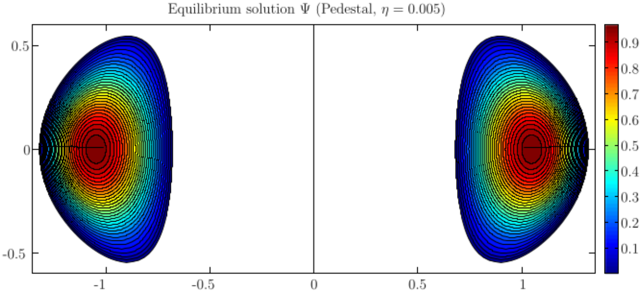
Introduction to MagnetoHydroDynamics (MHD)



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PART I: DESCRIBING A FUSION PLASMA

METHOD I: SELF-CONSISTENT PARTICLE PUSHING

- Intuitive idea: solve for the motion of all particles iteratively, combining Newton's law with Maxwell equations
- At each time step i, solve

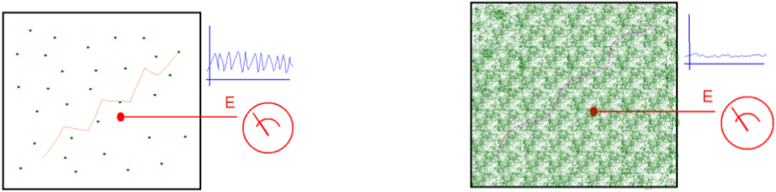
$$m\frac{d^2\mathbf{x}_k^{(i)}}{dt} = q_k\left(\mathbf{E}(\mathbf{x}_k)^{(i-1)} + \frac{d\mathbf{x}_k^{(i)}}{dt} \times \mathbf{B}^{(i-1)}(\mathbf{x}_k)\right) \qquad k = 1, \ldots, N$$

$$\left(\frac{\partial \mathbf{E}}{\partial t}\right)^i = c^2\nabla \times \mathbf{B}^{(i-1)} - \mu_0 c^2 \sum_{k=1}^{N} q_k \frac{d\mathbf{x}_k^i}{dt}\delta(\mathbf{x} - \mathbf{x}_k^i)$$

$$\left(\frac{\partial \mathbf{B}}{\partial t}\right)^i = -\nabla \times \mathbf{E}^i$$

- Even with best available numerical solvers, problem still not tractable even with the most powerful computers when $N \sim 10^{20} - 10^{22}$ as in magnetic fusion grade plasmas

Method II: Coarse-grain average in phase space



(From G. Lapenta's: https://perswww.kuleuven.be/ u0052182/weather/pic.pdf)

- For hot and diffuse systems with large # of particles, following every single particle is a waste of time and resources
- Replace the discrete particles with smooth distribution function $f(\mathbf{x}, \mathbf{v}, t)$ defined so that

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v}$$

is the # of particles in the infinitesimal six-dimensional phase-space volume $d\mathbf{x}d\mathbf{v}$.

Distribution function and Vlasov equation

- Macroscopic (fluid) quantities are velocity moments of f

$$n(\mathbf{x},t) = \iiint f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Density}$$

$$n\mathbf{V}(\mathbf{x},t) = \iiint \mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Mean flow}$$

$$\mathbf{P}(\mathbf{x},t) = m\iiint (\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V}) f d\mathbf{v} \qquad \text{Pressure tensor}$$

- Conservation of f along the phase-space trajectories of the particles determines the time evolution of f:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla f + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}} f = 0$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

$$\Rightarrow \qquad \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f = 0$$

This is the Vlasov equation

The Boltzmann equation

- In fusion plasmas, we separate, leading to the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

  This equation to be combined with Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

- Nonlinear, integro-differential, 6-dimensional PDE
- Describes phenomena on widely varying length ($10^{-5} - 10^{3}$ m) and time ($10^{-12} - 10^{2}$ s) scales
- Still not a piece of cake, and never solved as such for fusion plasmas

Moment approach

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t}\right)_c$$

- Taking the integrals $\iiint d\mathbf{v}$, $\iiint m\mathbf{v}d\mathbf{v}$ and $\iiint mv^2/2d\mathbf{v}$ of this equation, we obtain the exact **fluid equations**:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0 \qquad \text{Continuity}$$

$$mn\left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s\right) = q_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) - \nabla \cdot \mathbf{P}_s + \mathbf{R}_s \qquad \text{Momen}$$

$$\frac{d}{dt}\left(\frac{3}{2}p_s\right) + \frac{5}{2}p_s \nabla \cdot \mathbf{V}_s + \boldsymbol{\pi}_s : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = 0 \qquad \text{(Energy)}$$

