SULI 2016 @ PPPL * These notes accompany the slideshow * M. Kunz

3 Applications of Plasma Physics to Astrophysical Systems (amongst many...)

① Star Formation: the "magnetic-flux problem" and the "angular-momentum problem"

(some names from the 1950's – 1990's who pioneered this work: Lyman Spitzer, Leon Mestel, George Field, Frank Shu, Telemachos Mouschovias) (+ one fundamental paper by Chandrasekhar & Fermi)

② Accretion Disks: How to transport angular momentum when molecular viscosity is negligible

(some names from the 1970's – 1990's who made fundamental contributions, which changed the field: Nikolai Shakura, Rashid Sunyaev, Donald Lynden-Bell, Jim Pringle, John Papaloizou, Doug Lin, Peter Goldreich, Jeremy Goodman, Steve Balbus, John Hawley, Jim Stone, Charles Gammie)

③ Galaxy Clusters: When is a stratified atmosphere convectively stable?

(Convective stability goes back to Karl Schwarzschild (1906), Vilho Väisälä (1925), and David Brunt (1927) — also Chandrasekhar and Joseph Boussinesq, and Lord Rayleigh and Lewis Fry Richardson.

We'll concentrate on modern improvements to this theory for weakly collisional plasmas due to Balbus, Quataert, Kunz, etc.)

That the names above are all theorists says nothing about the great contributions from observers who established these problems, but rather

says something about the predilection of these notes' author.
Mea culpa.

① (a) Magnetic-Flux Problem:

Let's make the Sun. Take a $1\,M_\odot$ blob of the interstellar medium (ISM), whose density is $\sim 1\ \mathrm{cm}^{-3}$ and magnetic-field strength is $\sim 1\ \mu G$. The density of the Sun is $\sim 10^{24}\ \mathrm{cm}^{-3}$. If the magnetic flux were conserved during spherical contraction ($\Phi_B = B\pi r^2$) and mass were conserved as well ($nR^3 =$ constant), then $B \propto n^{2/3}$ and the magnetic field of the Sun would thus be $\left(\frac{10^{24}\ \mathrm{cm}^{-3}}{1\ \mathrm{cm}^{-3}}\right)^{2/3} \times (1\ \mu G)$

$\sim 10^{10}\ G$! (the actual solar field is $\sim 1\ G$, and this is after a vigorous solar dynamo has done its thing)

Having a phase of cylindrical contraction ($nR^2 \sim$ constant) helps, but clearly isn't enough. Substantial redistribution of magnetic flux must take place at some point in the star-formation process. This was realized early on (e.g., Babcock & Cowling 1953, p. 373). [that's "Cowling's theorem" Cowling]

To solve this problem, one must learn some non-ideal MHD, in which the magnetic field is allowed to slip through the bulk (neutral) plasma. There are 3 processes of flux redistribution / dissipation:

(1) Ohmic Dissipation ... this is well-known. If the plasma is a poor conductor, it cannot sustain the currents required to support the magnetic field (Ampère's law), and so the magnetic field diffuses through the (mobile and usually line-tied) electrons. ~~Fada~~ Faraday's law becomes

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}_{\rm ohm} = -c\vec{\nabla}\times(\underbrace{\eta}_{\text{resistivity}}\underbrace{\vec{J}}_{\text{current density } = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}})$$

$$= -\frac{c^2}{4\pi}\vec{\nabla}\times\left[\eta\vec{\nabla}\times\vec{B}\right]$$

$$\xrightarrow{\text{if } \eta \text{ is const.}} -\frac{c^2}{4\pi}\eta\left[\vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{B}}}_{0 \text{ (no monopoles)}}) - \nabla^2\vec{B}\right]$$

$$= \frac{c^2\eta}{4\pi}\nabla^2\vec{B} \leftarrow \text{a diffusion equation!}$$

Ohmic dissipation becomes important in star formation around densities $\underbrace{n_n}_{\text{neutrals}} \gtrsim 10^{12}\ \text{cm}^{-3}$, when enough column density has been built up to shield the protostar from its principal sources of ionization (e.g. cosmic rays).

(2) Ambipolar Diffusion ... this refers to the drift between neutrals (e.g. molecular hydrogen, neutral atoms like $Na^0$ and $Mg^0$, helium) and ions (e.g. $Na^+$, $Mg^+$, $K^+$, $HCO^+$), which are (nearly) tied to the magnetic field until $n_n \gtrsim 10^8\ \text{cm}^{-3}$: $\vec{v}_i - \vec{v}_n \neq 0$

Its form is greatly complicated in star-forming cores with myriad chemical species in the gas phase and ~1% dust grains by mass (e.g. Kunz + Mouschovias 2010a), but its essence can be understood with just one population of ions, which is interacting collisionally with the neutrals and which, being a charged species, is informed of the presence of $\vec{B}$ through the Lorentz force. Equations of motion for the ions (i) and neutrals (n) are, respectively,

$$m_i n_i \left( \frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i \cdot \vec{\nabla} \vec{u}_i \right) = -\vec{\nabla}\left(p_i + p_e\right) + \frac{\vec{J} \times \vec{B}}{c} + \vec{F}_{in} + \vec{F}_{en} + \ldots$$

$$m_n n_n \left( \frac{\partial \vec{u}_n}{\partial t} + \vec{u}_n \cdot \vec{\nabla} \vec{u}_n \right) = -\vec{\nabla} p_n + \vec{F}_{ni} + \vec{F}_{ne} + \ldots$$

e.g. $\vec{F}_{sn} = \frac{m_s n_s}{\tau_{sn}} (\vec{u}_n - \vec{u}_s) = -\vec{F}_{ns}$, $\tau_{sn}$ = collision timescale for s with n

~~There~~, Here, m is mass, n is # density, $\vec{u}$ is mean flow velocity, p is pressure, $\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B}$ is current density, $\vec{B}$ is magn. field, and $\vec{F}_{sk}$ is the friction force, resulting from collisions between species s and k. (We have taken the electrons to be massless — a good assumption.)

Note that the neutrals have no direct knowledge of the magnetic field. That knowledge comes only through collisions with the (~ line-tied) ions, which DO know about the magnetic field. By Newton's 3rd law, $\vec{F}_{ni} = -\vec{F}_{in}$ and $\vec{F}_{ne} = -\vec{F}_{en}$. Adding the two eqns. above together and using the fact that $\frac{n_i}{n_n} \lesssim 10^{-8}$ in protostellar cores gives

$$m_n n_n\left(\frac{\partial \vec{u}_n}{\partial t}+\vec{u}_n\cdot\vec{\nabla}\,\vec{u}_n\right)=-\nabla \underbrace{p}_{p_i+p_e+p_n}+\frac{\vec{J}\times\vec{B}}{c}$$

... the neutrals know about the Lorentz force! Weird!

Surely the fact that the neutrals, which comprise the majority of the plasma, don't directly know about $\vec{B}$ appears somewhere.

It does. Consider the induction eqn:

$$\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\times\left(\vec{u}\times\vec{B}\right)$$

What is this? It's certainly not the neutrals' velocity.

Let us assume the flux is frozen into the ions (and electrons).

