Heuristic derivations of basic plasma parameters

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1 Introductory Remarks

This lecture is intended to be a brief introduction to what we consider to be the principal "must know" characteristics of plasma. It is, in no way, intended to be a comprehensive discussion of the topic. For more advanced introductions to plasma physics, there are several good resources: eg. Introduction to Plasma Physics (F. Chen), Plasma Physics (R, Goldston, P. Rutherford). There are also free online lecture notes of Intro to Plasma courses: R. Fitzpatrick at UT Austin and R. Parker at MIT (linkable on the pdf of this document).

2 Plasma Characteristics

When gas becomes ionized it becomes a plasma. Typically, what we consider to be a plasma is actually not fully ionized. In many cases, only a small fraction of the gas is ionized. These are called (not surprisingly) weakly ionized plasmas, as opposed to fully ionized plasmas (deep in the sun or inside a magnetically confined fusion device). The degree of ionization is determined by the Saha Equation:

$$\frac{n_i}{n_n} \approx 2.4 \times 10^{21} \frac{T^{3/2}}{n_i} e^{-U_i/k_B T} \quad (1)$$

Where n_i and n_n are the density of the ions and the neutrals in $[m^{-3}]$, T is the gas temperature in Kelvin, k_B is Boltzmann's constant and U_i is the ionization energy, that is, the energy required to remove the outermost electron. As a comparison, at standard temperature and pressure, nitrogen has a degree of ionization of:

$$\frac{n_i}{n_n} \approx 10^{-122}. \quad (2)$$

As the temperature starts rising to the order of U_i (that is, to around a few thousands degrees K), the ionization becomes non-negligible and the gas becomes a plasma.

3 Review of basic mechanics equations

Disregarding magnetic forces, the basic equation of motion of a given particle of mass m_1 and electric charge q_1 when it comes a distance $r_{1,2}$ to another charged particle of mass m_2 and charge q_2 is given by the equation:

$$m_1\vec{a} = \Sigma\vec{F} = \vec{F}_G + \vec{F}_E = \left[-\frac{Gm_1m_2}{r_{1,2}^2} + \frac{q_1q_2}{4\pi\epsilon_0 r_{1,2}^2}\right]\hat{r} \tag{3}$$

where $\vec{F}_G$ is the gravitational attraction (hence the minus sign) and $\vec{F}_E$ is the electrical force. G and ϵ_0 are the gravitational constant and the permittivity of free space respectively. Assuming particle 1 is an electron and particle 2 is a Deuterium isotope, then the ratio between the forces is:

$$\frac{F_E}{F_G} = 1.1 \times 10^{39}, \tag{4}$$

therefore, for laboratory plasmas, gravitational forces can be disregarded and we can focus only on electric and magnetic forces, otherwise called the *Lorentz Force*.

For a particle of mass m and charge q moving with a velocity $\vec{v}$ through an electric and magnetic field of magnitudes $\vec{E}$ and $\vec{B}$ respectively, the equation of motion of the particle is:

$$\vec{F} = m\vec{a} = q\left[\vec{E} + \vec{v} \times \vec{B}\right] \tag{5}$$

This is the equation we will use when analyzing the mechanics of individual particles in the plasma.

4 Plasma thought experiment

Let's begin with a simple picture of a rectangular box of plasma which, as quasi-neutrality dictates, is composed of electrons and positive ions, as shown in Figure 1.

4.1 Plasma Frequency

Now suppose we are to move the center of mass of the electrons to the left (or negative direction in our $\hat{x}$ axis) a distance Δx. There is now an accumulation of electrons on the left and an accumulation of ions on the right. An electric field is therefore created which points away from the positive slab and towards the negative slab. In fact, if we imagine the distance between the positive and negative slabs to be very small compared to the area of the slabs, then the boundary conditions are too far from our points of interest and we can view this as an ideal parallel plate capacitor.