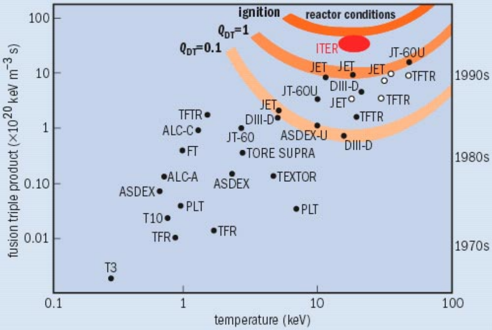
  with $\mathbf{P}_s = p_s \mathbf{I} + \boldsymbol{\pi}_s$.
- Closure problem: for each moment, we introduce a new unknown $\Rightarrow$ End up with too many unknowns
- Need to make approximations to close the moment hierarchy

Kinetic models vs Fluid models

- For some fusion applications/plasma regimes (heating and current drive, transport), kinetic treatment cannot be avoided
- Simplify and reduce dimensionality of the Vlasov equation with approximations:
  - Strong magnetization : Gyrokinetic equation
  - Small gyroradius compared to relevant length scales : Drift kinetic equation
  - Vanishing gyroradius : Kinetic MHD
- In contrast, fluid models are based on approximate expressions for higher order moments (off-diagonal entries in pressure tensor, heat flux) in terms of lower order quantities(density, velocity, diagonal entries in pressure tensor)
- We will now focus on the relevant regime and the approximations made to derive a widely used fluid model: the ideal MHD model

PART II: THE IDEAL MHD MODEL

Lawson Criterion and MHD



Condition for ignition: $\boxed{p\tau_E \geq 8\ bar.s}$ $T_{min} \sim 15\text{keV}$

- The maximum p is limited by the stability properties
  Job of MHD
- The maximum τ_E is determined by the confinement properties
  Job of kinetic models

Philosophy

- The purpose of ideal MHD is to study the macroscopic behavior of the plasma
- Use ideal MHD to design machines that avoid large scale instabilities
- Regime of interest
  - Typical length scale: the minor radius of the device $a \sim 1m$
    Wave number k of waves and instabilitities considered: $k \sim 1/a$
  - Typical velocities: Ion thermal velocity speed $v_T \sim 500km/s$
  - Typical time scale: $\tau_{MHD} \sim a/v_T \sim 2\mu s$
    Frequency ω_{MHD} of associated waves/instabilities $\omega_{MHD} \sim 500kHz$

EXAMPLE: VERTICAL INSTABILITY

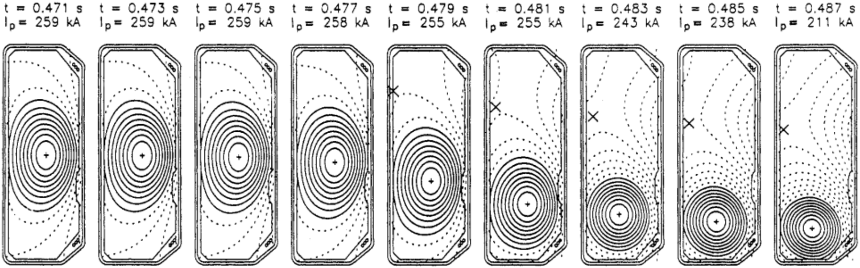


FIG. 2. Loss of vertical position control following a feedback cut at 0.470 s.

Figure from F. Hofmann *et al.*, Nuclear Fusion **37** 681 (1997)

Ideal MHD - Maxwell's equations

- $a \gg \lambda_D$, the distance over which charge separation can take place in a plasma
  $\Rightarrow$ On the MHD length scale, the plasma is neutral : $n_i = n_e$

- $\omega_{MHD}/k \ll c$ and $v_{T_i} \ll v_{T_e} \ll c$ so we can neglect the displacement current in Maxwell's equations:

$$
\begin{aligned}
& n_i = n_e \\
& \nabla \cdot \mathbf{B} = 0 \\
& \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
& \nabla \times \mathbf{B} = \mu_0 \mathbf{J}
\end{aligned}
$$

Ideal MHD - Momentum equation

- $a \gg \lambda_D$ and $a \gg r_{Le}$ (electron Larmor radius)
- $\omega_{MHD} \ll \omega_{pe}, \omega_{MHD} \ll \omega_{ce}$
- The ideal MHD model assumes that on the time and length scales of interest, the electrons have an infinitely fast response time to changes in the plasma
- Mathematically, this can be done by taking the limit $m_e \to 0$
- Adding the ion and electron momentum equation, we then get

$$\rho \frac{d\mathbf{V}}{dt} - \mathbf{J} \times \mathbf{B} + \nabla p = -\nabla \cdot (\boldsymbol{\pi}_i + \boldsymbol{\pi}_e)$$

  where $\rho = m_i n$ and $\mathbf{V}$ is the ion fluid velocity
- If the condition $v_{Ti}\tau_{ii}/a \ll 1$ is satisfied in the plasma

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \qquad \text{(Ideal MHD momentum equation)}$$

Ideal MHD - Electrons

- In the limit $m_e \to 0$, the electron momentum equation can be written as

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{1}{en} \left(\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e + \mathbf{R}_e \right)$$

