Single Particle Motion

Cami Collins

General Atomics

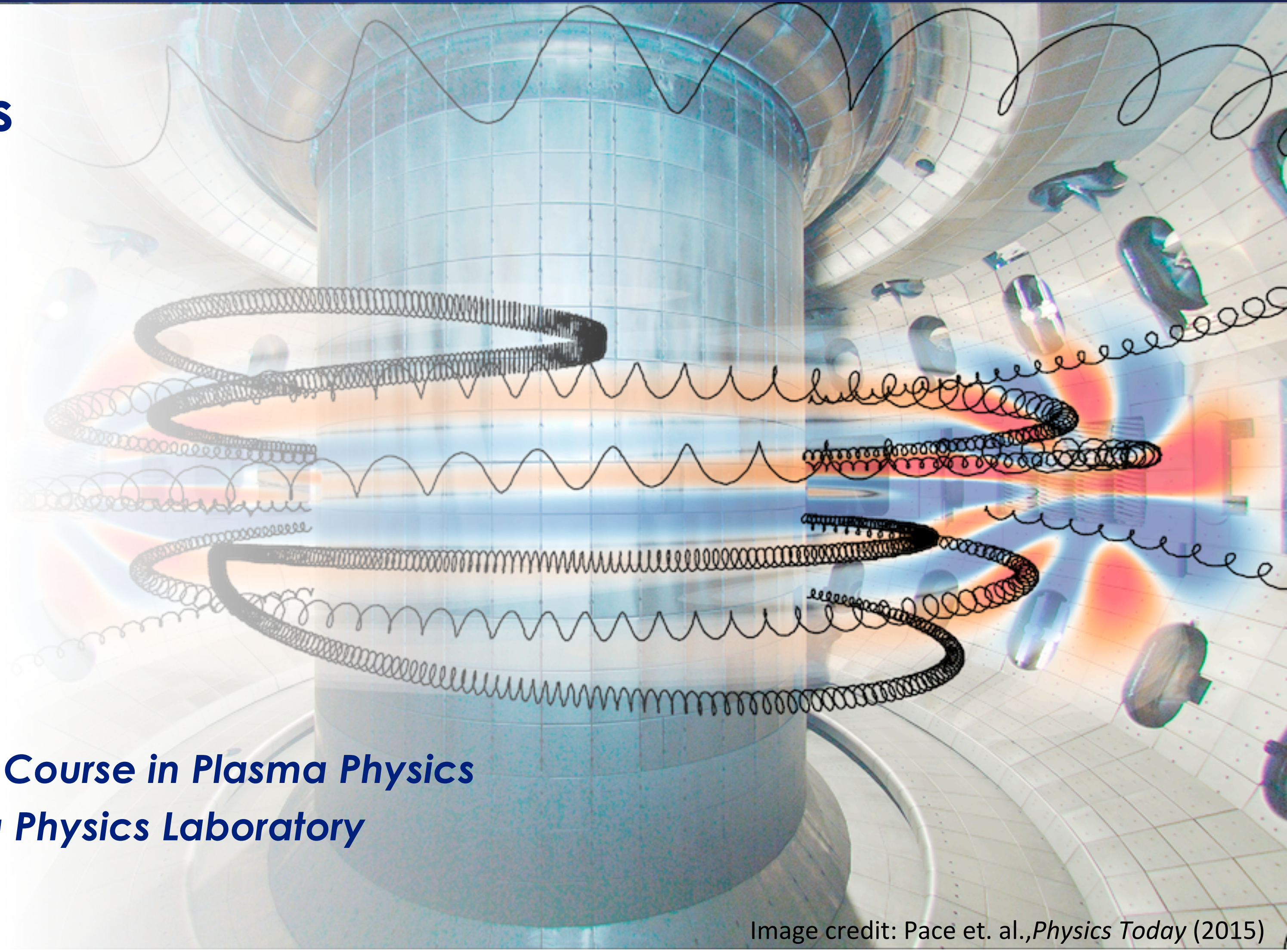


Image credit: Pace et. al.,*Physics Today* (2015)

Presented at the
SULI Introductory Course in Plasma Physics
Princeton Plasma Physics Laboratory
June 6, 2016

DIII-D NATIONAL FUSION FACILITY

GENERAL ATOMICS

My Path In Plasma Physics

scientist
(permanent student)

MONTANA STATE UNIVERSITY

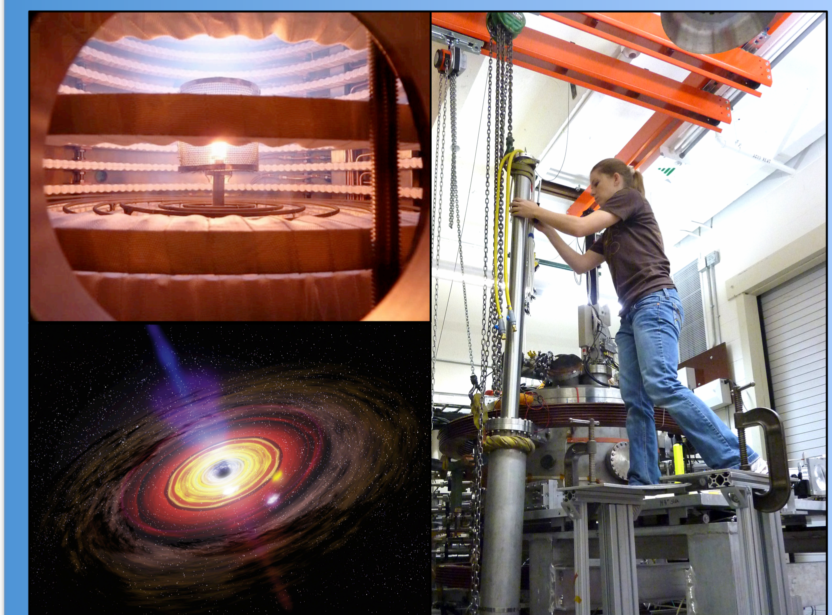
WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

UCIrvine
University of California, Irvine

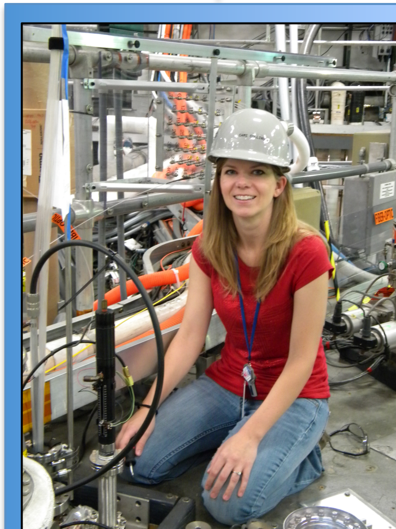
GENERAL ATOMICS

2006 **2007-2013** **2014-2016**

NUF student

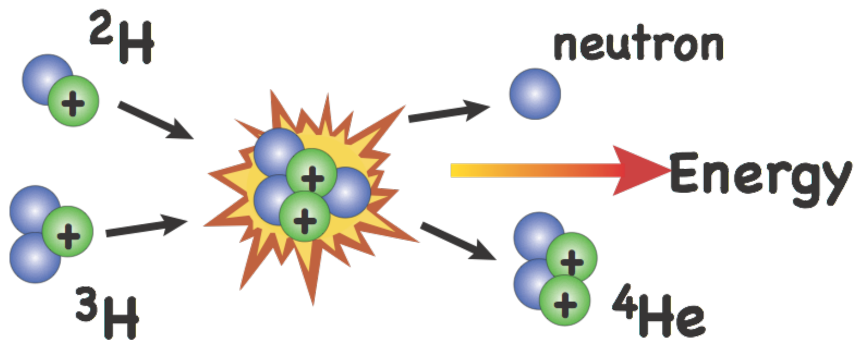


PhD student



postdoc
(~student)

Plasma Physics is the Basis for Fusion Research



- **Sustained fusion reactions require enough particles (density) that are energetic enough (temperature) and collide often enough (confinement time).**

- **The fusion triple product is the figure of merit:**

$$nT\tau_E \gtrsim 5\times 10^{21}\ \mathrm{keV\ s\ m^{-3}}$$

T~100-200 million K

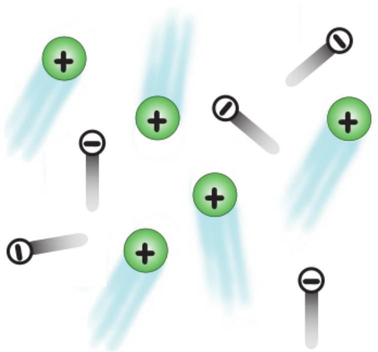
n~2-3x10^{20} ions/m^3

τ~1-2 s

D & T is a plasma at these temperatures

We can understand a lot about how fusion devices confine plasma by studying single particle motion.