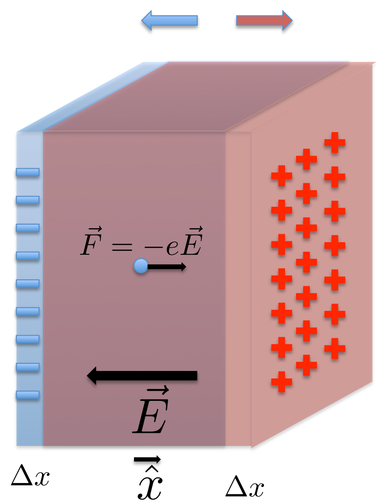

Figure 1: Moving the center of mass of the electrons with respect to the ions creates a restoring force

The electric field inside an ideal parallel plate capacitor is simply:

$$\vec{E} = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0} = \frac{(en_e A \Delta x)/A}{\epsilon_0} = \frac{en_e \Delta x}{\epsilon_0} \tag{6}$$

pointing in the negative direction, where σ is the surface charge density (charge per unit area) of the plate, or slab in this case, Q is the total charge of the slab and A is its area. Note that the electric field is uniform between the slabs and it does not depend on their area, only on their thickness and number density. The most common way of finding the electric field in a capacitor is done using Gauss' Law: $\nabla \cdot \vec{E} = \rho/\epsilon_0$, where, in our case, $\rho = en_e$ is the volume charge density. We won't go into detail here, but this is a very beautiful derivation which uses the symmetry of the system.
Now, if we have an electron in the middle of the box feeling the electric field, the force on this electron (which, as with all of the electrons in the slab, has been shifted in the $-\hat{x}$ direction), is:

$$\vec{F}_e = m_e \vec{a} = -e\vec{E} = \frac{e^2 n_e (-\vec{\Delta x})}{\epsilon_0} \rightarrow \vec{a} = -\frac{e^2 n_e}{m_e \epsilon_0} \vec{\Delta x} \tag{7}$$

Where I have incorporated the direction of the shift in to the Δx vector. But Equation 7 is simply that of a harmonic oscillator with frequency:

$$\omega_{pe} \equiv \sqrt{\frac{e^2 n_e}{m_e \epsilon_0}} \tag{8}$$

Not surprisingly this is the electron plasma frequency of the system. Analogously, for an ion of charge Ze and mass m_i, the ion plasma frequency can be defined as: $\omega_{pi} = \sqrt{(Z^2e^2n_i)/(m_i\epsilon_0)}$. Let's look at it in a little more detail: If you look at the thought experiment, for the same displacement, the total charge in each slab will increase as you increase the electron number density n_e, hence the force is stronger and our oscillation is faster. Also, for the same field, the acceleration on electrons is greater than that of ions because the same force ($e\vec{E}$) is excerpted on such disparate masses. This explains the inverse relation on mass.

4.2 Thermal Velocity

Now, let's forget about the thought experiment for a second and think about the individual moving particles in our system. As energy is given to the plasma (through external voltages, neutral particle bombardment, microwave heating, etc.), the particles will start accelerating and colliding with each other. The particles will all gain random velocities which, as can be derived from statistical mechanical arguments, follows the Maxwell-Boltzmann distribution function:

$$f_v(v_x, v_y, v_z, x, y, z) = n(x, y, z)\left(\frac{m}{2\pi kT}\right)^{3/2} e^{\left[-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}\right]} \tag{9}$$

where $f_v(v_x, v_y, v_z, x, y, z)$ is the probability density that a particle is located in the vicinity (in 6 dimension) of the velocity $\vec{v} = [v_x, v_y, v_z]$ and the location $\vec{r} = [x, y, z]$, $k = 1.38 \times 10^{-23} JK^{-1}$ is the Boltzmann constant, T is the temperature and m is the particle mass. The probability distribution is normalized such that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_v(v_x, v_y, v_z, x, y, z)dv_x dv_y dv_z = n(x, y, z)$$

What Equation 9 says is that the velocity of the particles are distributed in a Gaussian or bell curve with a width proportional to the temperature of the gas. Or, put it another way, the width of distribution in velocity space is what gives rise to the concept of temperature of a gas.