- This is called the generalized Ohm's law
- Different MHD models (resitive MHD, Hall MHD) keep different terms in this equation
- If $r_{Li}/a \ll 1$, $v_{Ti}\tau_{ii}/a \ll 1$, and $(m_e/m_i)^{1/2}(r_{Li}/a)^2(a/v_{Ti}\tau_{ii}) \ll 1$, the momentum equation becomes the ideal Ohm's law

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$$

- The ideal MHD plasma behaves like a perfectly conducting fluid

Energy equation

- Define the total plasma pressure $p = p_i + p_e$
- Add electron and ion energy equations
- Under the conditions $r_{Li}/a \ll 1$ and $v_{Ti}\tau_{ii}/a \ll 1$, this simplifies as

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0$$

- Equation reminiscent of $pV^\gamma = Cst$: the ideal MHD plasma behaves like a monoatomic ideal gas undergoing a reversible adiabatic process

Ideal MHD - Summary

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Valid under the conditions

$$\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{v_i \tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2\left(\frac{m_e}{m_i}\right)^{1/2}\frac{a}{v_{Ti}\tau_{ii}} \ll 1$$

Validity of the ideal MHD model (I)

- Are the conditions for the validity of ideal MHD

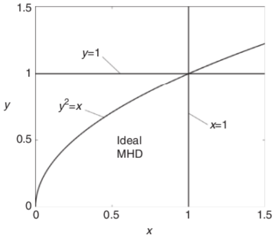
$$\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{v_i\tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2\left(\frac{m_e}{m_i}\right)^{1/2}\frac{a}{v_{Ti}\tau_{ii}} \ll 1$$

mutually compatible?

- Define $x = (m_i/m_e)^{1/2}(v_{Ti}\tau_{ii}/a)$, $y = r_{Li}/a$.

$x \ll 1$ (High collisionality) $y \ll 1$ (Small ion Larmor radius)

$y^2/x \ll 1$ (Small resistivity)

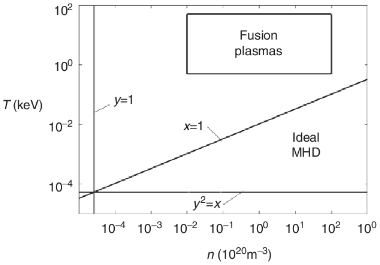


There exists a regime for which ideal MHD is justified (Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

Is that the regime of magnetic confinement fusion?

Validity of the ideal MHD model (II)

- Express three conditions in terms of usual physical parameters: n, T, a
- For tokamak-like pressures and $a = 1m$, we find:



The regime of validity of ideal MHD does NOT coincide with the fusion plasma regime (Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

The collisionality of fusion plasmas is too low for the ideal MHD model to be valid.

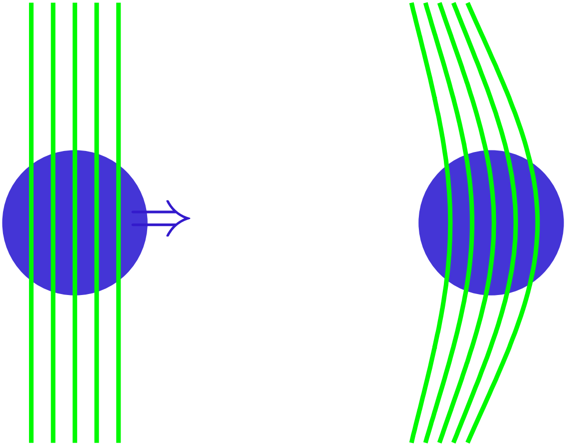
Is that a problem?

Validity of the ideal MHD model (III)

- It turns out that ideal MHD often does a very good job at predicting stability limits for macroscopic instabilities
- This is not due to luck but to subtle physical reasons
- One can show that collisionless kinetic models for macroscopic instabilities are more optimistic than ideal MHD
- This is because ideal MHD is accurate for dynamics perpendicular to the fields lines
- Designs based on ideal MHD calculations are conservative designs

Frozen in law

- $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$: in the frame moving with the plasma, the electric field is zero
- The plasma behaves like a perfect conductor
- The magnetic field lines are "frozen" into the plasma motion



MAGNETIC RECONNECTION

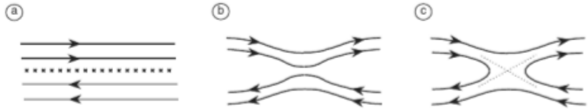
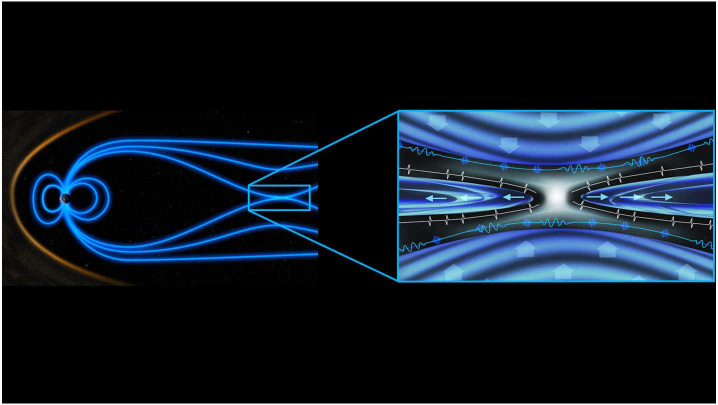


