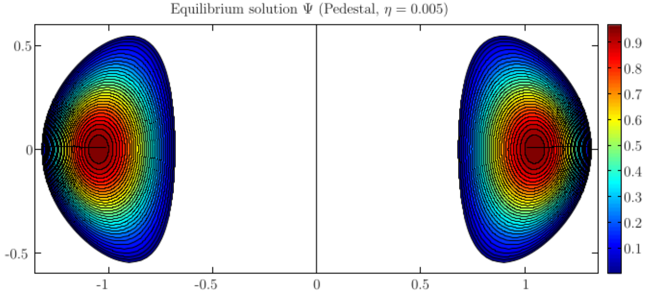
Introduction to MagnetoHydroDynamics (MHD)



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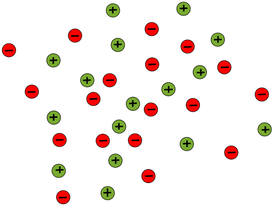
SULI Introductory Course in Plasma Physics, June 12, 2018

PART I: DESCRIBING A FUSION PLASMA

Method I: Self-consistent particle pushing



Natural idea: Move each particle according to $\mathbf{F}_p = m_p \mathbf{a}_p$

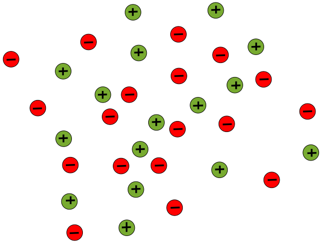
- **Difficulty 1**: There are MANY particles, $N \sim 10^{20} - 10^{22}$ in magnetic fusion grade plasmas
- **Difficulty 2**: $\mathbf{F}_p$ depends on the position and velocity of all the other particles. $\mathbf{F}_p$ is expensive to compute
  e.g.: for electrostatic electric field force

$$\mathbf{F}_p = q_p \sum_{j=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\mathbf{x}_j - \mathbf{x}_p|^2}$$

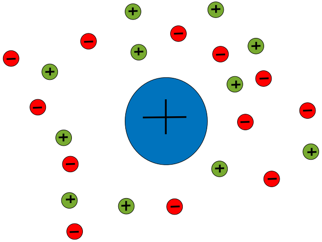
- Problem still not tractable even with the most powerful computers when $N \sim 10^{20} - 10^{22}$ and best algorithms

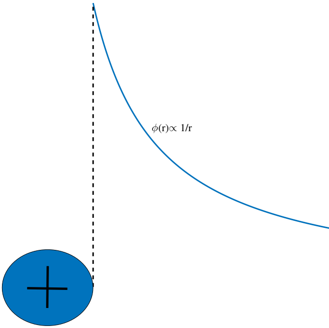
Debye shielding

Even if computers could solve this problem, should we ask them to?

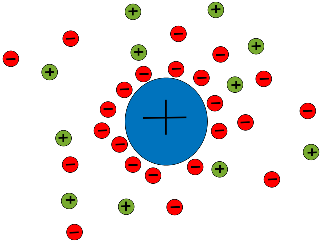


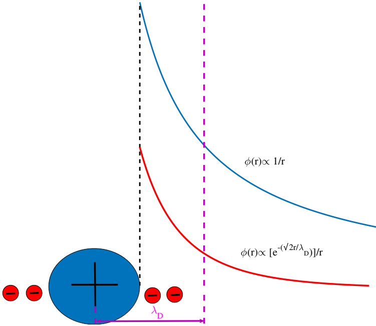
Debye shielding





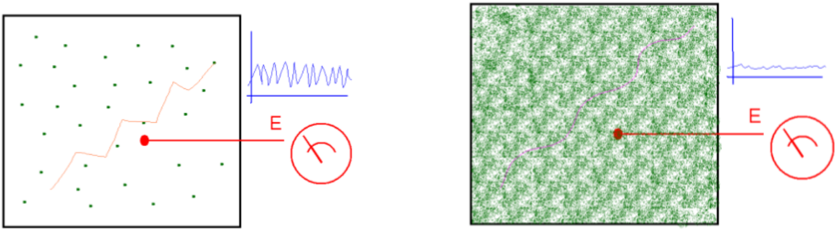
Debye shielding





- Local charge imbalance shielded within a few λ_D
- $\lambda_D = \frac{\epsilon_0 T}{e^2 n}$ is called the Debye length

Method II: For weakly coupled plasmas, coarse-grain average in phase space



- Weakly coupled plasma: large # of particles in any volume of size λ_D^3
- A large fraction of scientifically interesting plasmas are weakly coupled
- For weakly coupled plasmas, replace the discrete particles with smooth distribution function $f(\mathbf{x}, \mathbf{v}, t)$ defined so that

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v} = \#\text{ of particles in 6D phase-space volume } d\mathbf{x} d\mathbf{v}$$

Distribution function and Vlasov equation

- Macroscopic (fluid) quantities are velocity moments of f

$$n(\mathbf{x},t) = \iiint f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Density}$$

$$n\mathbf{V}(\mathbf{x},t) = \iiint \mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Mean flow}$$

$$\mathbf{P}(\mathbf{x},t) = m\iiint (\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})fd\mathbf{v} \qquad \text{Pressure tensor}$$

- Conservation of f along the phase-space trajectories of the particles determines the time evolution of f:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla f + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}} f = 0$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

$$\Rightarrow \qquad \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = 0$$

This is the Vlasov equation

The Boltzmann equation

- In fusion plasmas, we separate, leading to the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

This equation to be combined with Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