If we want to study the *speed* distribution of the particles, $v \equiv |v| \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$, we find the more typical form of the the Maxwell-Boltzmann distribution function:

$$f_v(v) = 4\pi n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{\left[-\frac{mv^2}{2kT}\right]} \tag{10}$$

an example of which is shown in Figure 2 (note that the density has been taken as homogeneous, n_0, to simplify the analysis). As is clear, the temperature of the gas is related to the width of the distribution as well as to the average speed $v_{mean} = \sqrt{3kT/m}$, as well as the most probable speed (the peak of the curve) $v_{peak} = \sqrt{2kT/m}$. There is, therefore, a characteristic speed of the particles which we call the *thermal speed* defined as:

$$v_t \equiv \sqrt{\frac{kT}{m}} \tag{11}$$

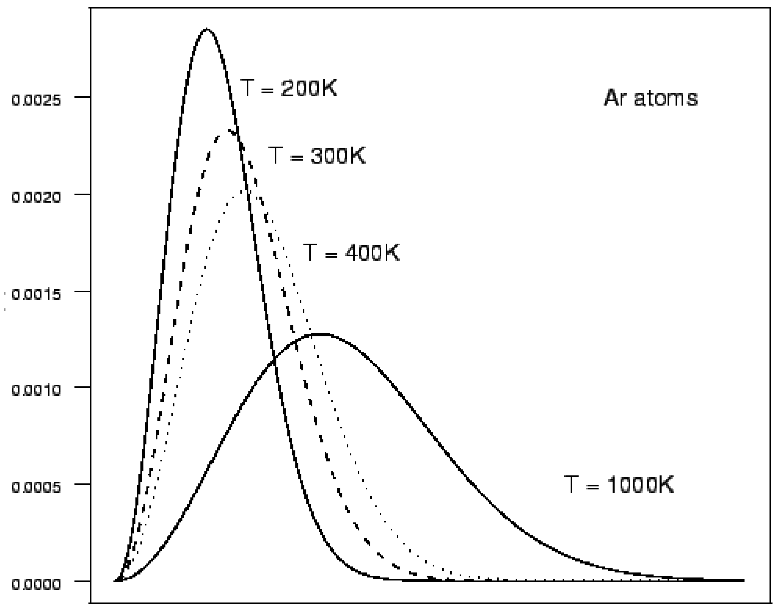

Figure 2: Maxwell-Boltzmann distribution function of Argon gas at different temperatures. The $\hat{x}$-axis is proportional to the *speed* of the atoms. Note that the area under the curve must be 1.

Now, the electrons and ions will have their own thermal speeds given by: $v_{te} = \sqrt{kT_e/m_e}$ and $v_{ti} = \sqrt{kT_i/m_i}$. How do these speeds typically compare?
Let's say an electron and an ion are getting energy from an electric field (which is most often the case) for a given amount of time t. The momentum gained by both particles is the same: $m_e v_e = m_i v_i = eEt$. so $v_e/v_i = m_i/m_e \gg 1$. This disparity translates to v_{te} and v_{ti} hence, if energy transfer between species is low, $T_e/T_i \propto m_i/m_e \gg 1$. Note that even if the particles have enough time to reach thermal equilibrium, which is more common in magnetically confined plasmas, $T_e = T_i$ still leads to $v_{te} \gg v_{ti}$. Hence, we always have much more thermal (faster) electrons than ions.