Image from *Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas* by J.P. Goedbloed and S. Poedts, Cambridge University Press (2004)

- Magnetic reconnection: a key phenomenon in astrophysical, space, and fusion plasmas
- Cannot happen according to ideal MHD
- Need to add additional terms in Ohm's law to allow reconnection: resistivity, off-diagonal pressure tensor terms, electron inertia, …

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} + \frac{1}{ne}\mathbf{J} \times \mathbf{B} - \frac{1}{ne}\nabla \cdot \mathbf{P}_e + \frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t}$$

- Associated instabilities take place on longer time scales than τ_{MHD}



PART III: MHD EQUILIBRIUM

EQUILIBRIUM STATE

- By equilibrium, we mean steady-state: $\partial/\partial t = 0$
- Often, for simplicity and/or physical reasons, we focus on static equilibria: $\mathbf{V} = \mathbf{0}$

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\mathbf{J} \times \mathbf{B} = \nabla p$$

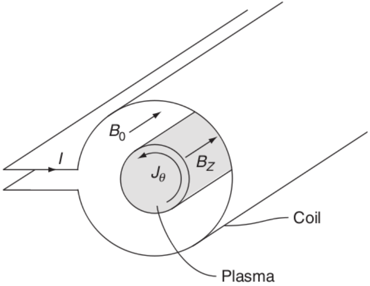
A more condensed form is

$$\nabla \cdot \mathbf{B} = 0 \qquad (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$$

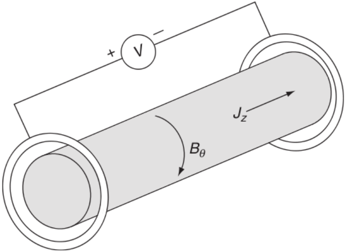
Note that the density profile does not appear

1D EQUILIBRIA (I)

θ-pinch



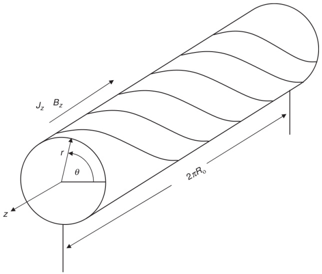
Z-pinch



Combine the two to get....

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

1D EQUILIBRIA (II)



Screw pinch

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Equilibrium quantities only depend on r
- Plug into $\nabla \cdot \mathbf{B} = 0$, $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ to find:

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

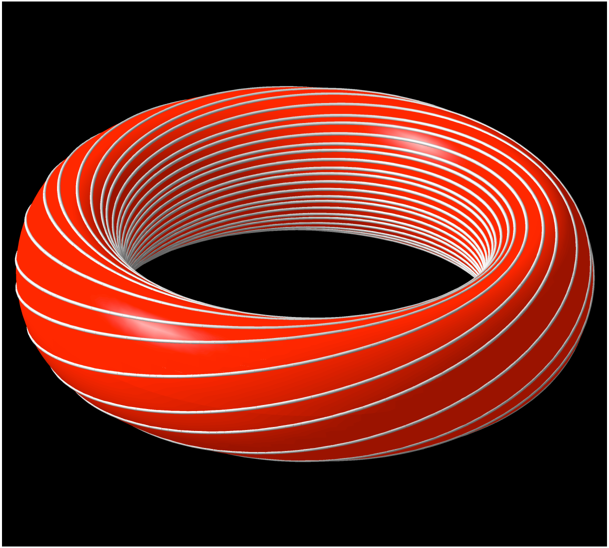
  Balance between plasma pressure, magnetic pressure, and magnetic tension
- Two free functions define equilibrium: e.g. B_z and p, or B_θ and B_z

General equilibria

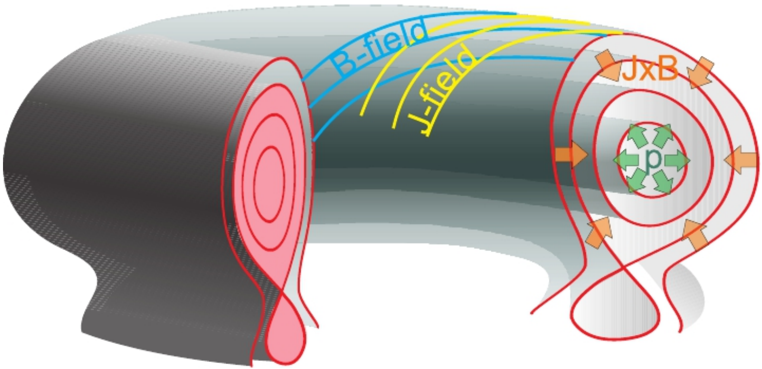
Equilibrium relation

$$\mathbf{J} \times \mathbf{B} = \nabla p$$

$$\begin{cases} (\mathbf{J} \times \mathbf{B} = \nabla p) \cdot \mathbf{B} \quad \Rightarrow \quad \mathbf{B} \cdot \nabla p = 0 \\ \text{Magnetic field is tangent to surfaces of constant pressure} \\ \\ (\mathbf{J} \times \mathbf{B} = \nabla p) \cdot \mathbf{J} = 0 \quad \Rightarrow \quad \mathbf{J} \cdot \nabla p = 0 \\ \text{Current density is tangent to surfaces of constant pressure} \end{cases}$$