Then
$$\frac{\partial\vec{B}}{\partial t}=\vec{\nabla}\times\left(\vec{u}_i\times\vec{B}\right)=\vec{\nabla}\times\left(\vec{u}_n\times\vec{B}\right)+\vec{\nabla}\times\left((\vec{u}_i-\vec{u}_n)\times\vec{B}\right)$$

$$=\vec{\nabla}\times\left(\vec{u}_n\times\vec{B}\right)+\vec{\nabla}\times\left[-\frac{\vec{F}_{in}\tau_{in}}{m_i n_i}\times\vec{B}\right]\quad\left(\text{from } F_{sn}=\frac{m_s n_s(\vec{u}_n-\vec{u}_s)}{\tau_{sn}}\right)$$

$$\simeq\vec{\nabla}\times\left(\vec{u}_n\times\vec{B}\right)+\vec{\nabla}\times\left[\frac{\vec{J}\times\vec{B}}{c}\frac{\tau_{in}}{m_i n_i}\times\vec{B}\right]\quad\left(\text{from force balance w/ } \frac{n_i}{n_n}\ll 1\right)$$

$$=\vec{\nabla}\times\left[\left(\vec{u}_n+\frac{\tau_{in}}{m_i n_i}\frac{\vec{J}\times\vec{B}}{c}\right)\times\vec{B}\right]$$

$\Rightarrow$ the neutrals diffuse through the magnetic field at a speed $\frac{\tau_{in}}{m_i n_i}\frac{(\vec{J}\times\vec{B})\times\vec{B}}{c}$ ← "Ambipolar diffusion" (<u>Non-linear</u>!!!)

This allows the bulk fluid to drift through the field lines, thus reducing the Mass-to-Flux ratio of the blob of gas.

(3) Hall effect.

You may have noticed... we were a bit cavalier about the electrons and ions in the previous section. There are instances when the electrons are tied to the field but the ions aren't. Then the flux drifts relative to the ions at a speed $\sim \vec{u}_i - \vec{u}_e \sim \frac{\vec{J}}{\underbrace{Z_i e}_{q_i = Z_i e} n_i} \sim \frac{c}{4\pi e n_i Z_i}(\vec{\nabla}\times\vec{B})$, so that

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{u}_e\times\vec{B}\right) = \vec{\nabla}\times\left(\vec{u}_i\times\vec{B}\right) + \vec{\nabla}\times\left((\vec{u}_e-\vec{u}_i)\times\vec{B}\right)$$

$$= \vec{\nabla}\times\left(\vec{u}_i\times\vec{B}\right) + \vec{\nabla}\times\underbrace{\left[-\frac{c}{4\pi e n_i Z_i}(\vec{\nabla}\times\vec{B})\times\vec{B}\right]}_{\text{Hall effect}}$$

This introduces dispersion, i.e. the Hall effect is not dissipative like Ohmic dissipation or ambipolar diffusion, but, from the standpoint of the ions, the flux is drifting (albeit in a funny way).

All three effects — Ohmic dissipation, ambipolar diffusion, Hall effect — deserve several lectures unto themselves, as each plays a vital role in the evolution of astrophysical plasmas (sometimes simultaneously, as in protoplanetary disks around ~1–10 au from the central protostar). Here, we're short on time, so I'll just summarize with a set of non-ideal MHD equations describing a plasma of electrons, ions, and (much more abundant) neutrals:

$$\frac{\partial \rho_n}{\partial t} + \vec{\nabla}\cdot(\rho_n \vec{u}_n) = 0$$ (continuity eqn. for neutral mass density $\rho_n = m_n n_n$)

momentum eqn.:

$$\underbrace{\frac{\partial(\rho_n \vec{u}_n)}{\partial t}}_{\frac{\partial}{\partial t}\text{ of neutrals' momentum}} + \vec{\nabla}\cdot(\rho_n \vec{u}_n \vec{u}_n)$$

$$= -\vec{\nabla} P_n - \rho_n \vec{\nabla}\psi + \frac{1}{c}\vec{J}\times\vec{B} + \ldots$$

Lorentz force; any additional external forces

$P_n$: neutrals' pressure (electron & ion pressure negligible because $\frac{n_i}{n_n} \sim \frac{n_e}{n_n} \ll 1$)

$\psi$: gravitational potential

force balance for electrons and ions (which are $\ll n_n$ in #):

$$0 = -e n_e \left(\vec{E} + \frac{\vec{u}_e \times \vec{B}}{c}\right) + \vec{F}_{en} + \vec{F}_{ei}$$

$$0 = Z_i e n_i \left(\vec{E} + \frac{\vec{u}_i \times \vec{B}}{c}\right) + \vec{F}_{in} + \vec{F}_{ie}$$

Ampère: $\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = Z_i e n_i (\vec{u}_i - \vec{u}_e)$

Faraday: $\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}$

gravity: $\nabla^2\psi = 4\pi G \rho_n$ EOS: $P_n = P_n(\rho_n, T_n)$ ($T_n$: neutrals' temperature)

(b) Angular - Momentum Problem:

A similar problem emerges from ang. mom. conservation. Take that $1\,M_\odot$ blob of ISM. The ISM is in the galaxy, of course, and so it has ang. mom. ($\Omega \simeq 10^{-15}\,s^{-1}$ is galactic rotation around where we are: "solar neighborhood") Try conserving ang. mom. while taking this blob and squeezing it a factor in radius of $\frac{r_{init}}{r_{final}} = \left(\frac{n_{final}}{n_{init}}\right)^{1/3} \simeq 10^8$

$$\ldots \; \Omega_{final} = \Omega_{init}\left(\frac{r_{init}}{r_{final}}\right)^2 = 10^{-15}\left(10^8\right)^2 = 10\,s^{-1}.$$ Yikes.

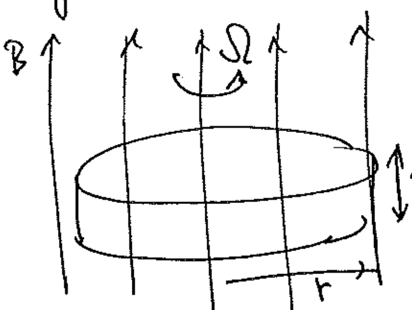
Gravity wouldn't be able to overcome centrifugal forces long before the Sun could form. In fact, forming the molecular clouds in which stars form might even be difficult: ratio of grav. pot. energy and rotational kin. energy of spherical blob of interstellar matter of density equal to mean interstellar density $\sim 1\,cm^{-3}$ and $\Omega \sim 10^{-15}$ rad/s is

$$\frac{|W_{grav}|}{2W_{rot}} = 2\pi\,\frac{G\rho}{\Omega^2} \sim 1$$

$\Rightarrow$ formation of clouds, regardless of mass, that involves contraction perpendicular to axis of rotation is ~~forb~~ forbidden if ang. mom. is conserved. (Mouschovias 1991a)

A beautiful piece of mathematics invoking magnetic tension to transport angular momentum and thereby "brake" protostellar-core rotation is given in Mouschovias & Paleologou (1979, 1980). Check it out. A simplified version is as follows:

Consider a rotating cylindrical disk, threaded by a strong magnetic field:




$(M = \rho_{disk}\, \pi r^2 2z)$

$(\Phi_B = B \pi r^2)$

Now, suddenly increase its ang. velocity (e.g. by cloud contraction). If the magnetic flux is frozen into the disk, the twist in the field lines will generate a wave — a torsional Alfvén wave — which will propagate away from the disk along the field lines. This wave will torque the material that it's propagating through. Once the wave traverses an amount of matter whose moment of inertia is equal to that of the disk, angular momentum will be bled from the disk and set into rotation the "external" material. A straightforward calculation captures this: $I_{disk} = \frac{1}{2} M r^2 = \frac{1}{2}(\rho_{disk}\, \pi r^2 2z)\, r^2$;

$I_{ext} = \frac{1}{2}(\rho_{ext}\, \pi r^2 2 z_1)\, r^2$, where $z_1 = v_A \tau$, $\tau$ ~ braking timescale;

$v_A$: Alfvén speed in external material

$$1 = \frac{I_{disk}}{I_{ext}} = \left(\frac{\rho_{disk}}{\rho_{ext}}\right)\left(\frac{z}{v_A \tau}\right) \rightarrow \boxed{\tau = \frac{z}{v_A}\frac{\rho_{disk}}{\rho_{ext}} = \left(\frac{\pi}{\rho_{ext}}\right)^{1/2} \frac{M}{\Phi_B}}$$

It turns out that magnetic braking is very efficient at reducing the ang. mom. of protostellar cores. Indeed, observations show that such cores rotate rigidly with relatively low specific angular momentum. This changes, of course, once the field is no longer frozen into the bulk plasma, or if the core contracts faster than the torsional Alfvén waves can propagate away.

## (2) Accretion Disks: Instability, Turbulence, & Enhanced Transport (see Balbus & Hawley 1998)

Before 1991, a long-standing problem in astrophysical fluid dynamics — indeed, astronomy in general — was how to transport angular momentum outwards in Keplerian disks of gas so that mass can be transported inwards and "accrete" onto the central object (whether it be a black hole, neutron star, protostar, etc.) The problem is that, hydrodynamically, Keplerian flows are quite stable (they're certainly linearly stable, and all evidence points to nonlinear stability as well... but there's no proof and so there continue to be adherents seeking hydrodynamic instabilities). In other words, fluid elements just don't like giving up their ang. mom. to others. The culprit is the Coriolis force, a surprisingly strong stabilizing agent.

(Indeed, shear flows w/o the Coriolis effect... like planar shear flows... quite easily disrupt.) Another issue is that the molecular viscosity, which might transport (angular) momentum purely by frictional means, is absolutely negligible in most all astrophysical fluids. Now, accretion disks accrete — we know this observationally: mass accretion rates have been observationally inferred in a variety of systems — and so what's a theorist to do? Posit some anomalous transport via (unknown) turbulence. This is the route taken in the classic Shakura & Sunyaev (1973) paper — assume turbulent transport, characterize it by a viscosity, and take that viscous stress to be proportional to the gas pressure (in a hydro. disk, this is the only option):

$$T_{R\varphi} \equiv \alpha_{ss} P$$

gas pressure

proportionality constant

R-φ component of the stress tensor, responsible for transporting φ momentum in the R direction

This led to the "alpha-disk" framework of accretion disks, ~~[illegible]~~ which has been extremely profitable, but woefully unsatisfying.

This changed in 1991.

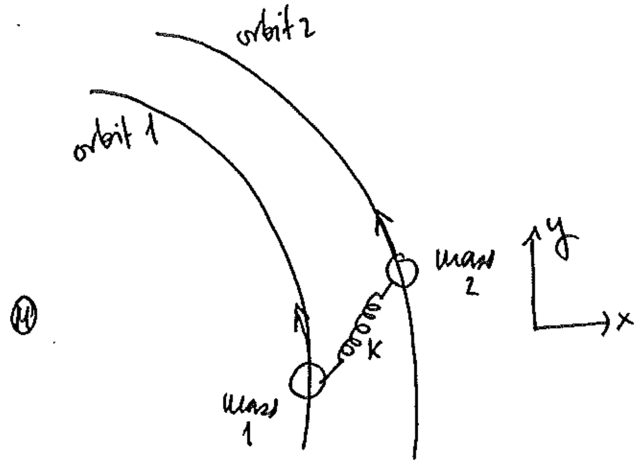
Balbus & Hawley, then both @ Univ. of Virginia, found, by a straightforward linear analysis and the use of 1990's super-computers, that a small but finite magnetic field is all that is required to linearly destabilize Keplerian flow. How could this be missed? ~~The~~ answer is complicated. The instability — at first known as the "Balbus-Hawley instability", but now goes by the more generic moniker "magnetorotational instability" or "MRI" — appeared in a little known Russian paper by Velikhov in 1959, and 2 years later it found its way into Chandrasekhar's classic text ~~on~~ on hydrodynamic and hydromagnetic stability. But there it appeared in a rather odd guise, at least to an astronomer thinking about accretion disks — Couette flow, i.e. rotational flow excited by placing a (conducting, in this case) fluid between two cylindrical walls rotating at different speeds. It wasn't until B&H rediscovered it and placed it in the astrophysical context that this instability became appreciated as a possible solution to the accretion problem. All that remained was to show that this instability, in its nonlinear phase, drives vigorous turbulence that can generate angular-momentum transport and consequent mass accretion at the observationally inferred rates.

Enter computational fluid dynamics, which, by now, is an entire industry unto itself. From 1991–1996, several axisymmetric and nonaxisymmetric local simulations were performed, confirming the initial suspicion that the MRI is responsible for the long-sought "$\alpha$" parameter, associated by Shakura & Sunyaev with anomalous (turbulent) transport of ang. mom. Nowadays, the focus is on (1) whether the MRI turbulence actually operates like an "$\alpha$" viscosity, with a stress proportional to gas pressure (doesn't appear so) and (2) what the MRI looks like in less ideal plasmas, like those near compact objects, or those with long collisional mean-free paths, or those that are poorly ionized and thus poorly conducting. It's a prosperous field now, and discoveries are still being made. But let's go back to the beginning... what <u>is</u> the MRI? For that, one simple orbital dynamics problem, and a discussion of MHD in a rotating frame. Let's start with orbital dynamics...

Consider two masses in orbit about a central mass, connected by a spring. The equations of motion are

$$\left.\begin{aligned} \ddot{x} - 2\Omega\dot{y} &= -\left[\frac{d\Omega^2}{d\ln R} + K^2\right]x \\ \ddot{y} + 2\Omega\dot{x} &= -K^2 y\,, \end{aligned}\right\} \quad \text{"Hill" equations (Hill 1878) w/ Hooke's law}$$

where $x$ is the radial direction in the disk, $y$ the azimuthal direction, $\Omega$ the ang. velocity of the disk @ $(x,y)$, and $\frac{d\Omega^2}{d\ln R}$ the local shear rate $\times 2\Omega$ (for Keplerian flows, it $= -3$); $K$ is the spring constant of the spring. Here's the setup pictorially:




What are the normal modes of this system? Take $x, y \sim e^{-i\omega t}$. Then

$$-\omega^2 x - 2\Omega(-i\omega y) = -\left[\frac{d\Omega^2}{d\ln R} + K^2\right]x$$

$$-\omega^2 y + 2\Omega(-i\omega x) = -K^2 y$$

$$\Rightarrow \begin{bmatrix} -\omega^2 + K^2 + \frac{d\Omega^2}{d\ln R} & 2\Omega i\omega \\ -2\Omega i\omega & -\omega^2 + K^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Solutions exist if determinant $= 0$.