- Nonlinear, integro-differential, 6-dimensional PDE – **Challenging**
- Describes phenomena on widely varying length ($10^{-5} - 10^{3}$ m) and time ($10^{-12} - 10^{2}$ s) scales
- Still not a piece of cake, and never solved as such for fusion plasmas

Moment approach

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

- Taking the integrals $\iiint d\mathbf{v}$, $\iiint m\mathbf{v}d\mathbf{v}$ and $\iiint mv^2/2d\mathbf{v}$ of this equation, we obtain the exact **fluid equations**:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0 \qquad \text{Continuity}$$

$$mn \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) - \nabla \cdot \mathbf{P}_s + \mathbf{R}_s \qquad \text{Momen}$$

$$\frac{d}{dt} \left(\frac{3}{2} p_s \right) + \frac{5}{2} p_s \nabla \cdot \mathbf{V}_s + \boldsymbol{\pi}_s : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = 0 \qquad \text{(Energy)}$$

with $\mathbf{P}_s = p_s \mathbf{I} + \boldsymbol{\pi}_s$.

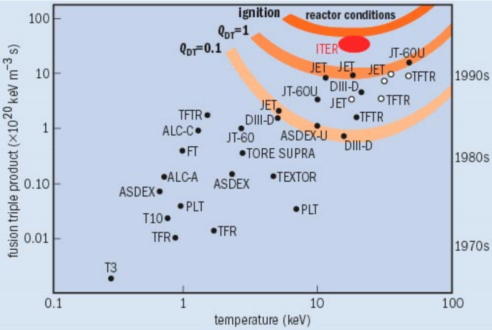
- Closure problem: for each moment, we introduce a new unknown $\Rightarrow$ End up with too many unknowns
- Need to make approximations to close the moment hierarchy

Kinetic models vs Fluid models

- For some fusion applications/plasma regimes (heating and current drive, transport), kinetic treatment cannot be avoided
- Simplify and reduce dimensionality of the Vlasov equation with approximations:
  - Strong magnetization : Gyrokinetic equation
  - Small gyroradius compared to relevant length scales : Drift kinetic equation
  - Vanishing gyroradius : Kinetic MHD
- In contrast, fluid models are based on approximate expressions for higher order moments (off-diagonal entries in pressure tensor, heat flux) in terms of lower order quantities(density, velocity, diagonal entries in pressure tensor)
- We will now focus on the relevant regime and the approximations made to derive a widely used fluid model: the ideal MHD model

PART II: THE IDEAL MHD MODEL

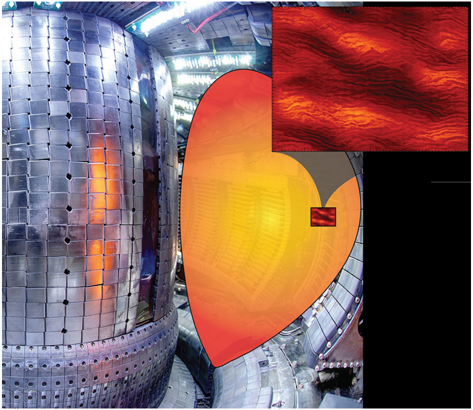
Lawson Criterion and MHD

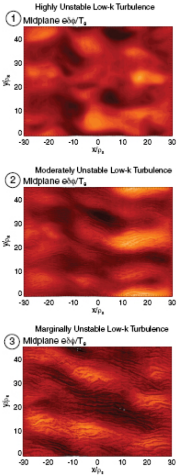


Condition for ignition: $\boxed{p\tau_E \geq 8\ bar.s}$ $T_{min} \sim 15\text{keV}$

- The maximum p is limited by the stability properties
  Job of MHD
- The maximum τ_E is determined by the confinement properties
  Job of kinetic models

Kinetic = Expensive simulations





Simulation run on NERSC Edison supercomputer

Each simulation required 17000 processors and $\sim$ 37 days ($\sim$ 15M CPU hours)

Work by N. Howard *et al.* (MIT PSFC)

Philosophy

- The purpose of ideal MHD is to study the macroscopic behavior of the plasma
- Use ideal MHD to design machines that avoid large scale instabilities
- Regime of interest
  - Typical length scale: the minor radius of the device $a \sim 1m$
    Wave number k of waves and instabilitities considered: $k \sim 1/a$
  - Typical velocities: Ion thermal velocity speed $v_T \sim 500km/s$
  - Typical time scale: $\tau_{MHD} \sim a/v_T \sim 2\mu s$
    Frequency ω_{MHD} of associated waves/instabilities $\omega_{MHD} \sim 500kHz$

Example: Vertical instability

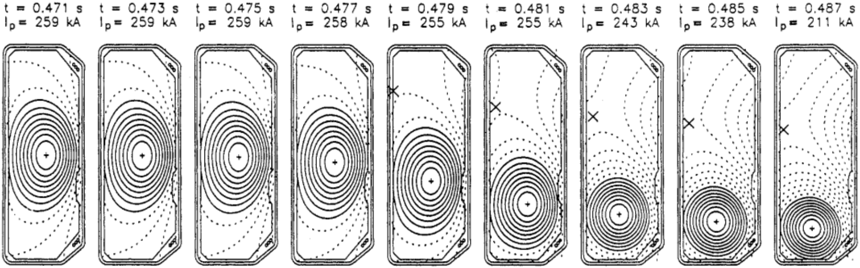


FIG. 2. Loss of vertical position control following a feedback cut at 0.470 s.