(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)



- The regions of constant pressure are nested toroidal surfaces
- Magnetic fields and currents lie on these nested surfaces

Grad-Shafranov equation

- Surfaces of constant pressure coincide with surfaces of constant magnetic flux Ψ
- These are given by the Grad-Shafranov equation

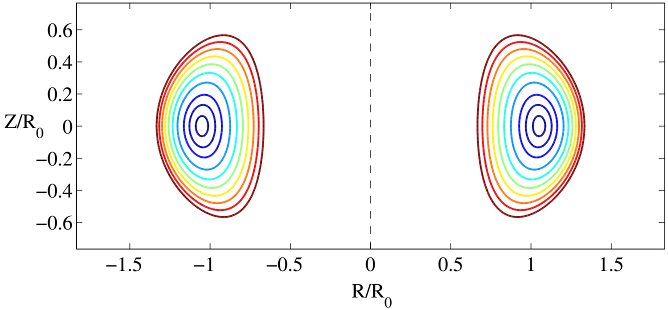
$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right)+\frac{\partial^2\Psi}{\partial Z^2}=-\mu_0 R^2\frac{dp}{d\Psi}-F\frac{dF}{d\Psi}$$

- Second-order, nonlinear, elliptic PDE. Derived independently by H. Grad[1] and V.D. Shafranov[2].
- The free functions p and F determine the nature of the equilibrium
- In general, the GSE has to be solved numerically

[1] *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy*, Vol. 31, p.190

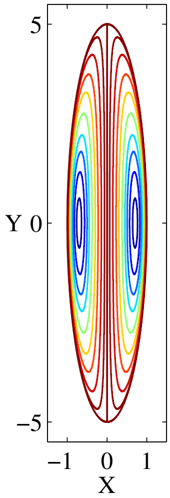
[2] *Sov. Phys. JETP* **6**, 545 (1958)

Examples (I)



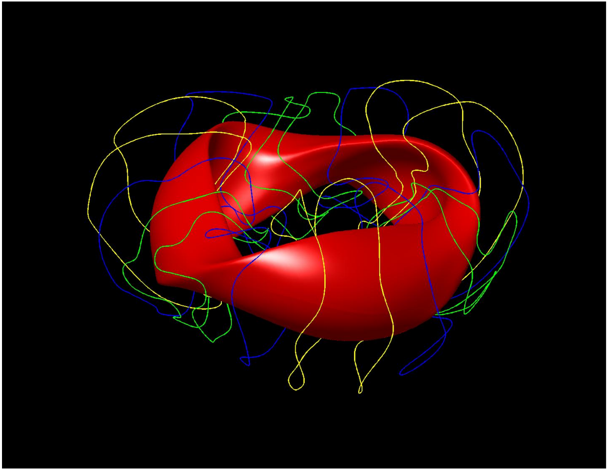
Grad-Shafranov equilibrium for JET tokamak

EXAMPLES (II)



Grad-Shafranov equilibrium for Field Reversed Configuration

3D EQUILIBRIA

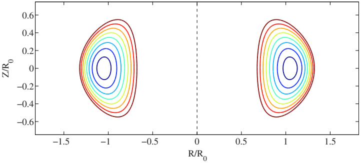


$$\partial/\partial\phi \neq 0$$

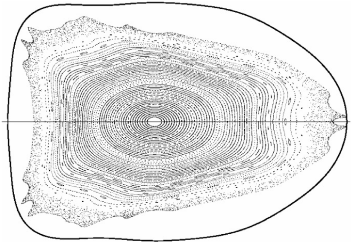
3D EQUILIBRIA

- Equilibrium equations $\nabla \cdot \mathbf{B} = 0\ ,\ (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ still hold
- Existence of nested toroidal surfaces not guaranteed anymore

Tokamak



Stellarator (3D)



- Computing 3D equilibria fast and accurately still a challenge
- Several existing codes, based on different assumptions/approximations and used to design and study stellarators: VMEC, PIES, SPEC, HINT, NSTAB

PART IV: MHD STABILITY

What do we mean by MHD stability?

- That the plasma is initially in equilibrium does not mean it is going to remain there
- The plasma is constantly subject to perturbations, small and large
- The purpose of stability studies is to find out how the plasma will react to these perturbations
  - Will it try to return to the initial steady-state?
  - Will it find a new acceptable steady-state?
  - Will it collapse?