$$\Downarrow$$

$$0 = (-\omega^2 + K^2)\left(-\omega^2 + K^2 + \frac{d\Omega^2}{d\ln R}\right) + (2\Omega i\omega)^2$$

$$\omega^4 - \omega^2\left(2K^2 + \frac{d\Omega^2}{d\ln R} + 4\Omega^2\right) + K^2\left[K^2 + \frac{d\Omega^2}{d\ln R}\right] = 0.$$

Solutions are $$\omega^2 = \frac{1}{2}\left(2K^2 + 4\Omega^2 + \frac{d\Omega^2}{d\ln R}\right) \pm \frac{1}{2}\sqrt{\left(2K^2 + 4\Omega^2 + \frac{d\Omega^2}{d\ln R}\right)^2 - 4K^2\left(K^2 + \frac{d\Omega^2}{d\ln R}\right)}$$

Now, for $\kappa^2=0$ (no spring), we have

$$\omega^2 = \underbrace{4\Omega^2 + \frac{d\Omega^2}{d\ln R}}_{\text{(epicyclic frequency)}^2} \quad \text{or} \quad \underbrace{\emptyset}_{\substack{\text{trivial displacement} \\ \text{(e.g. swap the 2 masses)}}}$$

Just stable epicycles due to Coriolis term, unless

$4\Omega^2 + \dfrac{d\Omega^2}{d\ln R} < 0$, which corresponds to the specific ang. momentum of the disk decreasing outwards. This situation leads to the Rayleigh instability — displacing a fluid element outwards while conserving its ang. mom. places it at a radius of lower angular momentum, and so it must continue to move outwards. The process runs away.

But now restore the spring. It is straightforward to show that the system is stable if and only if

$$\kappa^2 + \frac{d\Omega^2}{d\ln R} > 0.$$

Now take the spring constant to zero: $\dfrac{d\Omega^2}{d\ln R} > 0$! What changed?! We lost a $4\Omega^2$ term, which is stabilizing.

Now, a Keplerian flow $\left(\frac{d\Omega^2}{d\ln R} = -3\Omega^2\right)$ is potentially unstable! In fact, fastest growing mode has a growth rate $= \frac{1}{2}\left|\frac{d\Omega}{d\ln R}\right|$ at $K^2 = \left(1 - \frac{1}{4}\left|\frac{d\ln\Omega}{d\ln R}\right|\right)\left|\frac{d\ln\Omega}{d\ln R}\right|\Omega^2$.

This is an enormous growth rate. Unchecked, it results in a factor of $\gtrsim 10^4$ amplification in energy <u>per orbit</u>.

Replace the "spring" by a frozen-in magnetic field and you've got the MRI, with $K^2 = \underbrace{(\vec{k}\cdot\vec{v}_A)^2}_{\text{"magnetic tension"}}$

$\vec{k}$: wavevector

$\vec{v}_A$: Alfvén velocity $= \frac{\vec{B}}{\sqrt{4\pi\rho}}$

See Balbus & Hawley 1991, 1992ab, 1998 (review).

Essentially, the rich (in ang. mom.) get richer, and the poor (in ang. mom.) get poorer, the transfer being mediated via magnetic torques.

To do better requires using the MHD equations in a rotating frame. Acquiring them is an exercise in vector analysis. The nonlinear term $\vec{u}\cdot\vec{\nabla}\vec{u}$ is the trick... in curvilinear coordinates $(R,\varphi,z)$ you need to worry about differentiating unit vectors. Remember $\frac{\partial \hat{e}_R}{\partial\varphi} = \hat{e}_\varphi$ and $\frac{\partial \hat{e}_\varphi}{\partial\varphi} = -\hat{e}_R$? Now you do.

$$\text{Then } (\vec{u}\cdot\vec{\nabla}\vec{u}) = \vec{u}\cdot\vec{\nabla}\left(u_R\hat{R} + u_\varphi\hat{\varphi} + u_z\hat{z}\right)$$

$$= \hat{R}\,\vec{u}\cdot\vec{\nabla}u_R + \hat{\varphi}\,\vec{u}\cdot\vec{\nabla}u_\varphi + \hat{z}\,\vec{u}\cdot\vec{\nabla}u_z + u_R\frac{u_\varphi}{R}\underbrace{\frac{\partial\hat{R}}{\partial\varphi}}_{\hat{\varphi}} + \frac{u_\varphi^2}{R}(-\hat{R})$$

$$= \hat{e}_i\,\vec{u}\cdot\vec{\nabla}u_i + \underbrace{\frac{u_R u_\varphi}{R}\hat{\varphi}}_{\text{Coriolis}} - \underbrace{\frac{u_\varphi^2}{R}\hat{R}}_{\text{Centrifugal}}$$

Write $\vec{u} = \vec{v} + \underbrace{R\Omega\hat{\varphi}}_{\text{disk rotation}}$, with $\vec{\Omega} = \Omega(R)\hat{z}$. Then

$$\vec{u}\cdot\vec{\nabla}\vec{u} = \hat{e}_i\,\vec{v}\cdot\vec{\nabla}v_i + \hat{e}_i\,\Omega\frac{\partial v_i}{\partial\varphi} + \hat{\varphi}\,v_R\frac{d(R\Omega)}{dR}$$

$$+ \frac{v_R v_\varphi}{R}\hat{\varphi} + \frac{v_R R\Omega}{R}\hat{\varphi} - \frac{(v_\varphi + R\Omega)^2}{R}\hat{R}$$

$$= \hat{e}_i\left(\vec{v}\cdot\vec{\nabla}v_i + \Omega\frac{\partial v_i}{\partial\varphi}\right) - \frac{v_\varphi^2}{R}\hat{R} + \frac{v_R v_\varphi}{R}\hat{\varphi}$$

$$+ \hat{\varphi}\left(2\Omega v_R + v_R\frac{d\Omega}{d\ln R}\right) - \Omega^2 R\hat{R} - 2\Omega v_\varphi\hat{R}$$

$$\Rightarrow \boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} = \hat{e}_i \left( \frac{\partial v_i}{\partial t} + \vec{v}\cdot\vec{\nabla} v_i + \Omega \frac{\partial v_i}{\partial \varphi} \right) - \frac{v_\varphi^2}{R}\hat{R} + \frac{v_R v_\varphi}{R}\hat{\varphi} + \underbrace{2\vec{\Omega}\times\vec{v}}_{\text{Coriolis}} - \underbrace{\Omega^2 R \hat{R}}_{\text{Centrifugal}} + \underbrace{v_R \hat{\varphi} \frac{d\Omega}{d\ln R}}_{\text{Tidal}}}$$