4.3 Debye Lengths

Finally, we can go back to our thought experiment with the added knowledge of the plasma thermal speeds. The first, and easiest way, of arriving at the Debye length is to do a dimensional analysis of what we've already acquired. We've found a characteristic *[time]* in the ω_p, and we've found a characteristic speed, or *[length]/[time]* in v_t. Therefore, we can immediately deduce a characteristic length called the *Debye length*:

$$\lambda_D \equiv \frac{v_t}{\omega_p} = \frac{\sqrt{\frac{kT}{m}}}{\sqrt{\frac{q^2 n}{m\epsilon_0}}} = \sqrt{\frac{kT\epsilon_0}{q^2 n}} \tag{12}$$

More, specifically, the Debye length is defined as:

$$\lambda_{De} = \sqrt{\frac{kT_{eff}\epsilon_0}{e^2 n_e}} \tag{13}$$

where $T_{eff}^{-1} = T_e^{-1} + T_i^{-1}$ (derived from a more detailed analysis of Poisson's Equation). Nonetheless, when we can take the ions as stationary (particularly in weakly ionized cold plasmas), the Debye length is effectively taken as the electron Debye length:

$$\lambda_D = \sqrt{\frac{kT_e\epsilon_0}{e^2 n_e}}. \tag{14}$$

To get a more intuitive picture of what the Debye length is related to, we can go back to the thought experiment where we now have a picture of an electron that is subjected to a simple harmonic oscillator system. If we were to follow the motion of the electron in this simple picture, it would follow a harmonic motion of the form:

$$x = A\cos(\omega_{pe} t) \tag{15}$$

where I have disregarded any phase and I still haven't determined it's amplitude. Now how can we determine the amplitude A of oscillation? If we take the time derivative of Equation 15, we can find the velocity of the electron:

$$v = -A\omega_{pe}\sin(\omega_{pe} t) \tag{16}$$

But we know that the speed of the electrons is around v_{the} (of course, this is a characteristic speed), so we can use that as the constraint and we have

$$A\omega_{pe} = v_{te} \rightarrow A = v_{te}/\omega_{pe} \equiv \lambda_{De}. \tag{17}$$

Where we have recovered the result found from dimensional analysis.
So, what is the meaning of the Debye Length? One way of looking at it is this: The plasma wants to shield itself off from inhomogeneous charge buildup, therefore, as our thought experiment showed, the plasma will quickly reassemble to cancel the fields. Now, if the electric field comes from an externally applied source, e.g. a positive electrode in the plasma, the electrons will quickly try to shield it, but since they are moving so fast (v_{te}) and in all directions, there is a region close to the electrode where the electrons will escape (due to their own inertia) and not completely shield it. This region, where the electric fields are not completely shielded, is called the *sheath* and its length is of the order of the Debye length.

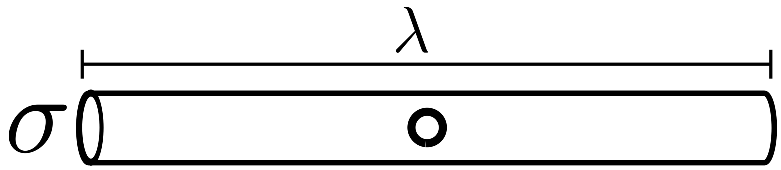

Figure 3: The volume associated with an individual electron is given by $\Delta V = 1/n_e$.

4.4 Collisional frequency

The final parameter to consider in an unmagnetized plasma is the frequency of collisions. I'll focus on electron-electron collisional frequency (noted hereafter as ν_e) but most of the dependencies and arguments are identical in ion-ion and electron-ion collisions.

The first step is to define two quantities: the time between collisions which is the inverse of the collisional frequency, $\Delta t = 1/\nu_e$ and the mean free path, λ_{mfp}. As we've seen before, the electrons are traveling at a characteristic speed of v_{the}, therefore, we can define:

$$\lambda_{mfp} = v_{the}\Delta t = v_{the}/\nu_e \tag{18}$$

as the distance traveled by an electron (on average) before it collides with another electron.