A mechanical analog

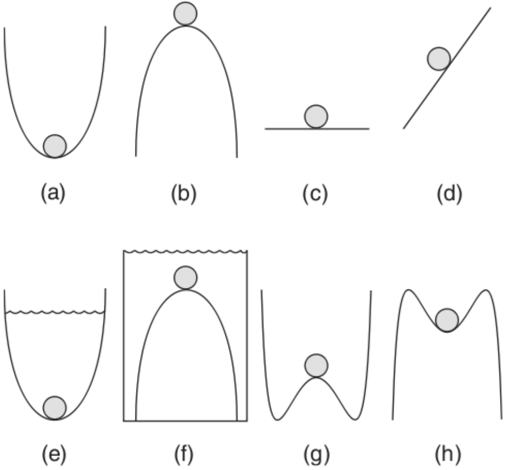


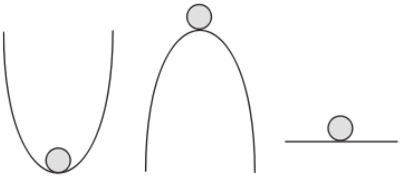
Figure from J.P. Freidberg, *Ideal MHD*, Cambridge University Press (2014)

Solving full nonlinear MHD equations

- Here is an idea to study stability of a magnetically confined plasma:
  - Choose a satisfying plasma equilibrium
  - Perturb it
  - Solve the full MHD equations with a computer
  - Analyze results
- Such an approach provides knowledge of the entire plasma dynamics
- There exist several numerical codes that can do that, for various MHD models (not only ideal): M3D, M3D-C1, NIMROD
- Computationally intensive
- Get more information than one needs?

Linear stability (I)

- Ideal MHD dynamics can be so fast and detrimental that one may often require linear stability for the equilibrium



- This can simplify the mathematical analysis tremendously
- Start with an MHD equilibrium:
  $\nabla \cdot \mathbf{B}_0 = 0\ ,\ (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \mu_0 \nabla p_0$
- Take full ideal MHD equations, and write $Q = Q_0(\mathbf{r}) + Q_1(\mathbf{r}, t)$ for each physical quantity, where Q_1 is considered very small compared to Q_0
- Drop all the terms that are quadratic or higher orders in the quantities Q_1 (linearization)

Linear stability (II)

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{J}_1 \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}_1 - \nabla p_1$$

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}_1 = 0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$$

$$\nabla \cdot \mathbf{B}_1 = 0$$

$$\mu_0 \mathbf{J}_1 = \nabla \times \mathbf{B}_1$$

- By design, the system is now linear in the unknown quantities ρ_1, $\mathbf{v}_1$, $\mathbf{J}_1$, $\mathbf{B}_1$, p_1
- Much easier to solve in a computer – or by hand!

Illustration: Waves in ideal MHD (I)

- Consider the stability of an infinite, homogeneous plasma:

$$\mathbf{B} = B_0 \overrightarrow{e_z}$$
$$\mathbf{J} = \overrightarrow{0}$$
$$p = p_0$$
$$\rho = \rho_0$$
$$\mathbf{v} = \mathbf{0}$$

- Write $\mathbf{v}_1 = \partial \boldsymbol{\xi}(\mathbf{r}) / \partial t$
- Expand $\boldsymbol{\xi}(\mathbf{r})$ as

$$\boldsymbol{\xi}(\mathbf{r}) = \hat{\boldsymbol{\xi}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

- Plug into linear system, and solve by hand
- Dynamics is anisotropic because of the magnetic field: $\mathbf{k}_\perp + k_{||}\mathbf{e}_z$

ILLUSTRATION: WAVES IN IDEAL MHD (II)

- Writing the expression for each component, we get the system

$$\begin{bmatrix} \omega^2 - k_\parallel^2 v_A^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - k_\perp^2 v_S^2 & -k_\parallel k_\perp v_S^2 \\ 0 & -k_\perp k_\parallel v_S^2 & \omega^2 - k_\parallel^2 v_S^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Two key velocities appear:

$$v_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \qquad v_S = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

v_A is called the *Alfvén velocity*, in honor of Hannes Alfvén, the Swedish scientist who first described MHD waves

v_s is the adiabatic sound speed

Illustration: Waves in ideal MHD (III)

$$\begin{bmatrix} \omega^2 - k_{\|}^2 v_A^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - k_{\perp}^2 v_S^2 & -k_{\|} k_{\perp} v_S^2 \\ 0 & -k_{\perp} k_{\|} v_S^2 & \omega^2 - k_{\|}^2 v_S^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

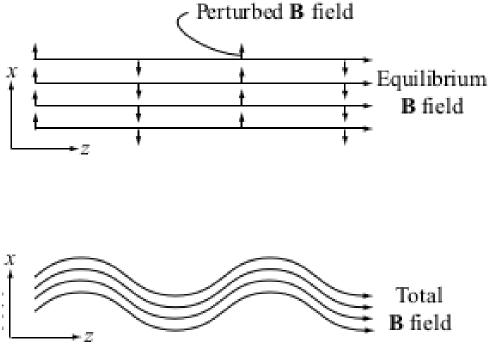
- For nontrivial solutions, determinant of the matrix should be 0
- This leads to the following three possibilities for ω^2:

$$\omega^2 = k_{\|}^2 v_A^2 \;,\; \omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 \pm \sqrt{1 - 4\frac{k_{\|}^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