Figure from F. Hofmann *et al.*, Nuclear Fusion **37** 681 (1997)

Ideal MHD - Maxwell's equations

- $a \gg \lambda_D$, the distance over which charge separation can take place in a plasma
  $\Rightarrow$ On the MHD length scale, the plasma is neutral : $n_i = n_e$

- $\omega_{MHD}/k \ll c$ and $v_{T_i} \ll v_{T_e} \ll c$ so we can neglect the displacement current in Maxwell's equations:

$$n_i = n_e$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Ideal MHD - Momentum equation

- $a \gg \lambda_D$ and $a \gg r_{Le}$ (electron Larmor radius)
- $\omega_{MHD} \ll \omega_{pe}, \omega_{MHD} \ll \omega_{ce}$
- The ideal MHD model assumes that on the time and length scales of interest, the electrons have an infinitely fast response time to changes in the plasma
- Mathematically, this can be done by taking the limit $m_e \to 0$
- Adding the ion and electron momentum equation, we then get

$$\rho \frac{d\mathbf{V}}{dt} - \mathbf{J} \times \mathbf{B} + \nabla p = -\nabla \cdot (\boldsymbol{\pi}_i + \boldsymbol{\pi}_e)$$

  where $\rho = m_i n$ and $\mathbf{V}$ is the ion fluid velocity
- If the condition $v_{Ti}\tau_{ii}/a \ll 1$ is satisfied in the plasma

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \qquad \text{(Ideal MHD momentum equation)}$$

IDEAL MHD - ELECTRONS

- In the limit $m_e \to 0$, the electron momentum equation can be written as

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{1}{en} \left(\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e + \mathbf{R}_e \right)$$

- This is called the generalized Ohm's law
- Different MHD models (resitive MHD, Hall MHD) keep different terms in this equation
- If $r_{Li}/a \ll 1$, $v_{Ti}\tau_{ii}/a \ll 1$, and $(m_e/m_i)^{1/2}(r_{Li}/a)^2(a/v_{Ti}\tau_{ii}) \ll 1$, the momentum equation becomes the ideal Ohm's law

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$$

- The ideal MHD plasma behaves like a perfectly conducting fluid

Energy equation

- Define the total plasma pressure $p = p_i + p_e$
- Add electron and ion energy equations
- Under the conditions $r_{Li}/a \ll 1$ and $v_{Ti}\tau_{ii}/a \ll 1$, this simplifies as

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0$$

- Equation reminiscent of $pV^\gamma = Cst$: the ideal MHD plasma behaves like a monoatomic ideal gas undergoing a reversible adiabatic process

Ideal MHD - Summary

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \\
&\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \\
&\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0 \\
&\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0} \\
&\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
&\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\
&\nabla \cdot \mathbf{B} = 0
\end{aligned}
$$

Valid under the conditions

$$
\left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{v_i \tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2 \left(\frac{m_e}{m_i}\right)^{1/2} \frac{a}{v_{Ti}\tau_{ii}} \ll 1
$$

VALIDITY OF THE IDEAL MHD MODEL (I)

- Are the conditions for the validity of ideal MHD

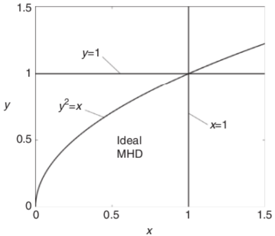
$$\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{v_i\tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2\left(\frac{m_e}{m_i}\right)^{1/2}\frac{a}{v_{Ti}\tau_{ii}} \ll 1$$

mutually compatible?

- Define $x = (m_i/m_e)^{1/2}(v_{Ti}\tau_{ii}/a)$, $y = r_{Li}/a$.

$x \ll 1$ (High collisionality) $y \ll 1$ (Small ion Larmor radius)

$y^2/x \ll 1$ (Small resistivity)

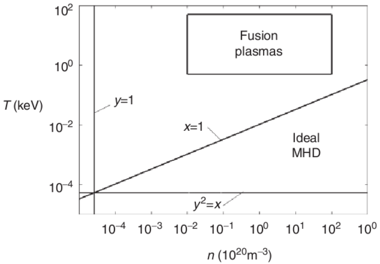


There exists a regime for which ideal MHD is justified (Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

Is that the regime of magnetic confinement fusion?

Validity of the ideal MHD model (II)

- Express three conditions in terms of usual physical parameters: n, T, a
- For tokamak-like pressures and $a = 1m$, we find:



The regime of validity of ideal MHD does NOT coincide with the fusion plasma regime (Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

The collisionality of fusion plasmas is too low for the ideal MHD model to be valid.

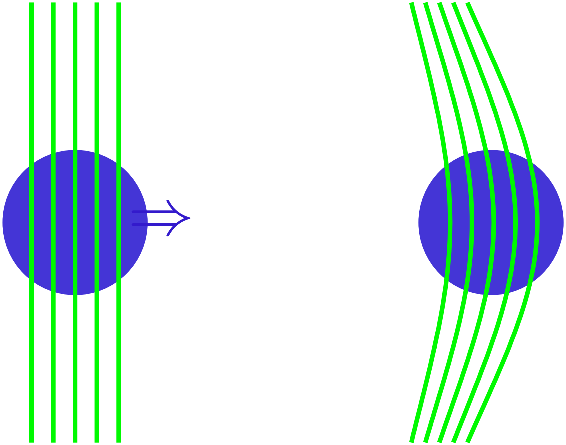
Is that a problem?