Typical velocity of a 100 million K ion:



$$kT = \frac{1}{2} m v_{\mathrm{th}}^2$$

$$v_{\mathrm{th}} \sim 6 \times 10^5 \ \mathrm{m/s}$$

Even with ~10^{20} *ions/m*^{3} , the ion would travel ~10 km before colliding with another

The ITER tokamak has R_{major}=6.2 m → ~40 m circumference

The trick: use magnetic fields

Outline of Single Particle Motion Topics

- **Gyromotion about a guiding center**
- **Forces can cause guiding center drift**
- **Real life consequences:**
  - Why do tokamaks have helical B fields?
  - What is a banana orbit?

References

- NRL Plasma Formulary
  www.nrl.navy.mil/ppd/content/nrl-plasma-formulary

- Introduction to Plasma Physics and Controlled Fusion by F. Chen

Charged Particles Feel The (Lorentz) Force

- **A particle with charge (q) moving with velocity ($\mathbf{v}$) in the presence of electric or magnetic fields will experience a force:**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

We know from Newton's second law of motion that force causes acceleration:

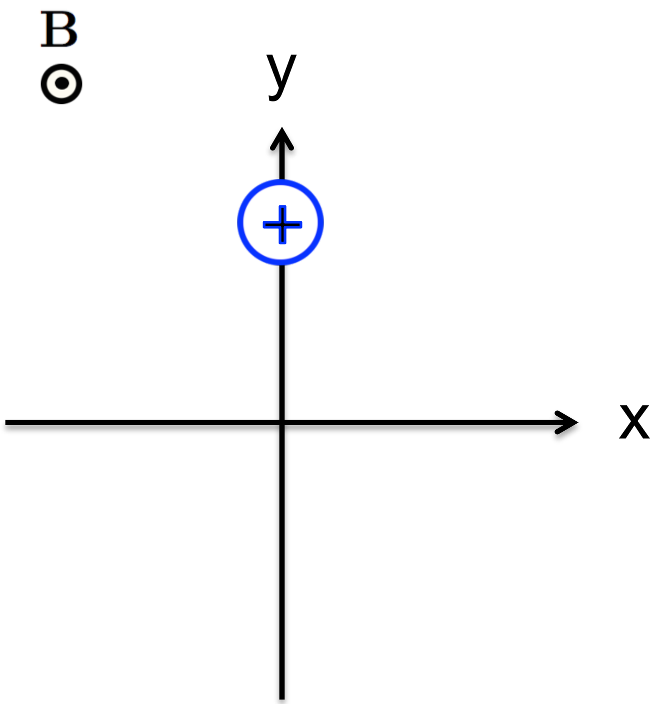
$$\mathbf{F} = m\mathbf{a}$$

A charged particle moving perpendicular to the magnetic field feels a force

How Does a Charged Particle Move in a Magnetic Field?

- **Consider the motion of a particle in a constant, uniform B field**

$$\mathbf{B} = B_z\hat{\mathbf{z}} \qquad \mathbf{E} = \mathbf{0}$$

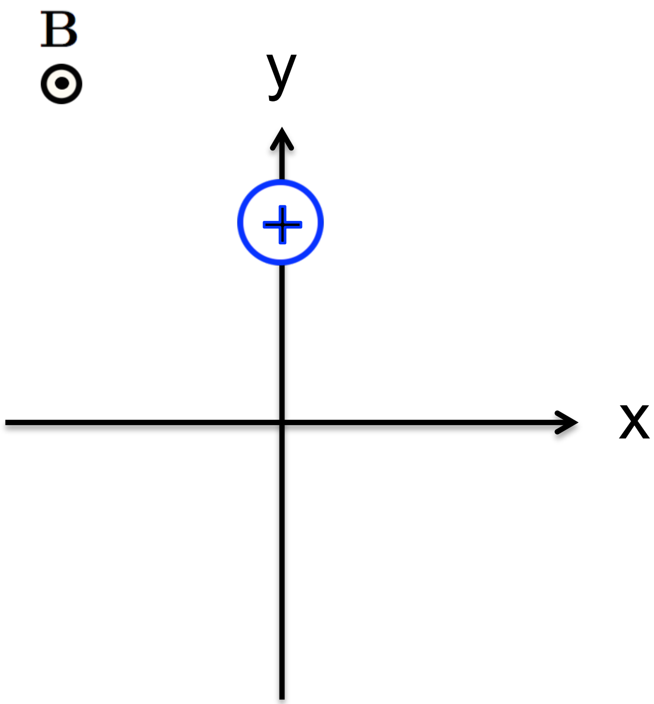


Then $$\mathbf{F} = q(\cancel{\mathbf{E}} + \mathbf{v} \times \mathbf{B})$$

So we can write

$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

Goal: Solve the Equations of Motion for a Charged Particle In A Magnetic Field



$$\frac{d\mathbf{v}}{dt} = \frac{q\mathbf{v} \times \mathbf{B}}{m}$$

Let's break this into components:

$$\dot{v}_x\hat{\mathbf{x}} + \dot{v}_y\hat{\mathbf{y}} + \dot{v}_z\hat{\mathbf{z}} = \frac{qv_yB_z\hat{\mathbf{x}} - qv_xB_z\hat{\mathbf{y}}}{m}$$

The 'dot' represents $\frac{d}{dt}$

Matching components:

$$\dot{v}_x = \frac{qv_yB_z}{m} \qquad \dot{v}_y = -\frac{qv_xB_z}{m}$$

$$\dot{v}_z = 0$$

Particles move freely along the field line

Take Another Time Derivative & Substitute to Obtain Differential Equations For Each Spatial Coordinate

$$\dot{v}_x = \frac{qv_yB_z}{m} \qquad \dot{v}_y = -\frac{qv_xB_z}{m}$$

$$\ddot{v}_x = \frac{q\dot{v}_yB_z}{m} \qquad \ddot{v}_y = -\frac{q\dot{v}_xB_z}{m}$$

Rewriting, we get

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y$$

These may remind you of the equations for a simple harmonic oscillator

Solve the Differential Equations

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x \qquad \ddot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_y$$

These differential equations can be solved using sines and cosines:

$$v_x = v_\perp \cos\left(\frac{|q|B_z}{m}t + \phi_0\right) \qquad v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right)$$

account for positive or negative *q*

$$v_\perp = \sqrt{(v_x^2 + v_y^2)}$$

the magnitude of the initial velocity perpendicular to ***B***

an arbitrary phase to match the initial velocity conditions

The Result: Circular Motion About A Guiding Center

$$v_x = v_\perp \cos\left(\frac{|q|B_z}{m}t + \phi_0\right) \qquad v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right)$$

Integrating, we obtain

$$x = \frac{mv_\perp}{|q|B_z}\sin\left(\frac{|q|B_z}{m}t + \phi_0\right) + x_0 \qquad y = \pm\frac{mv_\perp}{|q|B_z}\cos\left(\frac{|q|B_z}{m}t + \phi_0\right) + y_0$$

- **Charged particles undergo circular orbits about a guiding center** (x_0, y_0)

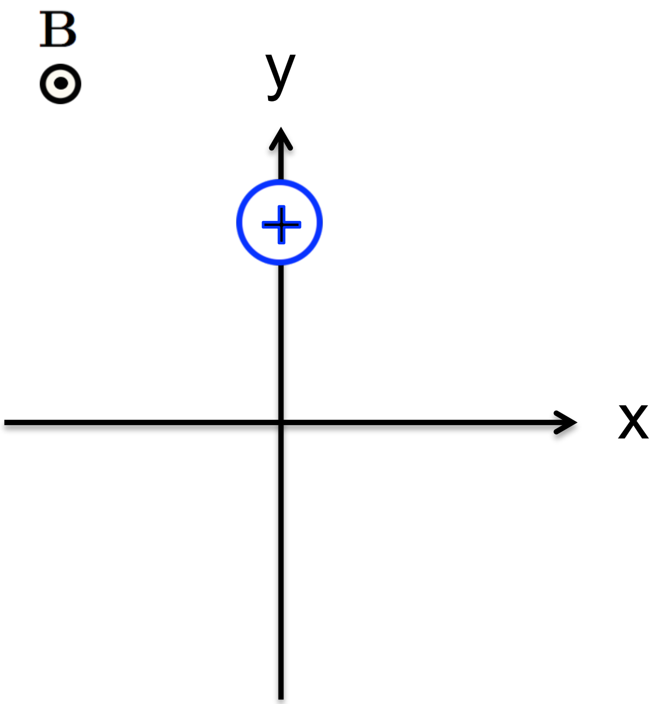
$$r_L \equiv \frac{mv_\perp}{|q|B}$$ **Larmor radius**

$$\omega_c \equiv \frac{|q|B}{m}$$ **Cyclotron frequency**

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin(\omega_c t + \phi_0) + x_0 \qquad y = \pm r_L \cos(\omega_c t + \phi_0) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a positively charged particle:

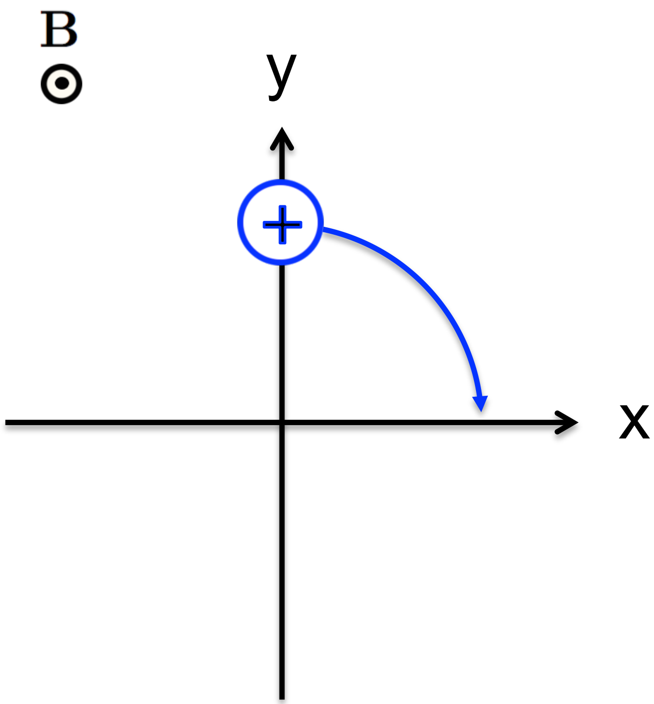
1. At $t = 0$,

$$x = 0 \qquad y = r_L$$

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin\left(\omega_c t + \phi_0\right) + x_0 \qquad y = \pm r_L \cos\left(\omega_c t + \phi_0\right) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a positively charged particle:

1. At $t = 0$,

$$x = 0 \qquad y = r_L$$

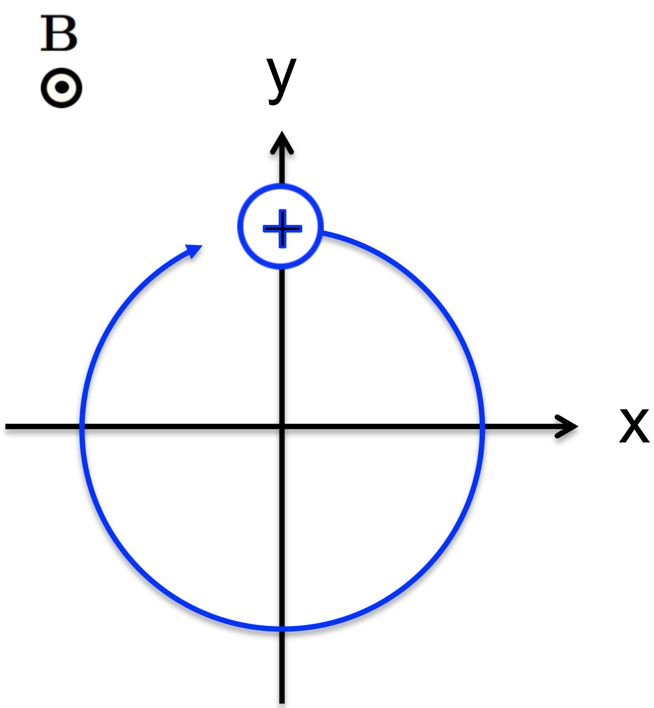
2. At $t = \frac{\pi}{2}\frac{1}{\omega_c}$,

$$x = r_L \qquad y = 0$$

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin\left(\omega_c t + \phi_0\right) + x_0 \qquad y = \pm r_L \cos\left(\omega_c t + \phi_0\right) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a positively charged particle:

1. At $t = 0$,

$$x = 0 \qquad y = r_L$$

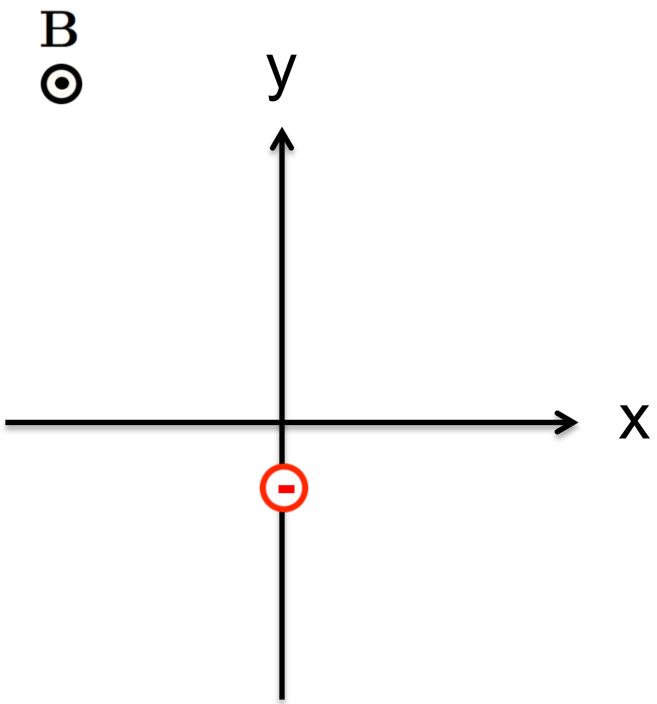
2. At $t = \frac{\pi}{2}\frac{1}{\omega_c}$,

$$x = r_L \qquad y = 0$$

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin(\omega_c t + \phi_0) + x_0 \qquad y = \pm r_L \cos(\omega_c t + \phi_0) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a negatively charged particle:

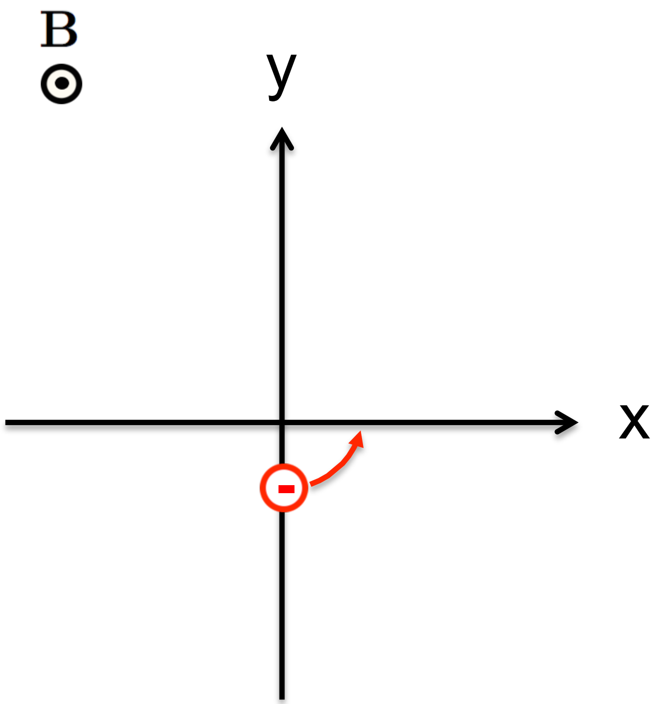
1. At $t = 0$,

$$x = 0 \qquad y = -r_L$$

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin(\omega_c t + \phi_0) + x_0 \qquad y = \pm r_L \cos(\omega_c t + \phi_0) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a negatively charged particle:

1. At $t = 0$,

$$x = 0 \qquad y = -r_L$$

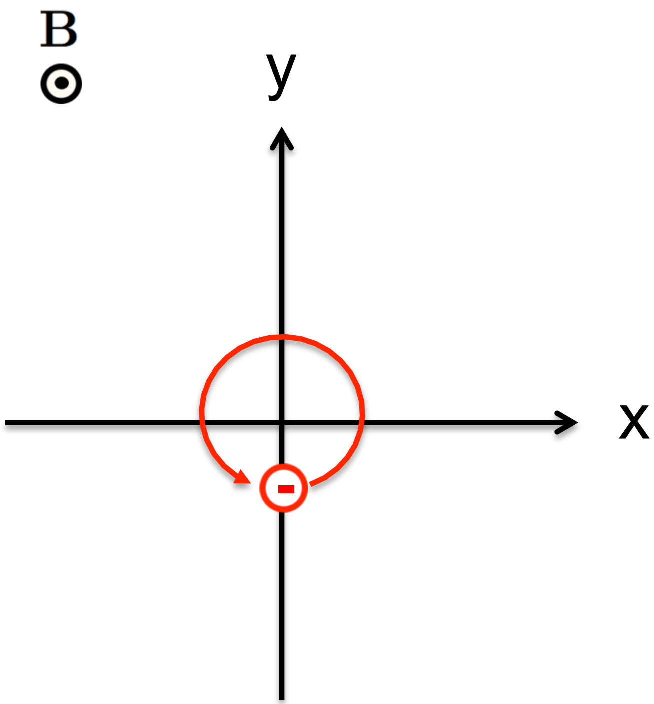
2. At $t = \frac{\pi}{2}\frac{1}{\omega_c}$,

$$x = r_L \qquad y = 0$$

Gyromotion of a Charged Particle In A Magnetic Field

$$x = r_L \sin(\omega_c t + \phi_0) + x_0 \qquad y = \pm r_L \cos(\omega_c t + \phi_0) + y_0$$