- One can see that $\omega^2 \geq 0 \Rightarrow$ The infinite homogeneous magnetized plasma is always MHD stable
- Some of these modes become unstable in magnetic fusion configurations, because of gradients and field line curvature

Shear Alfvén wave

- Branch $\omega^2 = k_{\|}^2 v_A^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

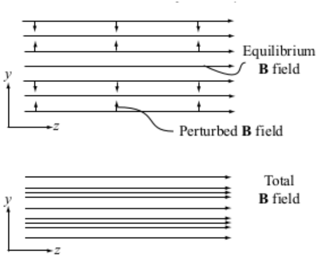
- Transverse wave
- Balance between plasma inertia and field line tension
- Incompressible $\Rightarrow$ often the most unstable MHD mode in fusion devices

Fast magnetosonic wave

- Fast magnetosonic wave given by

$$\omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 + \sqrt{1 - 4\frac{k_\parallel^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

- Simplifies in the limit $v_S^2 \ll v_A^2$: it becomes the compressional Alfvén wave, $\omega^2 = k^2 v_A^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

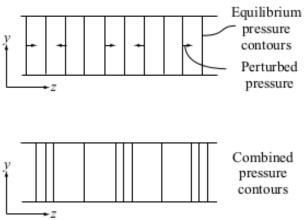
- Oscillation between plasma kinetic energy and magnetic compressional energy

Slow magnetosonic wave

- Slow magnetosonic wave given by

$$\omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 - \sqrt{1 - 4\frac{k_\parallel^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

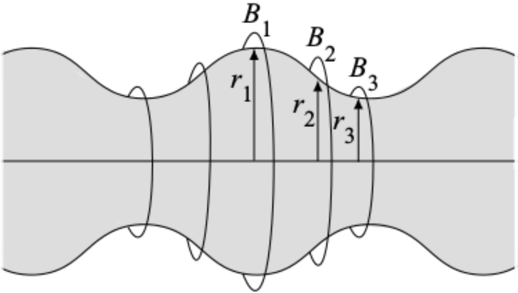
- Physics simplifies in the limit $v_S^2 \ll v_A^2$: it is then called the sound wave, with dispersion relation $\omega^2 = k^2 v_S^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Plasma motion parallel to field lines, compressible
- Oscillation between plasma kinetic energy and plasma internal energy (plasma pressure)

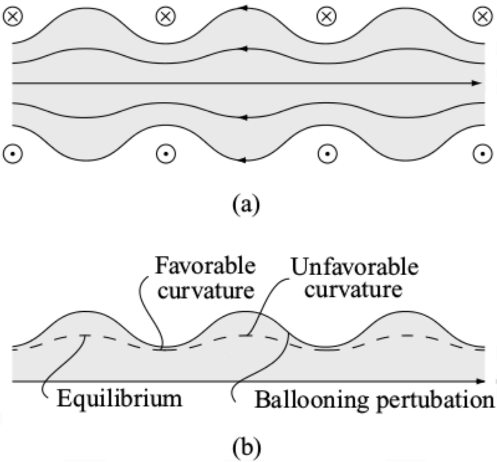
Common MHD instabilities (I)



Interchange instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

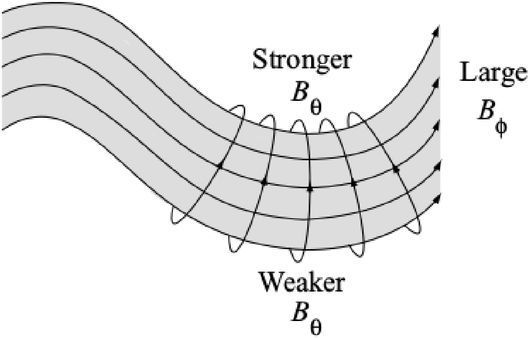
Common MHD instabilities (II)



Ballooning instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

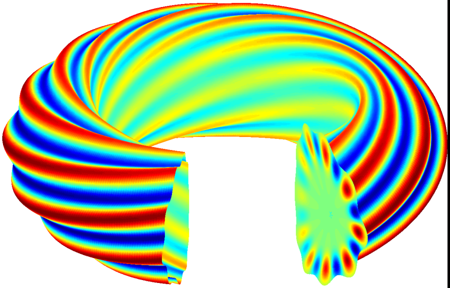
Common MHD instabilities (III)