Note further that
$$\frac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{B} = \frac{\partial \vec{B}}{\partial t} + \Omega \frac{\partial B_i}{\partial \varphi}\hat{e}_i + \vec{v}\cdot\vec{\nabla} B_i \hat{e}_i + \frac{v_\varphi B_R}{R}\hat{\varphi} - \frac{v_\varphi B_\varphi}{R}\hat{R} + \Omega\left(B_R\hat{\varphi} - B_\varphi \hat{R}\right)$$

and
$$\vec{B}\cdot\vec{\nabla}\vec{u} = \hat{e}_i \vec{B}\cdot\vec{\nabla} v_i + \frac{v_R B_\varphi}{R}\hat{\varphi} - \frac{v_\varphi B_\varphi}{R}\hat{R} - \Omega B_\varphi \hat{R} + \hat{\varphi} B_R \left(\Omega + \frac{d\Omega}{d\ln R}\right)$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{B} = \vec{B}\cdot\vec{\nabla}\vec{u} - \vec{B}\,\vec{\nabla}\cdot\vec{u} \quad \text{becomes}$$

$$\frac{\partial \vec{B}}{\partial t} + \left(\Omega \frac{\partial B_i}{\partial \varphi} + \vec{v}\cdot\vec{\nabla} B_i\right)\hat{e}_i + \frac{v_\varphi B_R}{R}\hat{\varphi} - \cancel{\frac{v_\varphi B_\varphi}{R}\hat{R}} + \Omega\left(\cancel{B_R\hat{\varphi}} - \cancel{B_\varphi \hat{R}}\right) = \hat{e}_i \vec{B}\cdot\vec{\nabla} v_i + \frac{v_R B_\varphi}{R}\hat{\varphi} - \cancel{\frac{v_\varphi B_\varphi}{R}\hat{R}} - \cancel{\Omega B_\varphi \hat{R}} + B_R\left(\cancel{\Omega} + \frac{d\Omega}{d\ln R}\right)\hat{\varphi}$$

$$\boxed{\hat{e}_i\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \varphi} + \vec{v}\cdot\vec{\nabla}\right)B_i + \left(\frac{v_\varphi B_R \hat{\varphi} - v_R B_\varphi \hat{\varphi}}{R}\right) = \hat{e}_i \vec{B}\cdot\vec{\nabla} v_i + B_R \frac{d\Omega}{d\ln R}\hat{\varphi}}$$

induction eqn. in rotating frame

Finally, $\vec{B}\cdot\vec{\nabla}\vec{B} = \hat{e}_i\,\vec{B}\cdot\vec{\nabla}B_i + \frac{B_\varphi}{R}\left(B_R\hat{\varphi} - B_\varphi\hat{R}\right)$. So... with

$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \Omega\frac{\partial}{\partial\varphi}$, we have our ideal MHD eqns:

$(\vec{g} = -\vec{\nabla}\Phi)$

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v} = 0$$

$$\frac{Dv_R}{Dt} - \frac{v_\varphi^2}{R} - 2\Omega v_\varphi - \Omega^2 R = -\frac{1}{\rho}\frac{\partial}{\partial R}\left(P + \frac{B^2}{8\pi}\right) - \frac{\partial\Phi}{\partial R} + \frac{\vec{B}\cdot\vec{\nabla}B_R}{4\pi\rho} - \frac{B_\varphi^2}{R}\frac{1}{4\pi\rho}$$

$$\frac{Dv_\varphi}{Dt} + \frac{v_R v_\varphi}{R} + 2\Omega v_R + v_R\frac{d\Omega}{d\ln R} = -\frac{1}{\rho}\frac{1}{R}\frac{\partial}{\partial\varphi}\left(P + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\vec{\nabla}B_\varphi}{4\pi\rho} + \frac{B_R B_\varphi}{4\pi\rho R}$$

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(P + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\vec{\nabla}B_z}{4\pi\rho} - \frac{\partial\Phi}{\partial z}$$

$$\frac{DB_R}{Dt} = \vec{B}\cdot\vec{\nabla}v_R \qquad \frac{DB_z}{Dt} = \vec{B}\cdot\vec{\nabla}v_z$$

$$\frac{DB_\varphi}{Dt} + \frac{v_\varphi B_R - v_R B_\varphi}{R} = \vec{B}\cdot\vec{\nabla}v_\varphi + B_R\frac{d\Omega}{d\ln R}$$

if you let $R \to \infty$ and $c_s^2 \equiv \frac{\gamma P}{\rho} \to \infty$, such that terms $\propto \frac{v^2}{R}$ and $\frac{B^2}{R}$ are dropped and you can assume $\vec{\nabla}\cdot\vec{v} = 0$ (incompressibility), then you have some nice equations for an accretion disk that you can solve for small perturbations $\vec{v} = \delta\vec{v}$, $\vec{B} = \vec{B}_0 + \delta\vec{B}$, $P = P_0 + \delta P$, with $\delta \sim e^{-i\omega t + i\vec{k}\cdot\vec{r}}$. These will also give MRI, but entail considerably more algebra.
(See B&H '91 and '98.)

To get you started, take $\vec{\nabla}P_0 = 0$ in the equilibrium and $-\Omega^2 R = -\frac{d\Phi}{dR} = g_R$. Linearized equations w/ $\vec{k} = k_R\hat{R} + k_z\hat{z}$ are:

$$i k_R \delta v_R + i k_z \delta v_z = 0$$

$$-i\omega \delta v_R - 2\Omega \delta v_\varphi = -\frac{i k_R}{\rho}\left(\delta P + \frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi}\right) + \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\delta B_R$$

$$-i\omega \delta v_\varphi + 2\Omega \delta v_R + \delta v_R \frac{d\Omega}{d\ln R} = \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\delta B_\varphi$$

$$-i\omega \delta v_z = -\frac{i k_z}{\rho}\left(\delta P + \frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi}\right) + \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho}\delta B_z$$

$$-i\omega \delta B_R = i\vec{k}\cdot\vec{B}_0 \delta v_R \qquad -i\omega \delta B_z = i\vec{k}\cdot\vec{B}_0 \delta v_z$$

$$-i\omega \delta B_\varphi = i\vec{k}\cdot\vec{B}_0 \delta v_\varphi + \delta B_R \frac{d\Omega}{d\ln R}$$

the rest is linear algebra...

③ Galaxy Clusters: When is a stratified plasma stable?

By now, you've come to appreciate that a plasma is more complex than a fluid. With that in mind, let's start with a fluid, and work our way from there.

Consider a stationary stratified atmosphere, with gravity pointing downwards: $\vec{g} = -g\hat{z}$. The hydrodynamic equations are

$\boxed{\text{Equilibrium: } -\frac{dP}{dz} = \rho g}$

(mass) $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

(momentum) $$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = -\vec{\nabla}P - \rho g\hat{z}$$

(internal energy) $$\frac{P}{\gamma-1}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln P\rho^{-\gamma} = 0$$

adiabatic index $= \left(\frac{f+2}{f}\right)$, where $f$ is # of degrees of freedom of a particle

essentially entropy conservation in Lagrangian (i.e. comoving) frame

These equations support sound waves and "internal" waves. This will be an exploration of the latter. To remove the former from the calculation, we assume that the sound speed is fast, so that such waves radiate away much faster than any gravitationally driven dynamics occurs. How fast? Fast enough that, for a disturbance of frequency $\omega$ and wavelength $\frac{2\pi}{k}$, the change in density $\delta\rho$ satisfies $\frac{\delta\rho}{\rho} \ll \left|\frac{k\,\delta u}{\omega}\right|$, where $\delta u$ is the change in fluid velocity. This is called the "Boussinesq approximation". It allows us to treat a compressible gas/fluid as incompressible effectively. Indeed, the continuity equation states that $-i\omega \frac{\delta\rho}{\rho} = -\vec{\nabla}\cdot\delta\vec{u} = -i\vec{k}\cdot\delta\vec{u}$,

and so our approximation means that we can assume $\vec{\nabla}\cdot\vec{u}=0$ to leading order. Leading order in what?? Mach number, $\frac{u}{c_s}$.