Validity of the ideal MHD model (III)

- It turns out that ideal MHD often does a very good job at predicting stability limits for macroscopic instabilities
- This is not due to luck but to subtle physical reasons
- One can show that collisionless kinetic models for macroscopic instabilities are more optimistic than ideal MHD
- This is because ideal MHD is accurate for dynamics perpendicular to the fields lines
- Designs based on ideal MHD calculations are conservative designs

Frozen in law

- $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$: in the frame moving with the plasma, the electric field is zero
- The plasma behaves like a perfect conductor
- The magnetic field lines are “frozen” into the plasma motion



MAGNETIC RECONNECTION

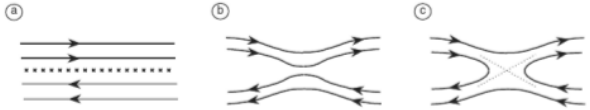
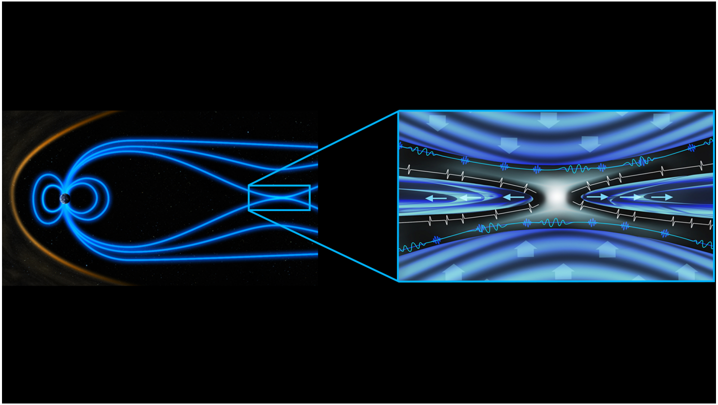


Image from *Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas* by J.P. Goedbloed and S. Poedts, Cambridge University Press (2004)

- Magnetic reconnection: a key phenomenon in astrophysical, space, and fusion plasmas
- Cannot happen according to ideal MHD
- Need to add additional terms in Ohm's law to allow reconnection: resistivity, off-diagonal pressure tensor terms, electron inertia, …

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} + \frac{1}{ne}\mathbf{J} \times \mathbf{B} - \frac{1}{ne}\nabla \cdot \mathbf{P}_e + \frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t}$$

- Associated instabilities take place on longer time scales than τ_{MHD}



PART III: MHD EQUILIBRIUM

Equilibrium state

- By equilibrium, we mean steady-state: $\partial/\partial t = 0$
- Often, for simplicity and/or physical reasons, we focus on static equilibria: $\mathbf{V} = \mathbf{0}$

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\mathbf{J} \times \mathbf{B} = \nabla p$$

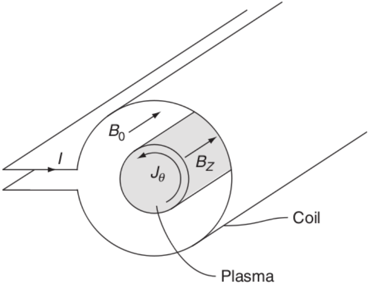
A more condensed form is

$$\nabla \cdot \mathbf{B} = 0 \qquad (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$$

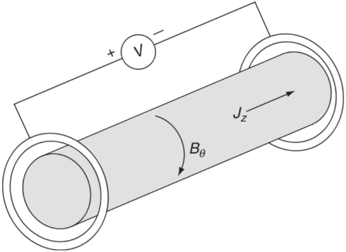
Note that the density profile does not appear

1D EQUILIBRIA (I)

θ-pinch



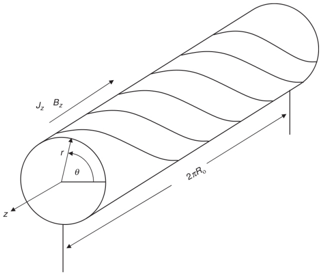
Z-pinch



Combine the two to get....

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

1D equilibria (II)



Screw pinch

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Equilibrium quantities only depend on r
- Plug into $\nabla \cdot \mathbf{B} = 0$, $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ to find:

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

Balance between plasma pressure, magnetic pressure, and magnetic tension

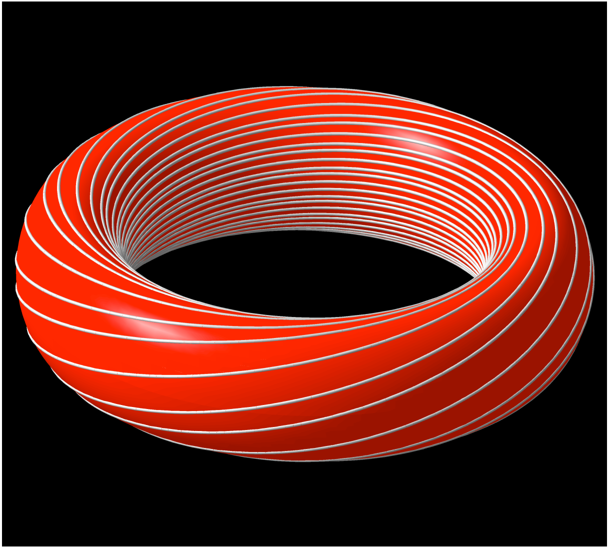
- Two free functions define equilibrium: e.g. B_z and p, or B_θ and B_z