Every electron has an associated volume in the system $\Delta V = 1/n_e$, that is, ΔV is the volume that is occupied by each electron. Therefore, we can define the electron-electron cross section, σ_{ee}, with the equation $\Delta V = \sigma_{ee}\lambda_{mfp}$, as shown in Figure 3. Therefore, using Equations 18 and 11:

$$\nu_e = n_e v_{the} \sigma_{ee}. \tag{19}$$

An estimate of the cross section can be made by observing that an electron-electron collision is really a Coulomb repulsion interaction. The distance of closest approach, b, between colliding electrons can be taken as the radius of the cross section, that is:

$$\sigma_{ee} \approx \pi b^2. \tag{20}$$

While the actual distance of closest approach between two colliding electrons depends on relative velocities and angles of approach (and a rigorous derivation would take all possible angles and velocities into account and weight them according to the distribution function), we can make a heuristic case and assume a typical configuration of an electron approaching a stationary electron head on with a speed of v_{the} as in Figure 4. As shown in the figure, in the center of mass frame, each electron is approaching the other with speeds $v_{the}/2$ and at closest approach, they are separated by b. From conservation of energy, assuming that

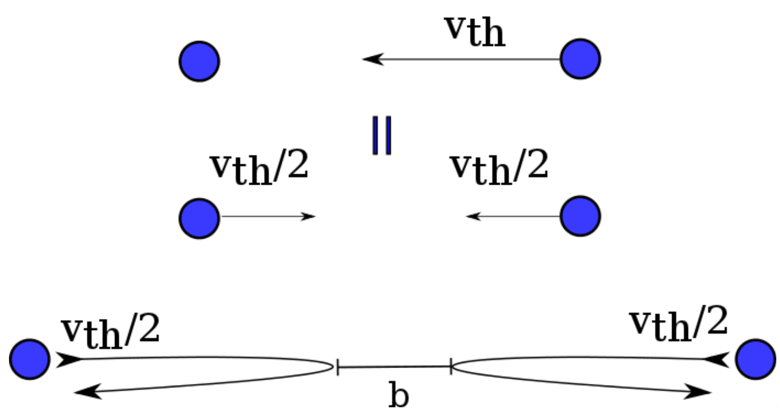

Figure 4: An electron with speed v_{the} colliding with a stationary one can be viewed in the center of mass frame and the distance of closest approach can be derived from conservation of energy.

the electrons are very far from each other at the beginning:

$$\frac{m_e(v_{the}/2)^2}{2} + \frac{m_e(v_{the}/2)^2}{2} = \frac{e^2}{4\pi\epsilon_0 b} \tag{21}$$

$$b = \frac{e^2}{\pi\epsilon_0 m_e v_{the}^2} = \frac{e^2}{\pi\epsilon_0 T_e}. \tag{22}$$

Not surprisingly, the larger the temperature, the closer the electrons can approach. Putting Equations 22 together with 20 and 19, we get the final result:

$$\nu_e \propto n_e \left(\sqrt{T_e/m_e}\right)\left(\frac{e^2}{T_e}\right)^2 = \frac{n_e e^4}{m_e^{1/2} T_e^{3/2}} \tag{23}$$

since we know that many assumptions have been made, we've disregarded the constants.
Since the *resistivity*, η, is proportional to the collisional frequency, we get the very important result:

$$\eta \propto T_e^{-3/2} \tag{24}$$

that is, the plasma becomes a better conductor as the temperature goes up. This dependance is of great importance in astrophysical plasmas as well as in tokamak plasmas.
As a point of comparison, the resistivity in a metal is well known to increase with temperature (contrary to the case in plasmas).