Let's take $\phi_0 = 0$ and $x_0 = y_0 = 0$



For a negatively charged particle:

1. At $t = 0$,

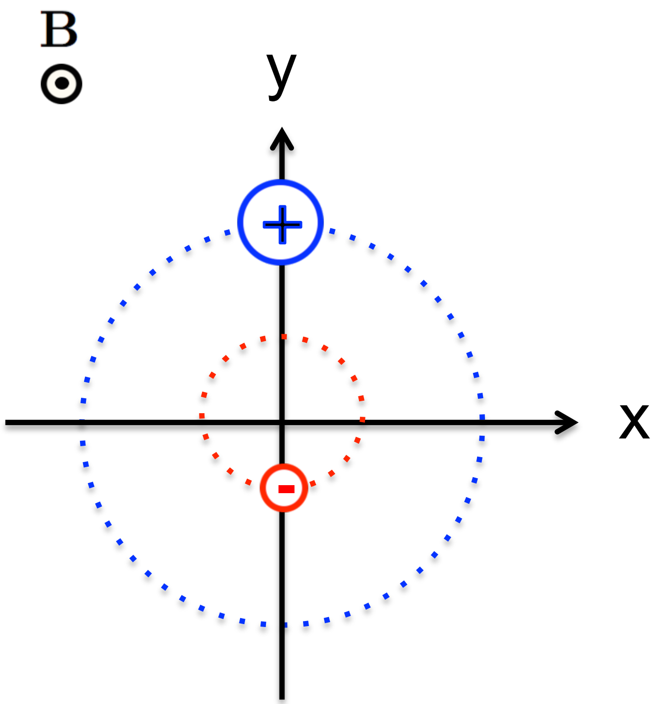
$$x = 0 \qquad y = -r_L$$

2. At $t = \frac{\pi}{2}\frac{1}{\omega_c}$,

$$x = r_L \qquad y = 0$$

Gyromotion of Ions vs. Electrons

- **The direction of gyromotion depends on the sign of the charge**
- **Ions generally have a much larger Larmor radius than electrons**



In ITER, for a typical deuterium ion with T_i=10 keV and B=5 Tesla would have

$$v_{Ti} = \sqrt{\frac{kT_i}{m_i}} \approx 700 \text{ km/s}$$

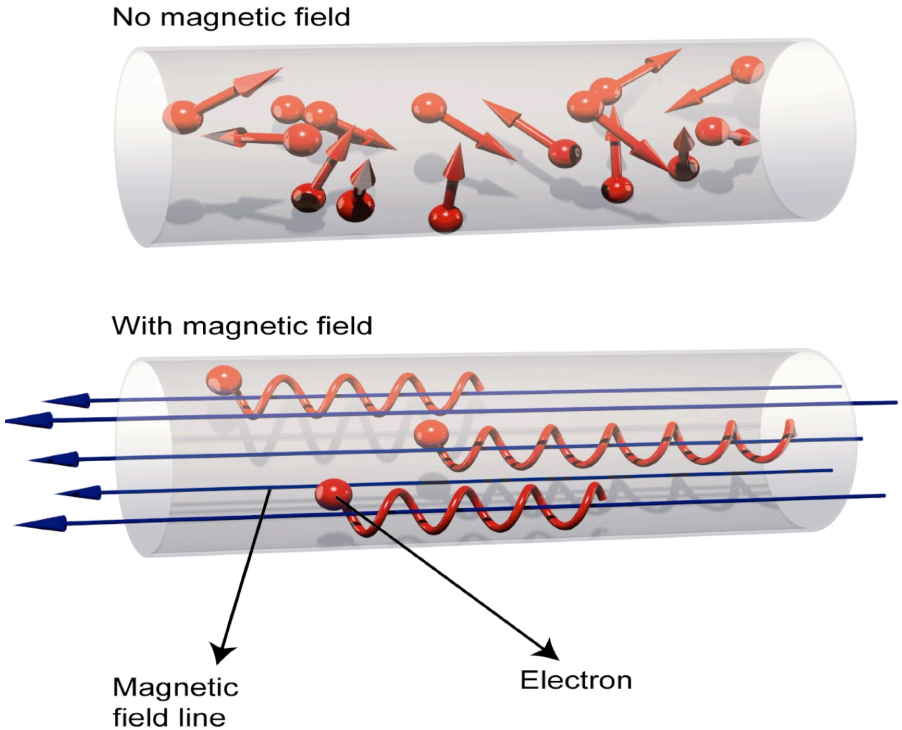
$$r_L \equiv \frac{mv_\perp}{|q|B} \approx 3 \text{ mm}$$

An electron with T_e=10 keV and B=5 Tesla has

$$r_L \approx 0.05 \text{ mm}$$ (60 times smaller)

Magnetic Confinement Devices Should Be Much Larger Than the Larmor Radius

$$\mathbf{r} = \left[r_L \sin\left(\omega_c t + \phi_0\right) + x_0\right]\hat{\mathbf{x}} + \left[r_L \cos\left(\omega_c t + \phi_0\right) + y_0\right]\hat{\mathbf{y}} + \left[v_{\|} t + z_0\right]\hat{\mathbf{z}}$$



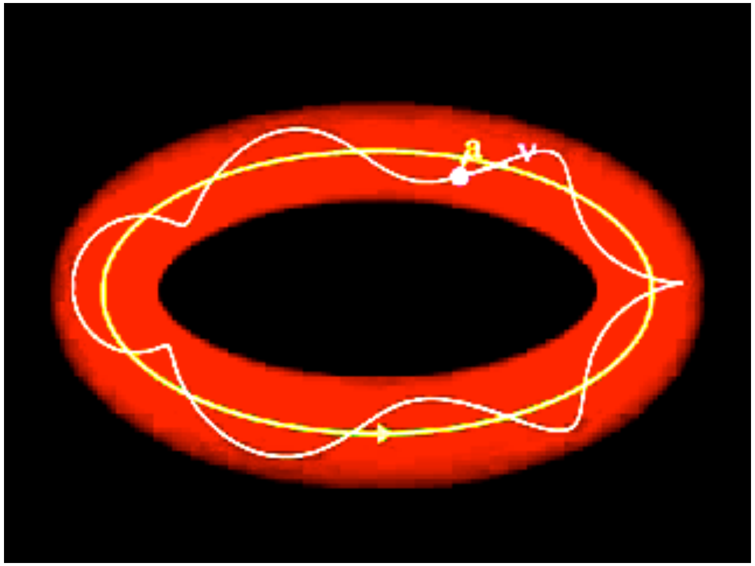
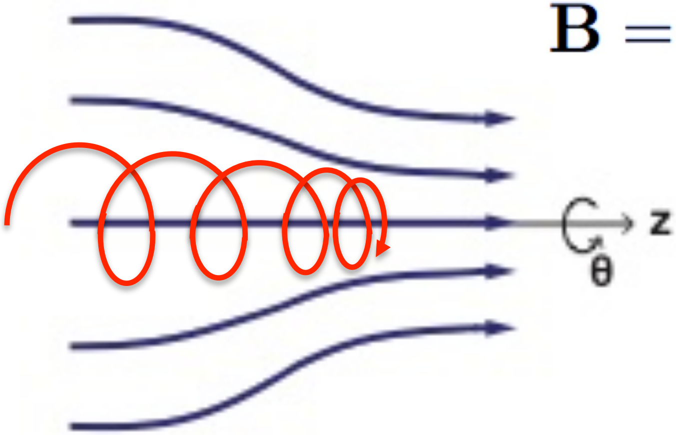


Image credit: http://iter.rma.ac.be/en/img/MagneticConfinement.jpg

Magnetic Mirrors



$$\mathbf{B} = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}} \qquad \mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

The B_r ends up causing additional acceleration in the z direction:

$$m\frac{dv_z}{dt} = -qv_\theta B_r$$

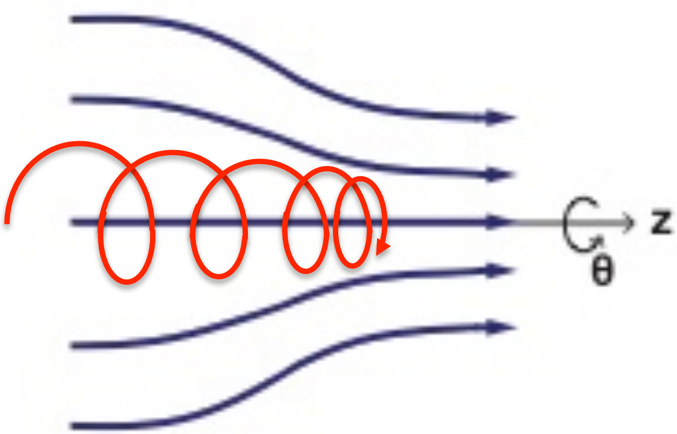
Result:
gyromotion + mirror force in the $-\hat{\mathbf{z}}$ direction

$$F_z = -\frac{mv_\perp^2}{2B}\frac{\partial B_z}{\partial z}$$

The magnetic moment is $\mu \equiv \dfrac{mv_\perp^2}{2B}$

mirror force $\mathbf{F}_{\|} = -\mu\nabla_{\|}B$

Magnetic Moment Is Conserved



$$\mathbf{F}_{\|} = -\mu \nabla_{\|} B \qquad \mu \equiv \frac{m v_{\perp}^2}{2B}$$