The idea is that a sufficiently slowly moving fluid element remains in close pressure equilibrium with its surroundings. To see how this plays out, let's just do the calculation with $\vec{\nabla}\cdot\vec{u}=0$ and see what we're missing...

For simplicity, take perturbations to have the form $e^{ikx-i\omega t}$. Then our equations are $k\,\delta u_x = 0$

$$\rho(-i\omega\,\delta u_x) = -ik\,\delta P \Rightarrow \delta P = 0.$$

$$\rho(-i\omega\,\delta u_y) = 0$$

$$\rho(-i\omega\,\delta u_z) = -\delta\rho\, g$$

$$\frac{P}{\gamma-1}(-i\omega)\Big(\underbrace{\cancel{\frac{\delta P}{P}}}_{0} - \gamma\frac{\delta\rho}{\rho}\Big) = -\frac{P}{\gamma-1}\,\delta u_z\,\frac{d\ln P\rho^{-\gamma}}{dz}$$

$$\Rightarrow \quad i\omega\gamma\frac{\delta\rho}{\rho} = -\frac{d\ln P\rho^{-\gamma}}{dz}\left(-\frac{\delta\rho}{\rho}\,g\,\frac{i}{\omega}\right)$$

$$\Rightarrow \quad \frac{\delta\rho}{\rho}\left[\omega^2 - \frac{g}{\gamma}\frac{d\ln P\rho^{-\gamma}}{dz}\right] = 0.$$

(No $k$! Nothing is propagating! We'll come back to this; it's basically a consequence of $\delta P = 0$ ... with exact pressure equilibrium being maintained everywhere, it's no wonder nothing is propagating.)

Solution: $$\boxed{\omega^2 = \frac{g}{\gamma}\frac{d\ln P\rho^{-\gamma}}{dz} = -\frac{1}{\gamma\rho}\frac{dP}{dz}\frac{d\ln P\rho^{-\gamma}}{dz} \equiv N^2}$$

$N$: "Brunt–Väisälä frequency"

Vertical displacements oscillate at a frequency $N$. These oscillations come about because an upward displacement (at constant entropy) places a fluid element into a region that is hotter, (if $N^2>0$) and so the element must sink back down into its original location. What if $N^2<0$? then $\omega$ is imaginary $\Rightarrow$ growth! This is convection, and $N^2 \geq 0$ is called the "Schwarzschild stability criterion". Just look at a pot of boiling water — there is more entropy closer to the stove than further away, and so upward displacements of water place high-entropy material (= low-density material, ~~by~~ by pressure balance) in lower-entropy surroundings.

(Good. Now, what about that wave propagation? Let's go back to our linearized equations and let $\vec{k} = k_x\hat{x} + k_z\hat{z}$. Then

Try this calculation w/o assuming $\vec{\nabla}\cdot\vec{u}=0$, and you'll see exactly what the Boussinesq approximation means. Answer: $(\omega^2 + g\frac{d\ln\rho}{dz}) = k^2a^2(1 - N^2/\omega^2)$ with $a^2 = \frac{\partial P}{\partial \rho}\big|_S$ = sound speed)

$$k_x \delta u_x + k_z \delta u_z = 0$$

$$\rho(-i\omega\, \delta u_x) = -i k_x\, \delta P$$

$$\rho(-i\omega\, \delta u_y) = 0$$

$$\rho(-i\omega\, \delta u_z) = -i k_z\, \delta P - \delta\rho\, g$$

$$\frac{P}{\gamma-1}(-i\omega)\left(\cancel{\frac{\delta P}{P}} - \gamma\frac{\delta\rho}{\rho}\right) = -\frac{P}{\gamma-1}\,\delta u_z\, \frac{d\ln P\rho^{-\gamma}}{dz}$$

small (prove!)

Algebra ... then we find $\boxed{\omega^2 = \frac{k_x^2}{k^2} N^2}$

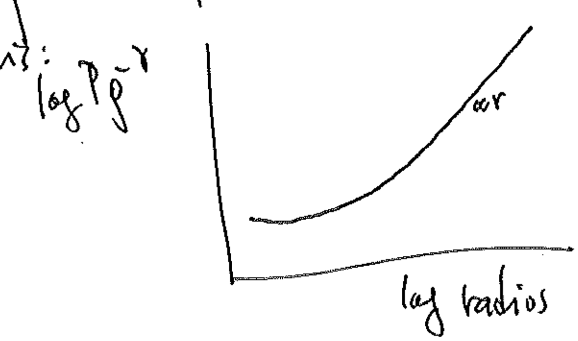
Now, these *are* waves, exhibiting not only ~~diffusion~~ dispersion $\left(\frac{\partial\omega}{\partial k} \neq \text{const.}\right)$ but anisotropy in propagation.

$$\frac{\partial\omega}{\partial k_x} = \frac{\omega}{k_x}\left(\frac{k_z^2}{k^2}\right) \neq \frac{\partial\omega}{\partial k_z} = -\frac{\omega}{k_z}\left(\frac{k_z^2}{k^2}\right)$$

Note that $\vec{k}\cdot\frac{\partial\omega}{\partial\vec{k}} = 0$ !!! Very different from a sound wave!

---

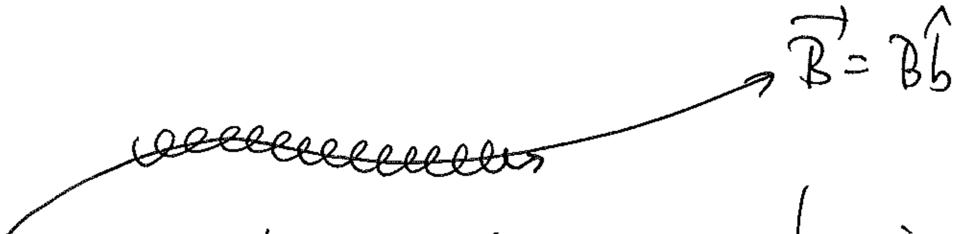
Stars like our Sun have a convectively unstable zone in their outer layers, with turbulence maintaining a value of $N^2$ close to zero but slightly negative (to sustain convection). Galaxy-cluster plasmas are observed to have entropy profiles like this:




← monotonically increasing. They should be convectively stable since $N^2 \geq 0$, right?

wrong...