4.5 Updated definition of a plasma

Now that we have discussed the important parameters of a non-magnetized plasma, let's update our definition of a plasma:

System	$n_e[m^{-3}]$	$T_e[eV]$	$\omega_{pe}[s^{-1}]$	$\lambda_D[m]$
Interstellar gas	10^6	1	10^5	10
Solar Wind	10^7	10	10^5	10
Van Allen belts	10^9	10^2	10^6	1
Ionosphere	10^{11}	10^{-1}	10^7	10^{-2}
Solar Corona	10^{13}	10^2	10^8	10^{-3}
Candle flame	10^{14}	10^{-1}	10^9	10^{-4}
Neon lights	10^{15}	1	10^9	10^{-4}
Gas Discharge	10^{18}	2	10^{11}	10^{-5}
Process Plasma	10^{18}	10^2	10^{11}	10^{-4}
Fusion Experiment	10^{19}	10^3	10^{11}	10^{-4}
Fusion Reactor	10^{20}	10^4	10^{12}	10^{-4}
Lightning	10^{24}	3	10^{14}	10^{-8}
Electrons in metal	10^{29}	10^{-2}	10^{16}	10^{-12}

Table 1: Plasma Frequency and Debye length for various systems

- $\omega \ll \omega_{pe}$: The system should respond quickly to changes (of order ω) that we impose on it.
- $L \gg \lambda_D$: The size of our system should be large enough so that the plasma can successfully shield electrostatic fields.

5 Plasma frequency and Debye length for various plasma systems

In Table 1 it's possible to view the wide range of density and temperature where plasma exists. The plasma frequency and Debye length has been calculated to give a sense of the characteristic parameters in the systems.

The *Electrons in metal* case leads to an interesting discussion, outlined in Feynman's Lectures on Physics VII 32-7 (linked in the pdf) where the reflection and transparency of metals to electromagnetic waves can be viewed through the lens of plasma.

Finally, the ω_{pe} of the ionosphere leads to distinct behavior between AM and FM radio waves. It explains the reflection of AM waves (where $\omega < \omega_{pe}$) and the penetration of FM waves (where $\omega > \omega_{pe}$).

6 Magnetized plasmas

Finally, we'll do a small introduction to what happens when we incorporate effects of magnetic fields on the plasma. As shown in Equation 5, the force of a

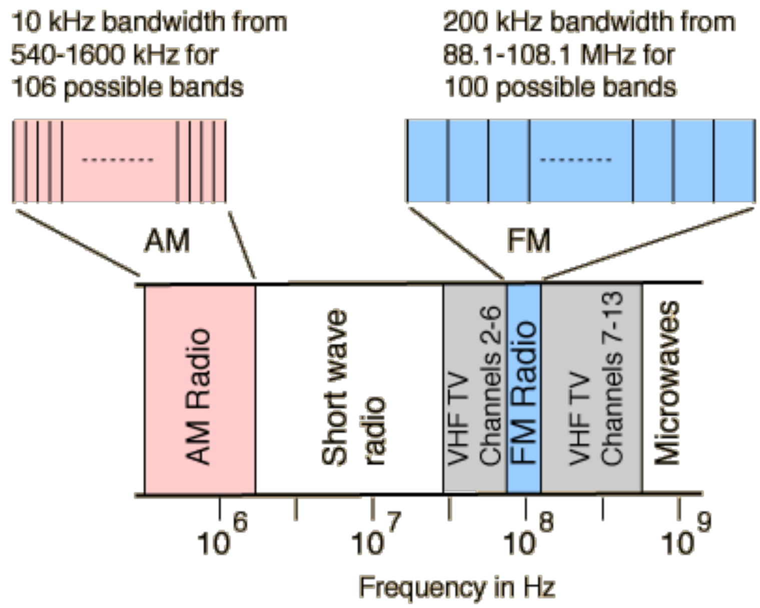

Figure 5: The AM spectrum is well below the $\approx 10MHz\ \omega_{pe}$ of the ionosphere, leading to their reflection. FM waves, at higher frequency, penetrate it.

particle which is moving in a magnetic field is of the form:

$$\vec{F} = m\vec{a} = q\vec{v} \times \vec{B} \rightarrow \vec{a} = \frac{q}{m}(\vec{v} \times \vec{B}) \tag{25}$$