The magnetic moment is a constant of motion

$$m \frac{dv_{\|}}{dt} = -\mu v_{\|} \frac{\partial B}{\partial s}$$

s is the coordinate along the field line

We can write $$\frac{dB}{dt} = \frac{\partial B}{\partial s} \frac{ds}{dt}$$

this is $v_{\|}$

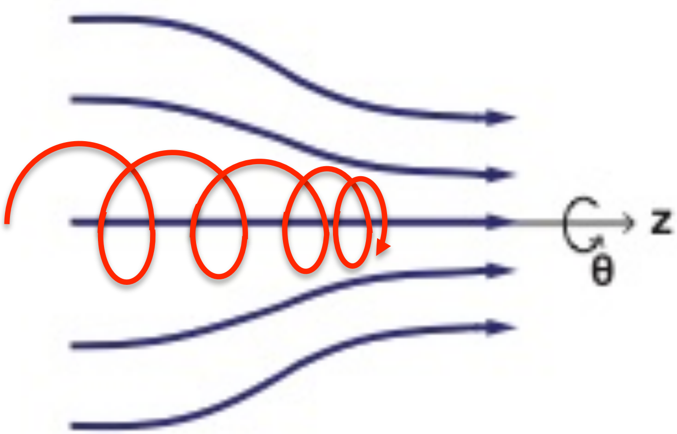
Then $$\frac{d}{dt}\left(\frac{1}{2} m v_{\|}^2\right) = -\mu \frac{dB}{dt}$$

We also have conservation of energy: $$\frac{d}{dt}\left(\frac{1}{2} m v_{\|}^2 + \frac{1}{2} m v_{\perp}^2\right) = 0$$

this is μB

$$\boxed{B \frac{d\mu}{dt} = 0}$$

More Insight Into Magnetic Mirrors



$$\mu \equiv \frac{mv_\perp^2}{2B} \qquad \frac{d\mu}{dt} = 0$$

1. As the particle moves to stronger B, $v_\perp$ must increase.

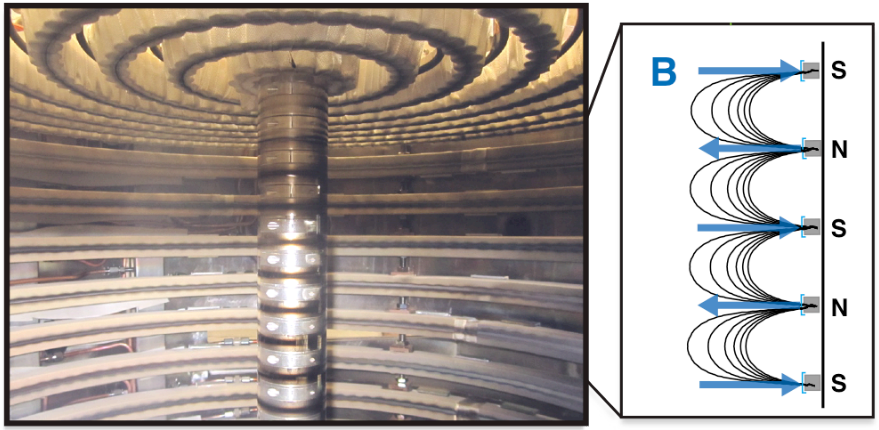
2. Since energy is conserved, $v_{||}$ must decrease.

3. If B is strong enough, $v_{||} \rightarrow 0$ and the particle is reflected.

$$E_o = \frac{1}{2}mv_{||}^2 + \frac{1}{2}mv_\perp^2 \qquad v_{||} = \pm\sqrt{\frac{2}{m}(E_o - \mu B)}$$

The particle is reflected when $E_o \leq \mu B$

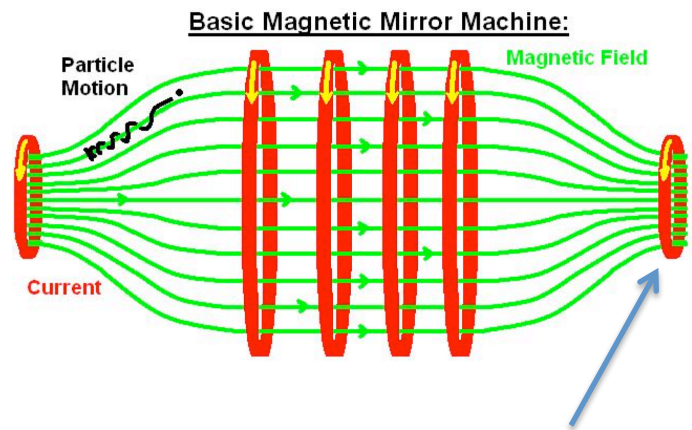
Magnetic Mirror Confinement In Action



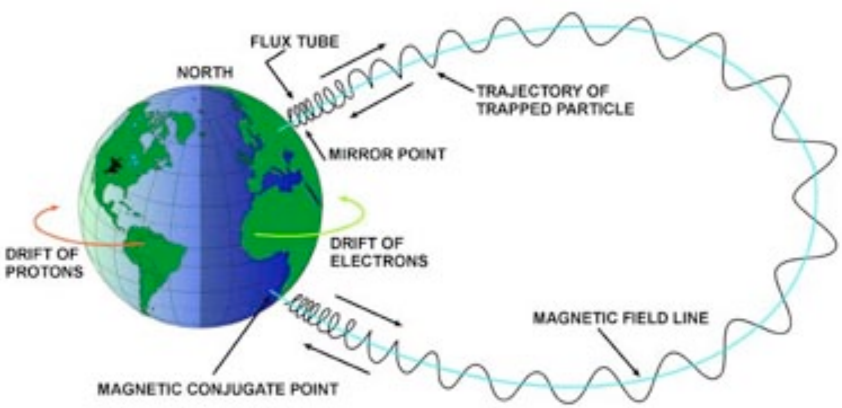
Multicusp Confinement Devices

Early Fusion Experiments

Ex: Tandem Mirror Experiment (LLNL,1980's) and other variants (Polywell devices)



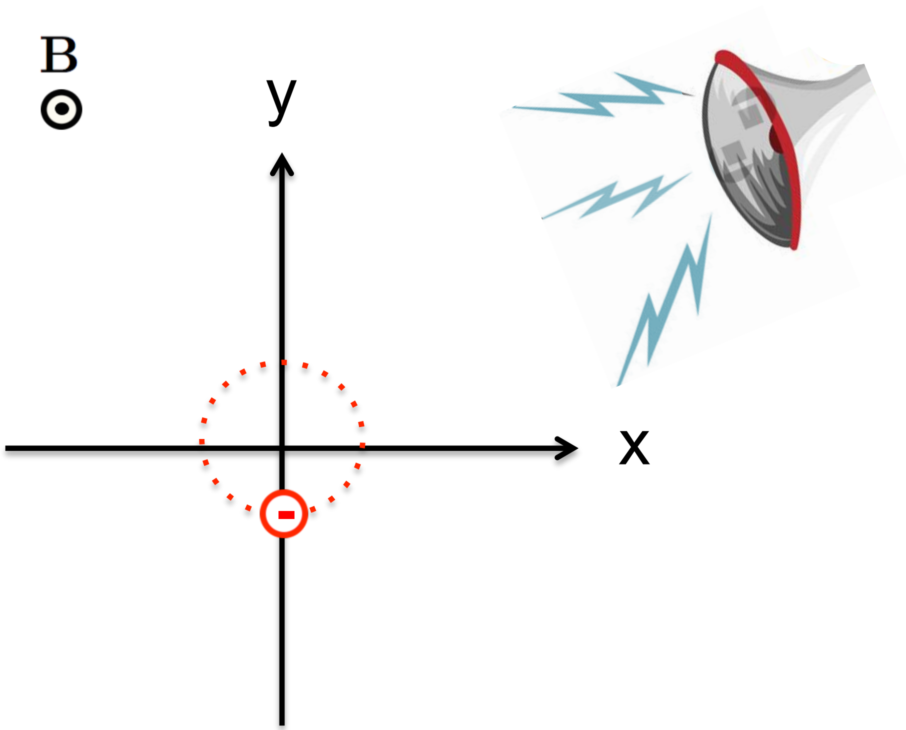
Particles with enough $v_{||}$ can still escape