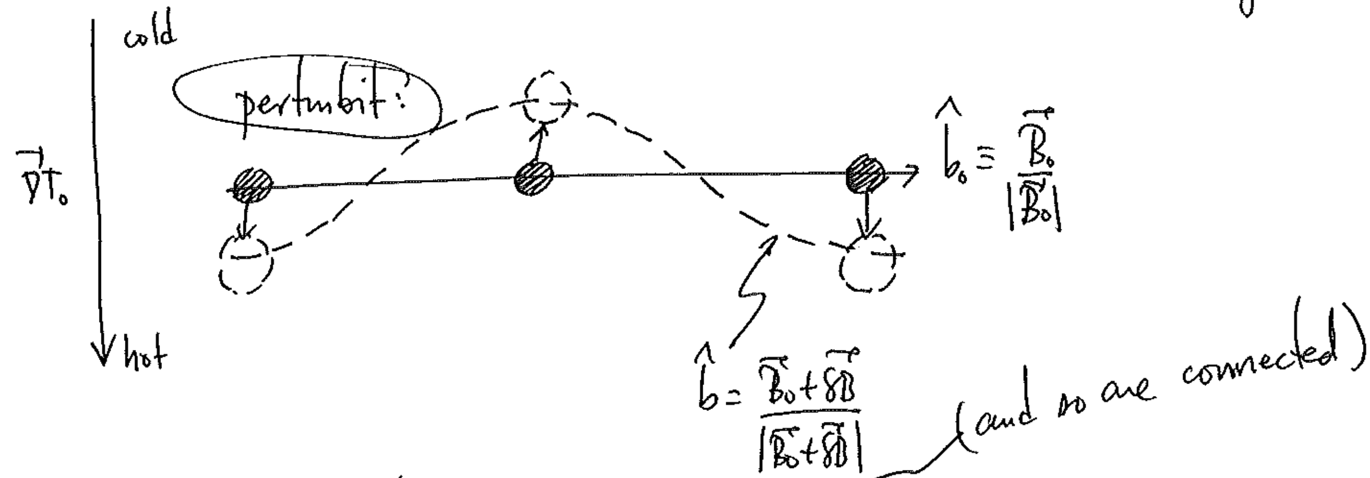
The problem is that it's quite difficult to have isentropic perturbations in a weakly collisional, magnetized plasma. Such plasmas don't look like ideal-MHD fluids, and the anisotropy introduced into the system by the magnetic field (recall that, in the ICM, $\rho_i \equiv \frac{v_{thi}}{\Omega_i} \sim 1$ npc $\lll \lambda_{mfp} \lesssim l$) spoils the ability of fluid elements to act independently of one another. This is a result of flux-freezing (resistivity is tiny!) plus field-aligned transport (Larmor radius is tiny!). For example:




consider an Alfvén wave propagating in a high-$\beta$ ($\gg 1$) plasma, of frequency $\omega$ and wavelength $\lambda = \frac{2\pi}{k_\parallel}$, with $\omega \ll \Omega_i$ and $k\rho_i \ll 1$. In a time $\sim \frac{1}{\omega}$,

and $k\lambda_{mfp} \gtrsim 1$

an ion can travel along the field a distance $\frac{v_{thi}}{\omega} \gg \lambda$, but, because its perpendicular motion is constrained by Larmor motion, it only travels a distance $\sim \rho_i$ across the perturbed field. This makes it very difficult for particles to interact across field lines, and so the transport of momentum and heat is almost entirely along the field. Because the field on these scales is frozen in, this transport-channeling field is always connected from one fluid element to another. Consider the following: take a field line $\perp$ to a temperature gradient:




The fluid elements go with the field! (and so are connected) If $k\lambda_{mfp} \lesssim 1$, then ~~par~~ particle-particle collisions along the perturbed field lines communicate thermodynamic information, and the perturbed field tends to be isothermal: $\hat{b}\cdot\vec{\nabla}T = 0$. If $k\lambda_{mfp} \gtrsim 1$, particles just free-stream along field lines and essentially accomplish the same thing. In neither case is this displacement adiabatic (i.e., preserving entropy), because neighboring fluid elements communicate w/ one another along frozen-in fields and equilibrate their temperatures. (This is all assuming the frequency of the displacement $\omega$ satisfies $\omega \ll k_\parallel v_{thi}$, usually true in a high-$\beta$ plasma for most of the interesting fluctuations.) Put differently, there is a <u>heat flux</u>, directed along the perturbed field, which short-circuits adiabatic evolution. Mathematically, ...

entropy equation:

$$\frac{p}{\gamma-1}\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)\ln p\rho^{-\gamma} = -\vec{\nabla}\cdot\vec{Q} \equiv -\vec{\nabla}\cdot\Big[-\underbrace{\chi}_{\text{conductivity} \sim n\lambda_{mfp} v_{th}}\underbrace{\hat{b}\hat{b}}_{=\vec{B}\vec{B}/B^2}\cdot\underbrace{\vec{\nabla}T}_{\text{temperature gradient}}\Big]$$

$\Rightarrow$ field-line-directed heat diffusion!

This makes perturbations with $\frac{\omega}{k_\parallel v_{th}} \ll k_\parallel \lambda_{mfp}$ approx. isothermal along field lines.

Let's revisit the convection problem with this physics, again with $\vec{k}=k\hat{x}$ but now with a horizontal magnetic field $\vec{B}=B_0\hat{x}$.

$$k\,\delta u_x = 0$$

$$\rho(-i\omega\delta u_x) = -ik\left(\delta p + \cancel{\frac{B_0\delta B_x}{4\pi}}\right) + \cancel{\frac{ikB_0}{4\pi}\delta B_x} \Rightarrow \delta p = 0$$

$$\rho(-i\omega\delta u_y) = \frac{ikB_0}{4\pi}\delta B_y$$

$$\rho(-i\omega\delta u_z) = \frac{ikB_0}{4\pi}\delta B_z - \delta\rho\, g$$

$$\frac{p}{\gamma-1}(-i\omega)\left(\cancel{\frac{\delta p}{p}}^{=0} - \gamma\frac{\delta\rho}{\rho}\right) + \frac{p}{\gamma-1}\delta u_z\frac{N^2\gamma}{g} = -k^2\chi_T\left(\frac{\delta T}{T}\right) + ik\chi_T\frac{\delta B_z}{B}\frac{d\ln T}{dz}$$

$$\frac{\delta T}{T} = \cancel{\frac{\delta p}{p}}^{=0} - \frac{\delta\rho}{\rho} = -\frac{\delta\rho}{\rho}$$

Flux freezing gives $\delta B_x = \frac{ikB_0 \delta u_x}{-i\omega} = 0$ $\qquad \delta B_z = \frac{ikB_0 \delta u_z}{-i\omega}$, so...

$$\delta B_y = \frac{ikB_0 \delta u_y}{-i\omega}$$

$$\Rightarrow \left(\frac{\gamma}{\gamma-1}\right) i\omega \frac{\delta\rho}{\rho} + \left(\frac{\gamma}{\gamma-1}\right)\frac{N^2}{g}\delta u_z = +k^2 \frac{\chi T}{p}\frac{\delta\rho}{\rho} - ik^2 \frac{\chi T}{p}\frac{\delta u_z}{\omega}\frac{d\ln T}{dz}$$

$$\left[i\omega - \left(\frac{\gamma-1}{\gamma}\right)\frac{\chi T}{p}k^2\right]\frac{\delta\rho}{\rho} = -\delta u_z\left[\left(\frac{\gamma-1}{\gamma}\right) ik^2\frac{\chi T}{p}\frac{1}{\omega}\frac{d\ln T}{dz} + \frac{N^2}{g}\right]$$

and $\qquad -i\omega\,\delta u_z = \frac{ikB_0}{4\pi\rho}\left(\frac{ikB_0}{-i\omega}\right)\delta u_z - \frac{\delta\rho}{\rho}g$

$$\Rightarrow (\omega^2 - k^2 v_{A0}^2)\,\delta u_z = -i\omega g\frac{\delta\rho}{\rho} \qquad w/ \quad v_{A0}^2 \equiv \frac{B_0^2}{4\pi\rho}$$