General equilibria

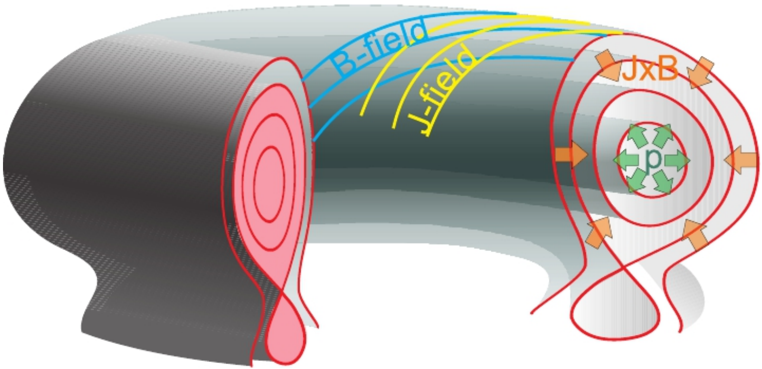
Equilibrium relation

$$\mathbf{J} \times \mathbf{B} = \nabla p$$

$$\begin{cases} (\mathbf{J} \times \mathbf{B} = \nabla p) \cdot \mathbf{B} \quad \Rightarrow \quad \mathbf{B} \cdot \nabla p = 0 \\ \text{Magnetic field is tangent to surfaces of constant pressure} \\ \\ (\mathbf{J} \times \mathbf{B} = \nabla p) \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \mathbf{J} \cdot \nabla p = 0 \\ \text{Current density is tangent to surfaces of constant pressure} \end{cases}$$



(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)



- The regions of constant pressure are nested toroidal surfaces
- Magnetic fields and currents lie on these nested surfaces

Grad-Shafranov equation

- Surfaces of constant pressure coincide with surfaces of constant magnetic flux Ψ
- These are given by the Grad-Shafranov equation

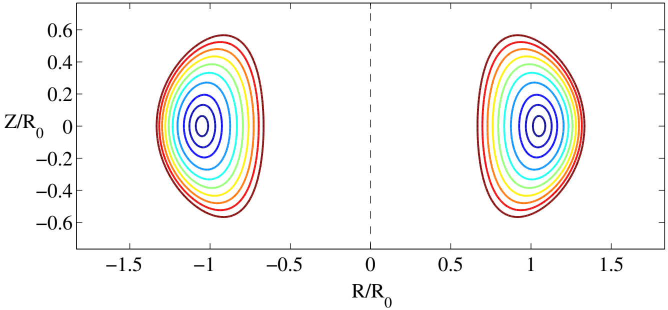
$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)+\frac{\partial^2\Psi}{\partial Z^2}=-\mu_0 R^2\frac{dp}{d\Psi}-F\frac{dF}{d\Psi}$$

- Second-order, nonlinear, elliptic PDE. Derived independently by H. Grad[1] and V.D. Shafranov[2].
- The free functions p and F determine the nature of the equilibrium
- In general, the GSE has to be solved numerically

[1] *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy*, Vol. 31, p.190

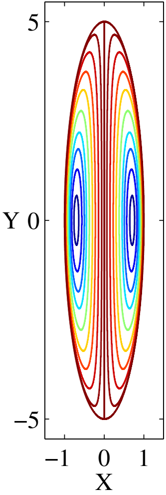
[2] *Sov. Phys. JETP* **6**, 545 (1958)

EXAMPLES (I)



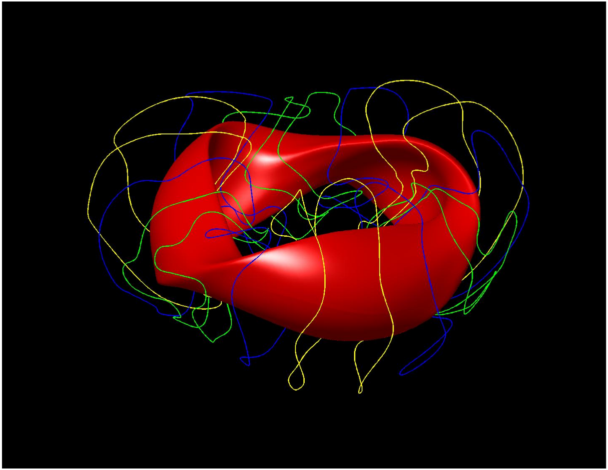
Grad-Shafranov equilibrium for JET tokamak

Examples (II)



Grad-Shafranov equilibrium for Field Reversed Configuration

3D EQUILIBRIA

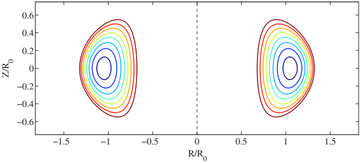


$$\partial/\partial\phi \neq 0$$

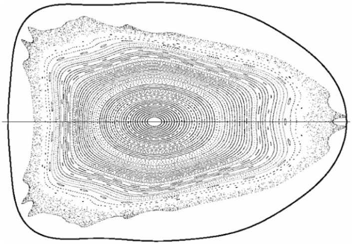
3D EQUILIBRIA

- Equilibrium equations $\nabla \cdot \mathbf{B} = 0\ ,\ (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ still hold
- Existence of nested toroidal surfaces not guaranteed anymore

Tokamak



Stellarator (3D)



- Computing 3D equilibria fast and accurately still a challenge
- Several existing codes, based on different assumptions/approximations and used to design and study stellarators: VMEC, PIES, SPEC, HINT, NSTAB

PART IV: MHD STABILITY

What do we mean by MHD stability?

- That the plasma is initially in equilibrium does not mean it is going to remain there
- The plasma is constantly subject to perturbations, small and large
- The purpose of stability studies is to find out how the plasma will react to these perturbations
  - Will it try to return to the initial steady-state?
  - Will it find a new acceptable steady-state?
  - Will it collapse?