Suppose a positively charged particle of mass m and charge q is moving in the plane of the paper with velocity $\vec{v}$ and there is a magnetic field $\vec{B}$ pointing into the paper. As shown in Figure 6, the force, hence the acceleration of the particle is always pointing towards a center of motion and the particle draws a circular orbit in the plane. From Equation 25, the magnitude of the acceleration is $a = qvB/m$. But we know from kinematics that if a particle is rotating around a fixed point, the acceleration must be centripetal and the magnitude should be:

$$a = \frac{v^2}{r} = \frac{qvB}{m} \rightarrow r = \frac{vm}{qB} \tag{26}$$

If the particle that is rotating is an electron (ion) with speed v_{te} (v_{ti}) then the radius of rotation is called the electron (ion) gyro-radius or Larmor radius and the equations are as follows:

$$\rho_e = \frac{m_e v_{te}}{eB}\ ,\ \rho_i = \frac{m_i v_{ti}}{ZeB} \tag{27}$$

Finally, we can figure out the frequency of rotation of an electron or ion that is rotating at thermal speeds: $v_t = \omega_c \rho$. These frequencies are very important in

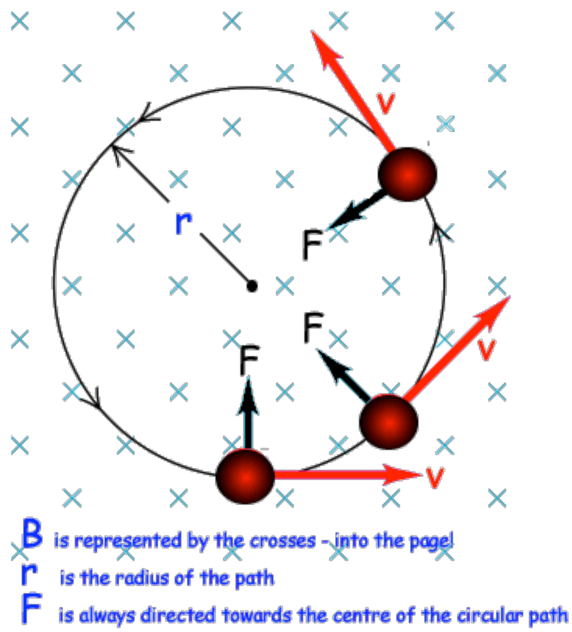

Figure 6: Trajectory of a positively charged particle moving with a velocity $\vec{v}$ where there is a magnetic field pointing into the page.

magnetized plasma physics and are called electron and ion gyro-frequencies (or cyclotron frequencies):

$$\omega_{ce} = \frac{v_{te}}{\rho_e} = \frac{v_{te}}{\frac{m_e v_{te}}{eB}} = \frac{eB}{m_e} \ , \ \omega_{ci} = \frac{ZeB}{m_i} \tag{28}$$

If the particles are not confined to the plane perpendicular to the magnetic fields but can move in three dimensions, the particles move freely in the direction parallel to the magnetic field but are confined to move in circular orbits perpendicular to the fields, therefore, they trace spiral orbits around the magnetic fields, as shown in Figure 7.

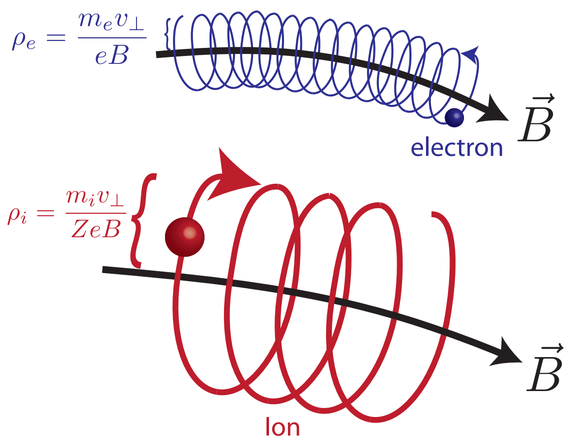

Figure 7: In three dimensions, particles follow spiral trajectories around magnetic fields.