Combine these: $(\omega^2 - k^2 v_{A0}^2)\left[\frac{ik^2}{\omega}\frac{\chi T}{p}\frac{\gamma-1}{\gamma}\frac{d\ln T}{dz} + \frac{N^2}{g}\right]^{-1} = +i\omega g\left[i\omega - \frac{\gamma-1}{\gamma}\frac{\chi T}{p}k^2\right]^{-1}$

$$\Rightarrow (\omega^2 - k^2 v_{A0}^2)(i\omega - \omega_{cond}) = +i\omega g\left[i\frac{\omega_{cond}}{\omega}\frac{d\ln T}{dz} + \frac{N^2}{g}\right]$$

with $\omega_{cond} \equiv k^2 \frac{\chi T}{p}\frac{\gamma-1}{\gamma}$.

$$\Rightarrow \boxed{\frac{i\omega}{\omega_c}\left(\omega^2 - k^2 v_{A0}^2 - N^2\right) = \omega^2 - k^2 v_{A0}^2 - g\frac{d\ln T}{dz}}$$

- without stratification, we have Alfvén waves $\omega = \pm k v_{A0}$, $\left(\frac{\delta B_y}{B_0} = \mp \frac{\delta u_y}{v_A}\right)$
"pseudo-Alfvén waves" $\omega = \pm k v_{A0}$ (i.e. Boussinesq slow modes), $\left(\frac{\delta B_z}{B_0} = \mp \frac{\delta u_z}{v_A}\right)$
and the "entropy" mode. $\omega = -i\omega_c$.

- slow conduction ($\omega/\omega_c \gg 1$) gives $\omega = \pm\sqrt{k^2 v_{A0}^2 + N^2}$ for $\delta B_z$-polarized slow modes; $\omega = \pm k v_{A0}$ for $\delta B_y$-polarized Alfvén waves. The former are on Brunt–Väisälä oscillations, with magnetic tension ($k^2 v_{A0}^2$) as a stabilizing agent.

- Fast conduction ($\omega/\omega_c \ll 1$) gives $\omega = \pm\sqrt{k^2 v_{A0}^2 + g\frac{d\ln T}{dz}}$;
Bouyancy frequency has changed! $\frac{d\ln P\rho^{-\gamma}}{dz} \to \frac{d\ln T}{dz}$. Moreover, note that a plasma can go unstable for $g\frac{d\ln T}{dz} < 0$ at long enough wavelengths such that $k^2 v_{A0}^2 \ll \left|g\frac{d\ln T}{dz}\right|$. This is now called "magnetothermal instability" (MTI), and was found by Balbus (2000, 2001). Since galaxy-cluster outskirts show $\frac{d\ln T}{dz} < 0$, it is thought that MTI is operating there.
The idea here is that rapid conduction gives isothermal displacements, not adiabatic ones, and so a comparison must be made to the surrounding temperature — not entropy. Conduction just needs to be

fast and directed along perturbed field lines. See Balbus (2000, 2001) for further details.

Another consequence of $\rho_i \ll$ any interesting scale is that the pressure tensor is (close to) diagonal in a frame oriented with $\vec{B}$: $\overleftrightarrow{P} = p_\perp(\overleftrightarrow{I} - \hat{b}\hat{b}) + p_\parallel \hat{b}\hat{b}$

Why would $p_\perp \neq p_\parallel$? Well, that's because the collisional mean free path in clusters is long, and so collisions are insufficient to enforce an isotropic Maxwellian distribution function. But what are $p_\perp$ and $p_\parallel$? When $\lambda_{mfp} \gg \rho_i$ but $\lambda_{mfp}$ is still $\ll \underset{\text{scales of interest}}{l}$, Braginskii (1965) says

$$p_\perp - p_\parallel \simeq \frac{3p}{\nu_{coll.}} \frac{d \ln B\rho^{-2/3}}{dt}$$

collisions drive $p_\perp - p_\parallel \to 0$ (i.e. they isotropize the distribution function), but why does $\frac{d}{dt}$ of $B$ and $\rho$ produce $p_\perp \neq p_\parallel$? This comes from adiabatic invariance of particles:

$$\mu = \frac{mv_\perp^2}{2B} \quad \text{and} \quad J = \oint m v_\parallel \, dl \sim m v_\parallel \left(\frac{B}{n}\right)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\frac{p_\perp}{nB} \sim \text{constant} \qquad\qquad \frac{p_\parallel B^2}{n^3} \sim \text{constant}$$

These are statements of flux conservation inside a Larmor orbit (i.e. if $B$ changes, the perpendicular speed of the particle must change so that the Larmor ~~radius~~ orbit encloses the same amount of magnetic flux) and canonical momentum conservation in a magnetic mirror, averaged over all particles.

(equivalent to ang. mom. conservation of a Larmor orbit of radius $\rho \sim \frac{v_\perp}{\Omega} \propto \frac{v_\perp}{B}$)

So... adiabatic invariance produces pressure anisotropy ($p_\perp \neq p_\parallel$) and collisions relax it ($p_\perp - p_\parallel \to 0$). This manifests as an anisotropic viscous stress in the plasma:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = -\vec{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi} + \vec{\nabla}\cdot\left[\left(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right)(p_\perp - p_\parallel)\right]$$

$$= \vec{\nabla}\cdot\left[\left(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right)\frac{3p}{\nu_{\text{coll.}}}\frac{d\ln B\rho^{-2/3}}{dt}\right]$$

$$= \vec{\nabla}\cdot\left[\left(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right)\frac{3p}{\nu_{\text{coll.}}}\left(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right) : \vec{\nabla}\vec{u}\right]$$ (by flux freezing and continuity eqn.)

two derivatives of velocity times pressure times collision timescale

$\Rightarrow$ viscosity!

Only affects parallel gradients of parallel momentum, since particle-particle communication is stifled across $\vec{B}$.

See Kunz 2011 for how this affects buoyancy in intracluster medium.

Lessons:

(1) Magnetic fields change stability and transport in astrophysical plasmas. Be careful with hydrodynamics!

(2) Have an astrophysical problem to solve? Consider adding a magnetic field! Never underestimate its influence!

(3) BUT, be sure your equations are applicable. Is the system fully ionized? Is it collisional? Ask first, compute second.

(4) Even though $\beta \equiv \frac{8\pi n T}{B^2}$ might be very large, the plasma is still likely magnetized, and so you should care what $\vec{B}$ is doing to the transport properties of the plasma (even though it may exert no dynamical effect through tension/pressure).

(5) They are dwindling, but there are still gems in linear theory. Master it and learn how to interpret results physically. Then, if you're so inclined, use a computer to solve nonlinear evolution.

for the

What I missed: Rayleigh-Taylor & Parker instabilities, Kelvin-Helmholtz instability, thermal (Field) instability, gravitational equilibria and stability, high-energy astro (e.g. shocks, Fermi acceleration, cosmic rays), reconnection, Alfvénic turbulence theory ..........