A mechanical analog

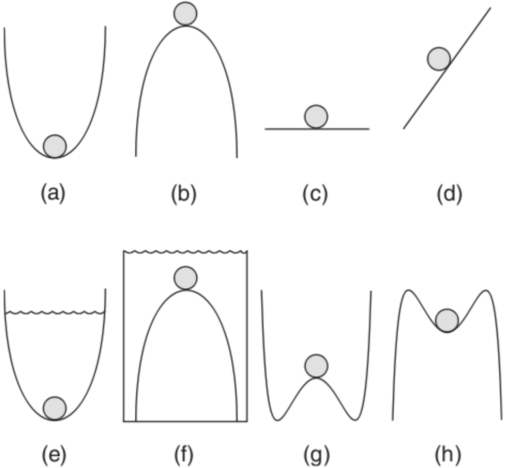


Figure from J.P. Freidberg, *Ideal MHD*, Cambridge University Press (2014)

Solving full nonlinear MHD equations

- Here is an idea to study stability of a magnetically confined plasma:
  - Choose a satisfying plasma equilibrium
  - Perturb it
  - Solve the full MHD equations with a computer
  - Analyze results
- Such an approach provides knowledge of the entire plasma dynamics
- There exist several numerical codes that can do that, for various MHD models (not only ideal): M3D, M3D-C1, NIMROD
- Computationally intensive
- Get more information than one needs?

MHD stability: Illustration

Vertical displacement event

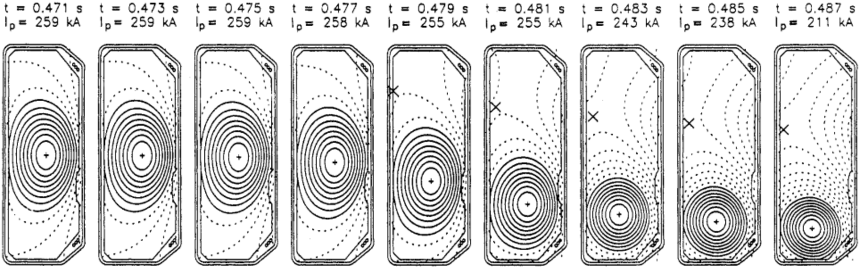
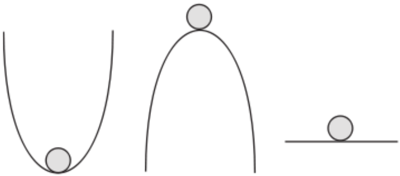


FIG. 2. Loss of vertical position control following a feedback cut at 0.470 s.

Figure from F. Hofmann *et al.*, Nuclear Fusion **37** 681 (1997)

Linear stability (I)

- Ideal MHD dynamics can be so fast and detrimental that one may often require linear stability for the equilibrium



- This can simplify the mathematical analysis tremendously
- Start with an MHD equilibrium:
  $\nabla \cdot \mathbf{B}_0 = 0\ ,\ (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \mu_0 \nabla p_0$
- Take full ideal MHD equations, and write $Q = Q_0(\mathbf{r}) + Q_1(\mathbf{r}, t)$ for each physical quantity, where Q_1 is considered very small compared to Q_0
- Drop all the terms that are quadratic or higher orders in the quantities Q_1 (linearization)

LINEAR STABILITY (II)

$$
\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0 \\
&\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{J}_1 \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}_1 - \nabla p_1 \\
&\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}_1 = 0 \\
&\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) \\
&\nabla \cdot \mathbf{B}_1 = 0 \\
&\mu_0 \mathbf{J}_1 = \nabla \times \mathbf{B}_1
\end{aligned}
$$

- By design, the system is now linear in the unknown quantities ρ_1, $\mathbf{v}_1$, $\mathbf{J}_1$, $\mathbf{B}_1$, p_1
- Much easier to solve in a computer – or by hand!

Illustration: Waves in ideal MHD (I)

- Consider the stability of an infinite, homogeneous plasma:

$$\mathbf{B} = B_0 \overrightarrow{e_z}$$
$$\mathbf{J} = \overrightarrow{0}$$
$$p = p_0$$
$$\rho = \rho_0$$
$$\mathbf{v} = \mathbf{0}$$

- Write $\mathbf{v}_1 = \partial \boldsymbol{\xi}(\mathbf{r})/\partial t$
- Expand $\boldsymbol{\xi}(\mathbf{r})$ as

$$\boldsymbol{\xi}(\mathbf{r}) = \hat{\boldsymbol{\xi}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

- Plug into linear system, and solve by hand
- Dynamics is anisotropic because of the magnetic field: $\mathbf{k}_\perp + k_\parallel \mathbf{e}_z$

Illustration: Waves in ideal MHD (II)

- Writing the expression for each component, we get the system

$$\begin{bmatrix} \omega^2 - k_{\|}^2 v_A^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - k_{\perp}^2 v_S^2 & -k_{\|} k_{\perp} v_S^2 \\ 0 & -k_{\perp} k_{\|} v_S^2 & \omega^2 - k_{\|}^2 v_S^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Two key velocities appear:

$$v_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \qquad v_S = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

v_A is called the *Alfvén velocity*, in honor of Hannes Alfvén, the Swedish scientist who first described MHD waves

v_s is the adiabatic sound speed

Illustration: Waves in ideal MHD (III)

$$\begin{bmatrix} \omega^2 - k_{\|}^2 v_A^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - k_{\perp}^2 v_S^2 & -k_{\|} k_{\perp} v_S^2 \\ 0 & -k_{\perp} k_{\|} v_S^2 & \omega^2 - k_{\|}^2 v_S^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

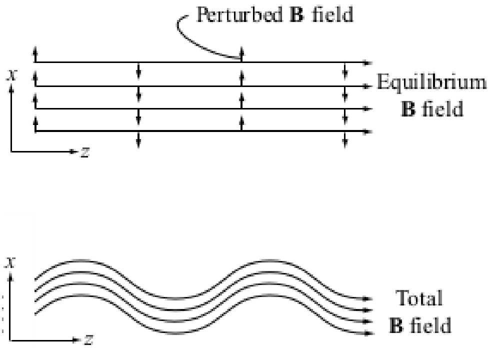
- For nontrivial solutions, determinant of the matrix should be 0
- This leads to the following three possibilities for ω^2:

$$\omega^2 = k_{\|}^2 v_A^2 \ , \ \omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right) \left[1 \pm \sqrt{1 - 4\frac{k_{\|}^2}{k^2} \frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