Charged particles can be trapped by Earth's magnetic field

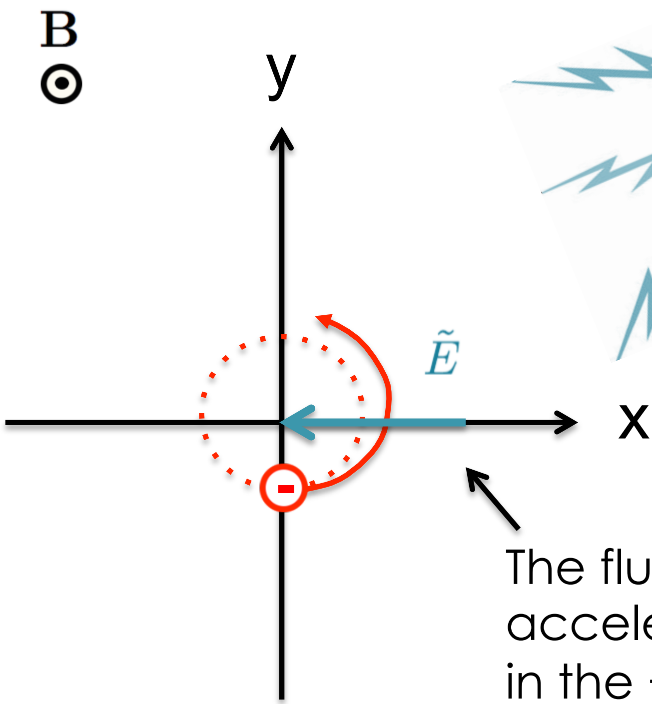
Time-varying Electric and Magnetic Fields Can Be Used To Accelerate & Heat Particles

- **A high frequency electro-magnetic field can be used to accelerate electrons or ions.**



Time-varying Electric and Magnetic Fields Can Be Used To Accelerate & Heat Particles

- **A high frequency electro-magnetic field can be used to accelerate electrons or ions.**
- **Particle gains energy as the applied electric field component oscillates at the cyclotron frequency ("in-phase" with the gyro-orbit).**



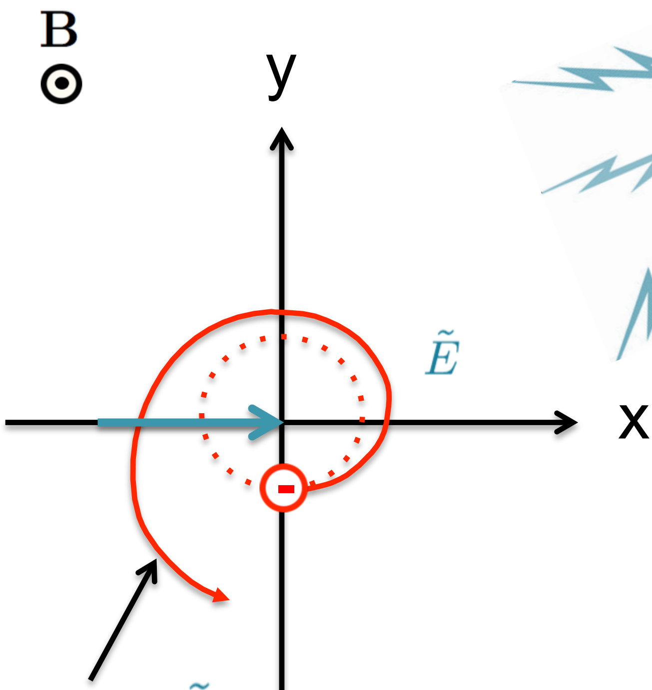
Cyclotron frequency

$$\omega_c \equiv \frac{|q|B}{m}$$

The fluctuating $\tilde{E}$
accelerates the electron
in the +x direction $\tilde{\mathbf{F}} = -e\tilde{\mathbf{E}}$

Time-varying Electric and Magnetic Fields Can Be Used To Accelerate & Heat Particles

- **A high frequency electro-magnetic field can be used to accelerate electrons or ions.**
- **Particle gains energy as the applied electric field component oscillates at the cyclotron frequency ("in-phase" with the gyro-orbit).**



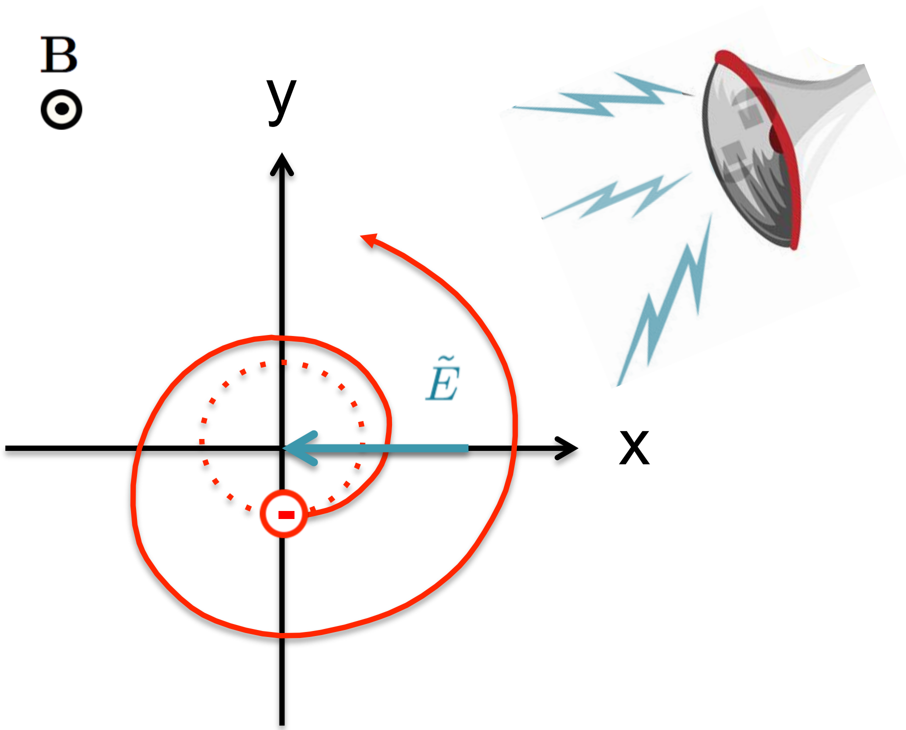
Now $\tilde{E}$ accelerates the electron in the -x direction

Cyclotron frequency

$$\omega_c \equiv \frac{|q|B}{m}$$

Time-varying Electric and Magnetic Fields Can Be Used To Accelerate & Heat Particles

- **A high frequency electro-magnetic field can be used to accelerate electrons or ions.**
- **Particle gains energy as the applied electric field component oscillates at the cyclotron frequency ("in-phase" with the gyro-orbit).**

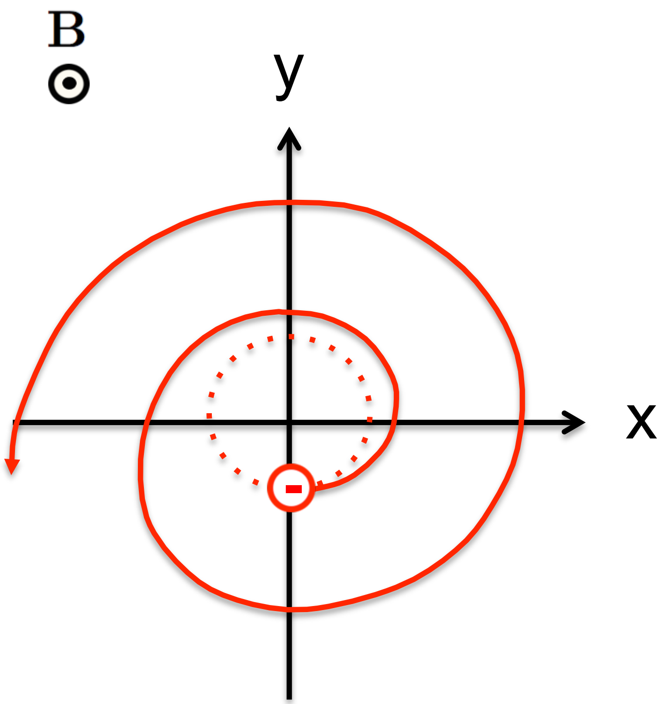


Cyclotron frequency

$$\omega_c \equiv \frac{|q|B}{m}$$

The Cyclotron Frequency is Important for Cyclotron Resonance Heating

- **A high frequency electro-magnetic field can be used to accelerate electrons or ions.**
- **Particle gains energy as the applied electric field component oscillates at the cyclotron frequency ("in-phase" with the gyro-orbit).**



