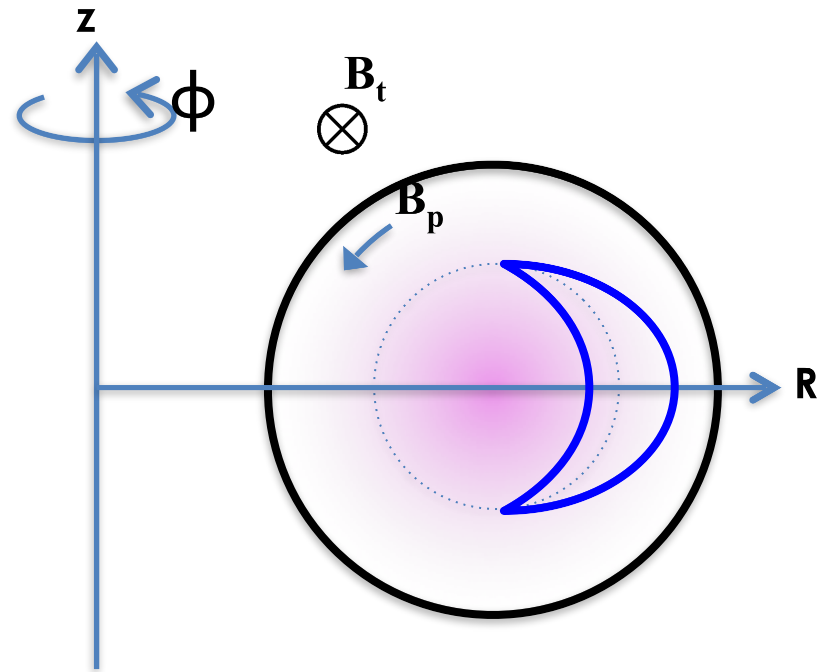
Will be returned to in lectures by D. Battaglia, F. Laggner, and A. Bader!

There Are Two Main Classes of Particle Orbits In Tokamaks

$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c}\times\mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$



Passing

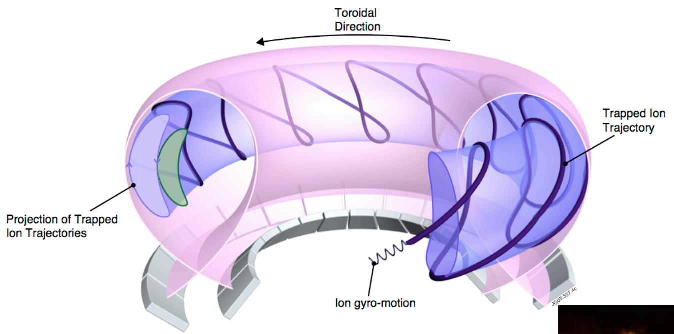


Trapped

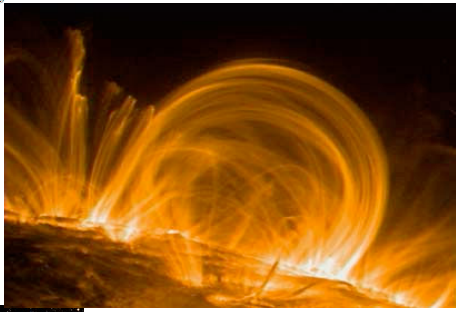
Particles with sufficient $\mathbf{v}_{||}$ will follow the helical magnetic field around the torus

Particles with lower $\mathbf{v}_{||}$ are reflected as they encounter stronger B and therefore execute "banana" orbits as they precess around the torus B

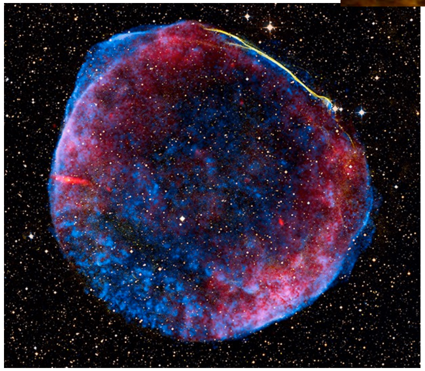
Conclusions - single particle and gyro motion in plasmas



- Magnetic fields needed to *confine* hot collisionless fusion plasmas.
- Guiding center undergoes ExB, gradB and curvature drifts



- Cross-field confinement structures space, solar, and astrophysical plasmas, too



- Magnetic fields mediate collective behavior of plasmas, via drifts

- Work hard and soak it in. Have a good summer!