- One can see that $\omega^2 \geq 0 \Rightarrow$ The infinite homogeneous magnetized plasma is always MHD stable
- Some of these modes become unstable in magnetic fusion configurations, because of gradients and field line curvature

Shear Alfvén wave

- Branch $\omega^2 = k_{\|}^2 v_A^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

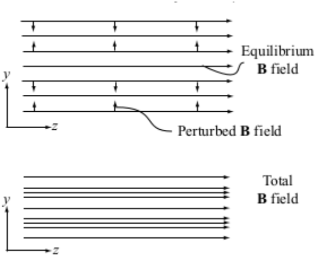
- Transverse wave
- Balance between plasma inertia and field line tension
- Incompressible $\Rightarrow$ often the most unstable MHD mode in fusion devices

Fast magnetosonic wave

- Fast magnetosonic wave given by

$$\omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 + \sqrt{1 - 4\frac{k_\parallel^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

- Simplifies in the limit $v_S^2 \ll v_A^2$: it becomes the compressional Alfvén wave, $\omega^2 = k^2 v_A^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

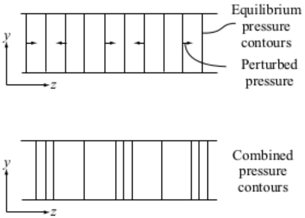
- Oscillation between plasma kinetic energy and magnetic compressional energy

Slow magnetosonic wave

- Slow magnetosonic wave given by

$$\omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 - \sqrt{1 - 4\frac{k_\parallel^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

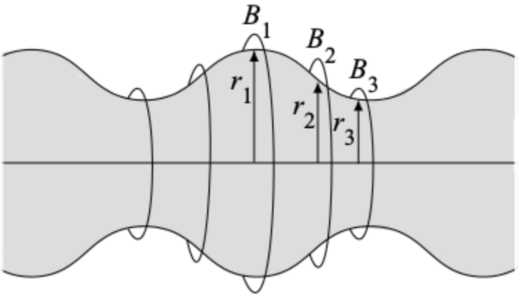
- Physics simplifies in the limit $v_S^2 \ll v_A^2$: it is then called the sound wave, with dispersion relation $\omega^2 = k^2 v_S^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Plasma motion parallel to field lines, compressible
- Oscillation between plasma kinetic energy and plasma internal energy (plasma pressure)

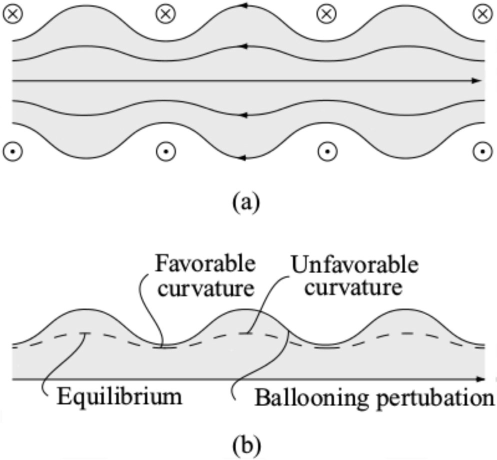
Common MHD instabilities (I)



Interchange instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

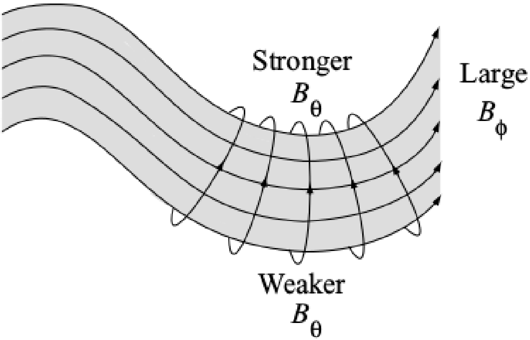
Common MHD instabilities (II)



Ballooning instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

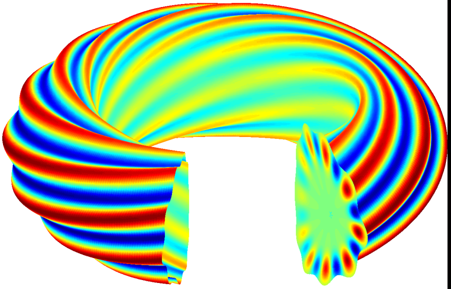
Common MHD instabilities (III)

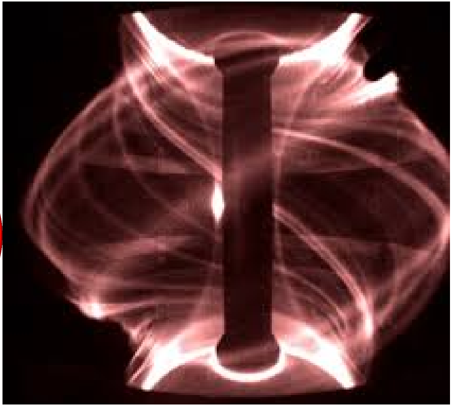


Kink instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

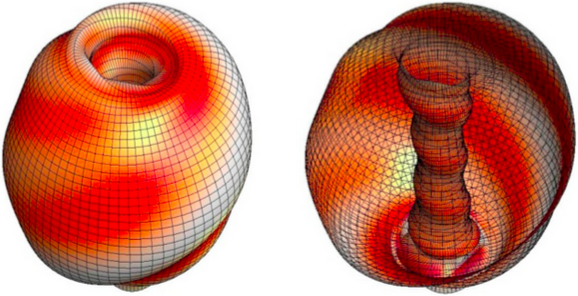
Ballooning modes in tokamaks





(Right image from *http://www.ccfe.ac.uk/assets/Documents/AIPCONFPROC103p174.pdf* by J.W. Connor *et al.*)