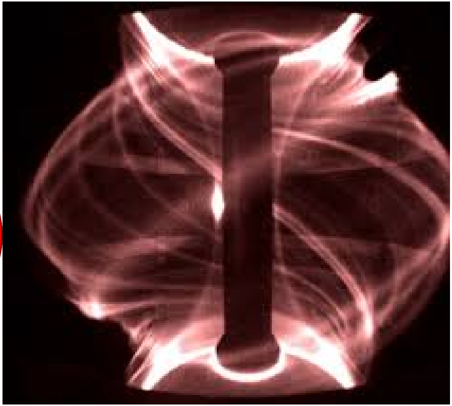


Kink instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

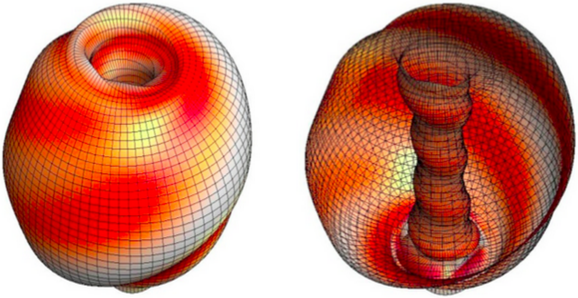
Ballooning modes in tokamaks





(Right image from *http://www.ccfe.ac.uk/assets/Documents/AIPCONFPROC103p174.pdf* by J.W. Connor *et al.*)

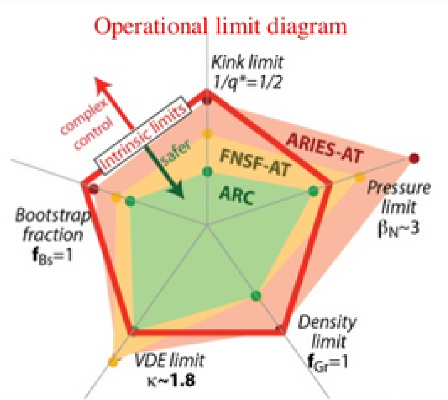
KINK BALLOONING MODES IN TOKAMAKS



Reconstruction of experimentally observed kink ballooning mode

(Figure from *U.S. Burning Plasma Organization eNews February 28, 2013 (Issue 69)* by S.A. Sabbagh *et al.*)

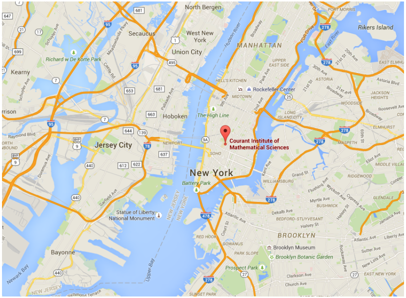
Key role of MHD in reactor design

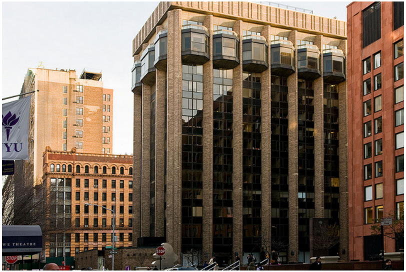


- Maximum achievable pressure set by ballooning modes
- Maximum achievable pressure set by kink modes
- Maximum elongation set by vertical instability

POST SCRIPTUM: THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES (CIMS) AT NYU

CIMS IN MANHATTAN





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Applied Math

Ranked in 2014, part of Math

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Specialty Applied Math

Rank	School name
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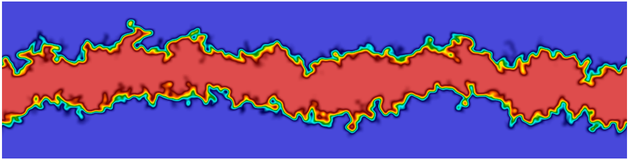
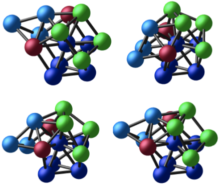
#2	University of California—Los Angeles Los Angeles, CA
#3	California Institute of Technology Pasadena, CA
#4	Massachusetts Institute of Technology Cambridge, MA
#5 Tie	Brown University Providence, RI
#5 Tie	University of Minnesota—Twin Cities Minneapolis, MN

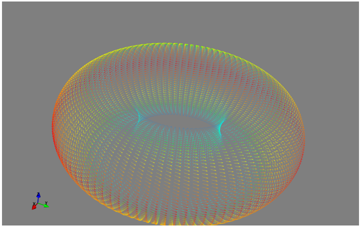
- Abel prize in 2005, 2007, 2009 and 2015
- 18 members of the National Academy of Sciences

  5 members of the National Academy of Engineering

- Specialization in applied math, scientific computing, mathematical analysis
- Particular emphasis on Partial Differential Equations
- PhD programs in Mathematics, Atmosphere and Ocean Science, Computational Biology
- Masters of Science in Mathematics, Masters of Science in Scientific Computing, Masters of Science in Data Science, Masters of Science in Math Finance
- $\sim$ 60 faculty
  $\sim$ 100 PhD students







MFD division at CIMS

- Founded by Harold Grad in 1954
- 3 faculty, 3 post-docs, 2 graduate students
- Work on MHD, wave propagation, kinetic models
  Analytic "pen and paper" work
  Development of new numerical solvers
- Collaboration with colleagues specialized in scientific computing, computational fluid dynamics, stochastic calculus, etc.
- Funding currently available for PhD students
- Feel free to contact me if you have any questions