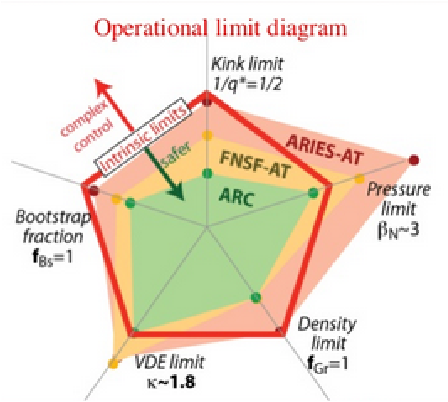
Kink ballooning modes in tokamaks



Reconstruction of experimentally observed kink ballooning mode

(Figure from *U.S. Burning Plasma Organization eNews February 28, 2013 (Issue 69)* by S.A. Sabbagh *et al.*)

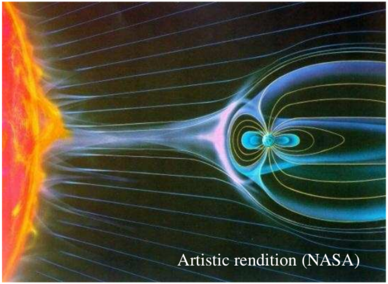
KEY ROLE OF MHD IN REACTOR DESIGN



- Maximum achievable pressure set by ballooning modes
- Maximum achievable pressure set by kink modes
- Maximum elongation set by vertical instability

MHD BEYOND FUSION: SOLAR PHYSICS

(Thanks to Jean C. Perez, FIT)



Artistic rendition (NASA)

Credit: M. Druckmller (Brno University of Technology)

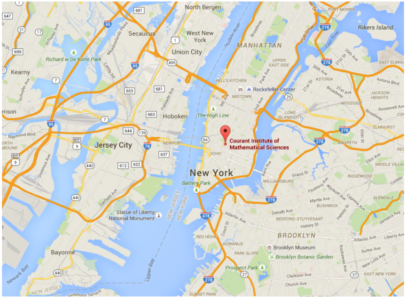
- Solar wind
- Solar corona
- Space weather

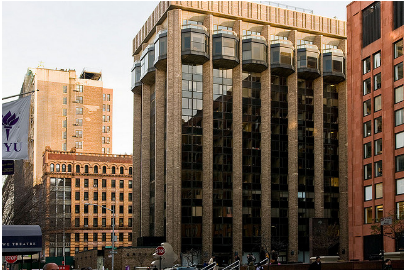
The ITER of solar physics: The Parker Solar Probe

- SPP will be first mission to visit the Sun
- Distance of closest approach: 9.5 solar radii
- Better understand:
  - Coronal heating and solar wind acceleration
  - Production, evolution and transport of solar energetic particles
- Expected launch date 2018

POST SCRIPTUM: THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES (CIMS) AT NYU

CIMS IN MANHATTAN





Home > Graduate Schools > Math > Applied Math

Applied Math

Ranked in 2014, part of Math

The applied math discipline is geared toward students who hope to use their mathematical prowess in business organizations, government agencies and other job sites. These are the best graduate schools for applied math.

Specialty: Applied Math

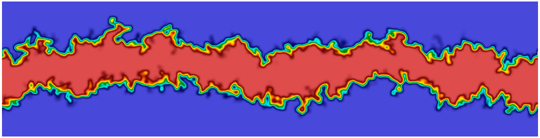
Rank	School name
#1	New York University New York, NY
#2	University of California—Los Angeles Los Angeles, CA
#3	California Institute of Technology Pasadena, CA
#4	Massachusetts Institute of Technology Cambridge, MA
#5 Tie	Brown University Providence, RI
#5 Tie	University of Minnesota—Twin Cities Minneapolis, MN

- Abel prize in 2005, 2007, 2009 and 2015
- 18 members of the National Academy of Sciences

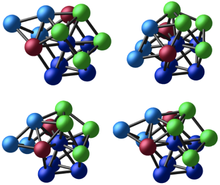
  5 members of the National Academy of Engineering

- Specialization in applied math, scientific computing, mathematical analysis
- Particular emphasis on Partial Differential Equations
- PhD programs in Mathematics, Atmosphere and Ocean Science, Computational Biology
- Masters of Science in Mathematics, Masters of Science in Scientific Computing, Masters of Science in Data Science, Masters of Science in Math Finance
- $\sim$ 60 faculty
  $\sim$ 80 PhD students

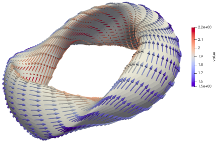
Microfluidics



Soft matter physics



Plasma physics



MFD division at CIMS

- Founded by Harold Grad in 1954
- 2 faculty, 4 post-docs, 1 graduate student, 1 undergraduate student
- Work on MHD, wave propagation, kinetic models
  Analytic “pen and paper” work
  Development of new numerical solvers
- Collaboration with colleagues specialized in scientific computing, computational fluid dynamics, stochastic calculus, etc.
- Funding currently available for PhD students
- Feel free to contact me if you have any questions