Cyclotron frequency

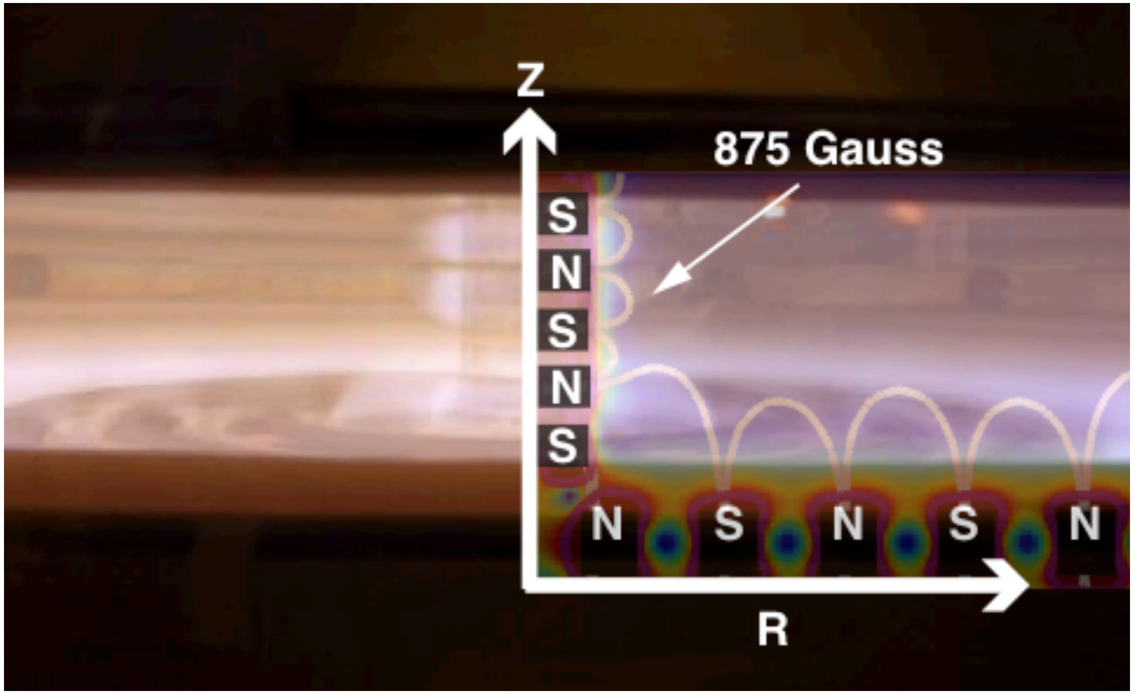
$$\omega_c \equiv \frac{|q|B}{m}$$

Ex: For an electron, what *B* corresponds to 2.45 Ghz (microwave oven frequency)?

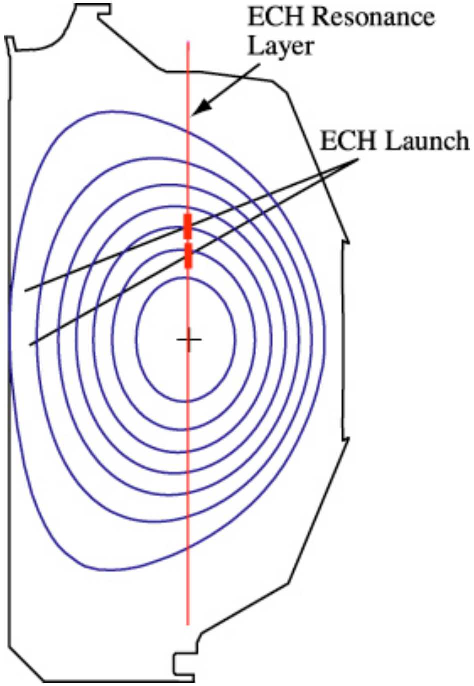
$$f = \frac{\omega_{ce}}{2\pi} = 2.45 \text{ GHz} \Rightarrow B = 875 \text{ Gauss}$$

Example of Cyclotron Heating in Action

Electron Cyclotron Heating In A Plasma Experiment at UW-Madison (2.45 GHz, B=875 Gauss)



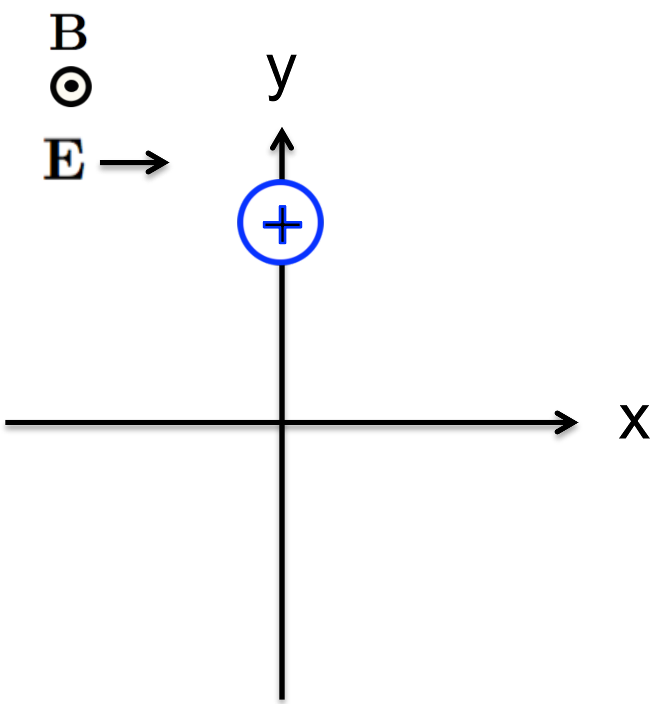
In the DIII-D tokamak, use 110 GHz second harmonic heating (B~2 Tesla)



Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

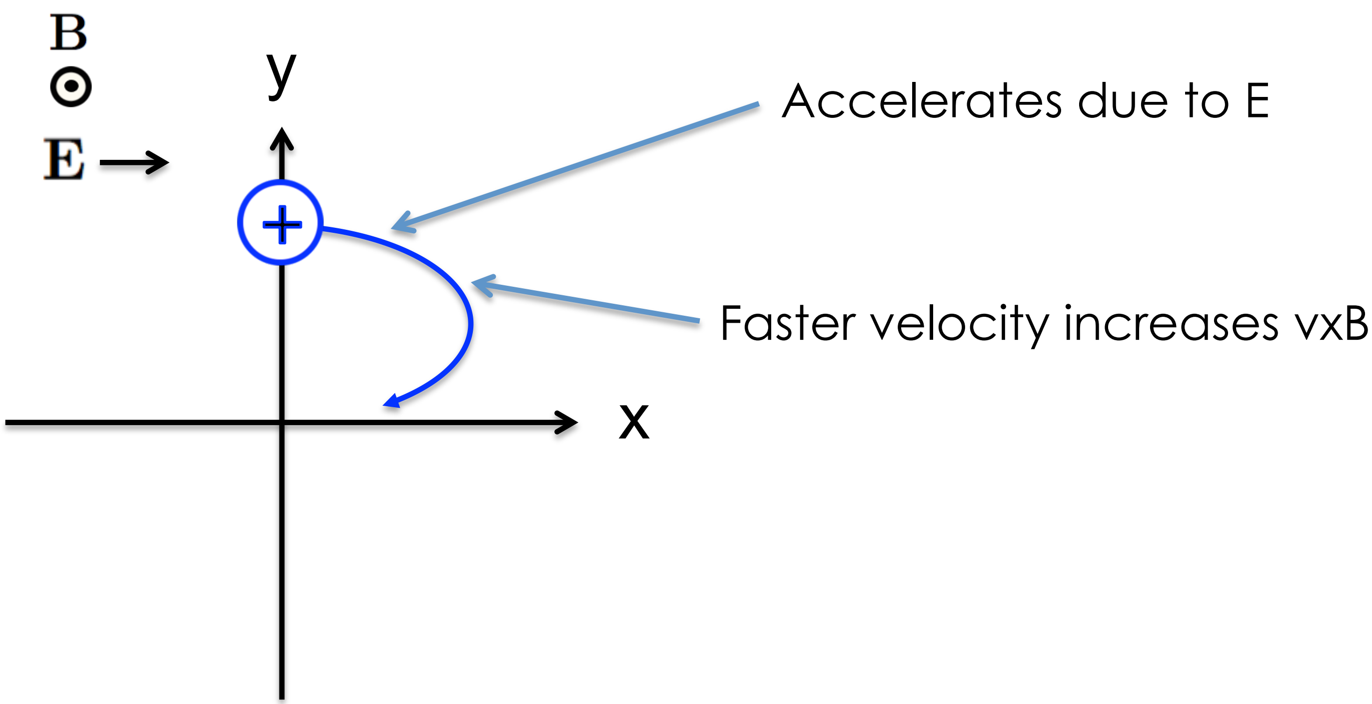
$$\mathbf{E} = E_x \hat{\mathbf{x}} \qquad \mathbf{B} = B_z \hat{\mathbf{z}}$$



Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

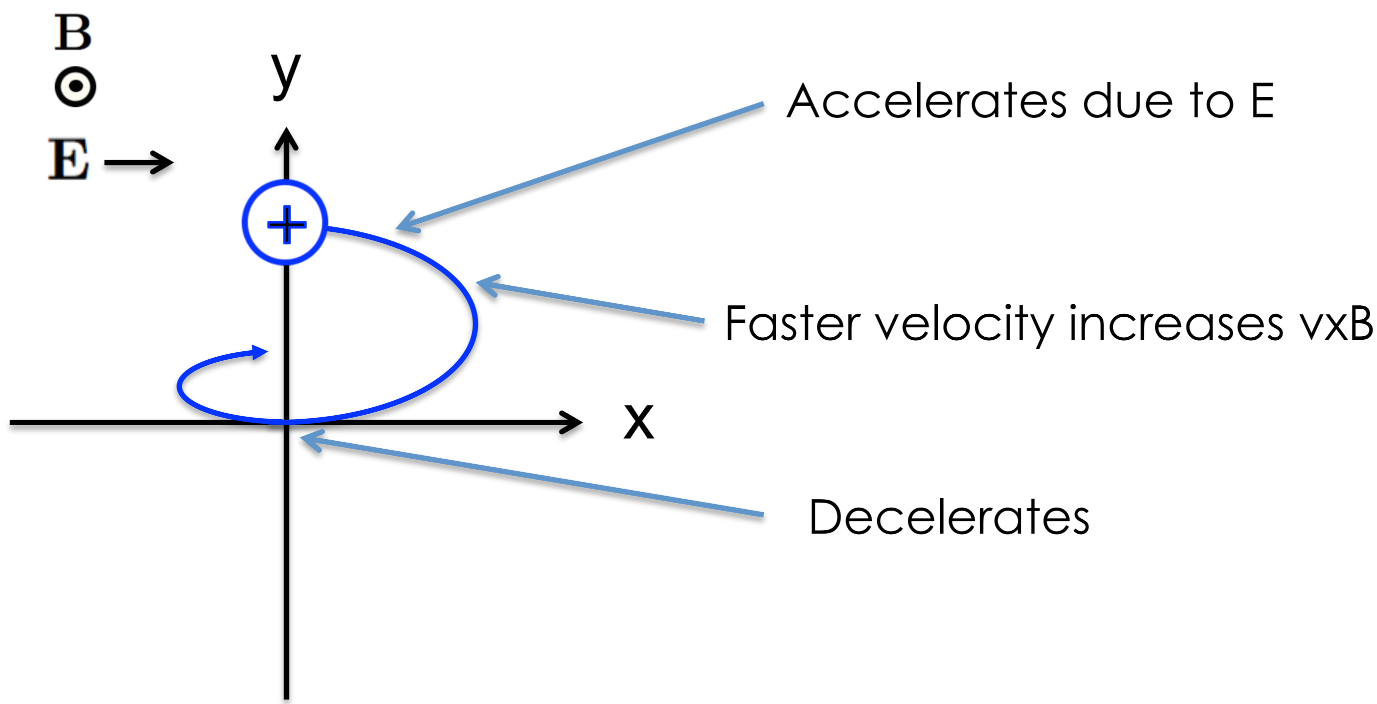
$$\mathbf{E} = E_x \hat{\mathbf{x}} \qquad \mathbf{B} = B_z \hat{\mathbf{z}}$$



Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

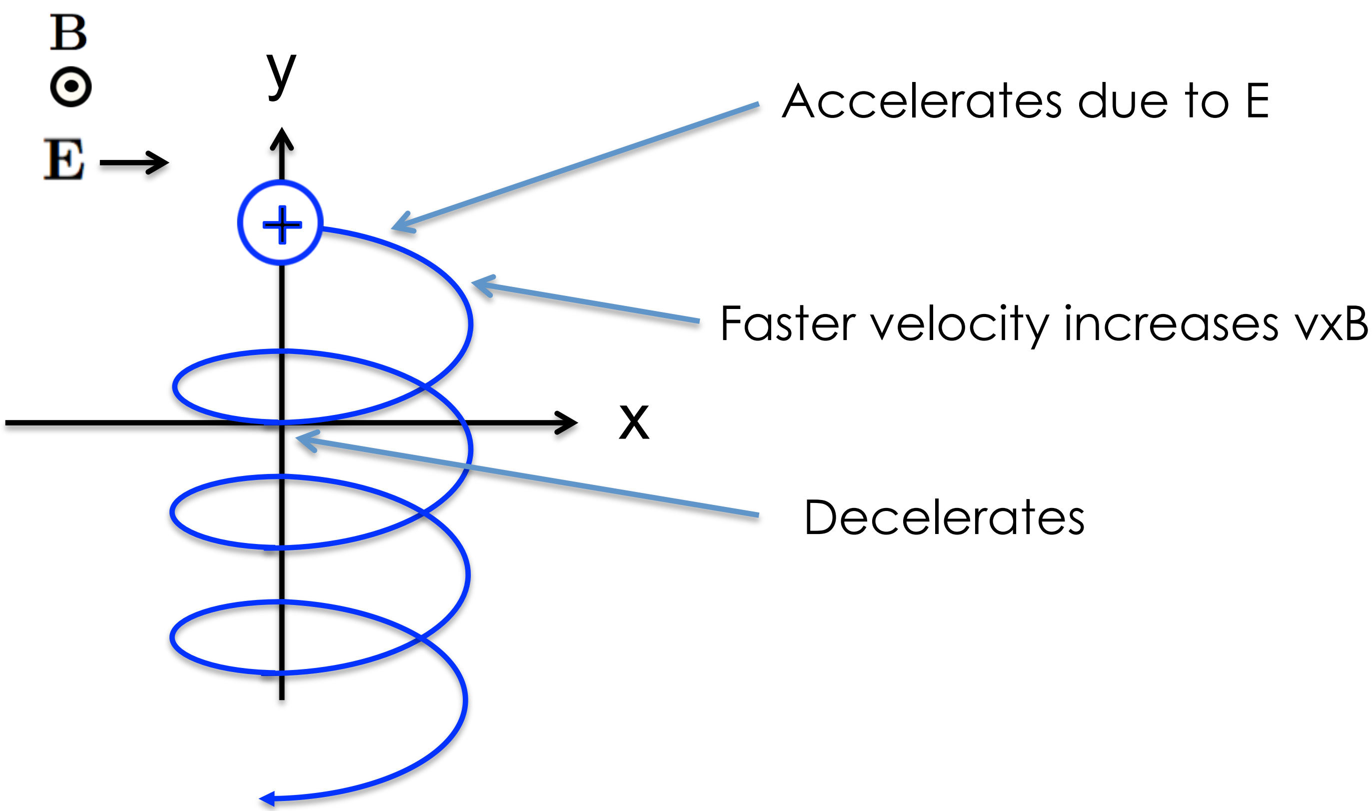
$$\mathbf{E} = E_x\hat{\mathbf{x}} \qquad \mathbf{B} = B_z\hat{\mathbf{z}}$$



Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

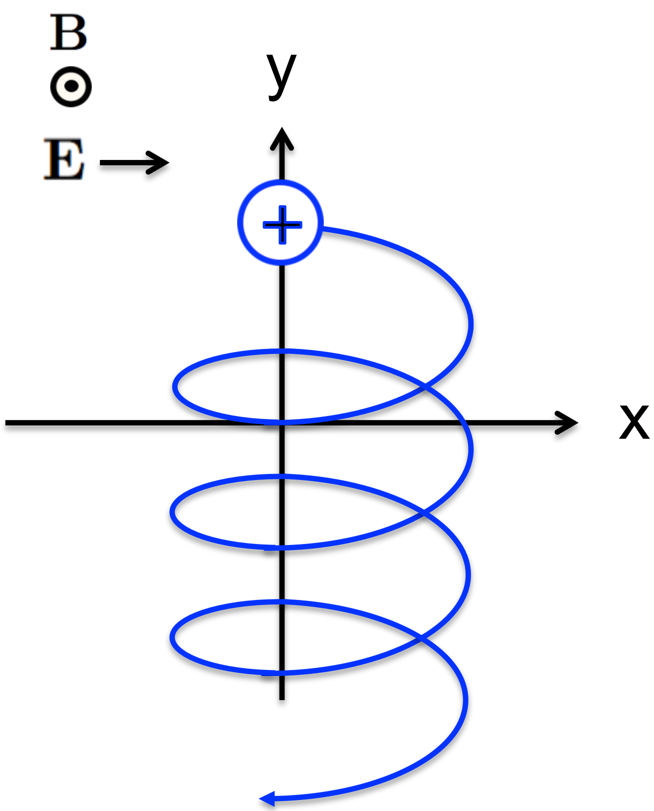
$$\mathbf{E} = E_x \hat{\mathbf{x}} \qquad \mathbf{B} = B_z \hat{\mathbf{z}}$$



Next Simplest Case to Analyze: Constant, Uniform Electric Field Perpendicular to Magnetic Field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{E} = E_x \hat{\mathbf{x}} \qquad \mathbf{B} = B_z \hat{\mathbf{z}}$$



$$\dot{v}_x = \frac{q}{m}(v_y B_z + E_x)$$

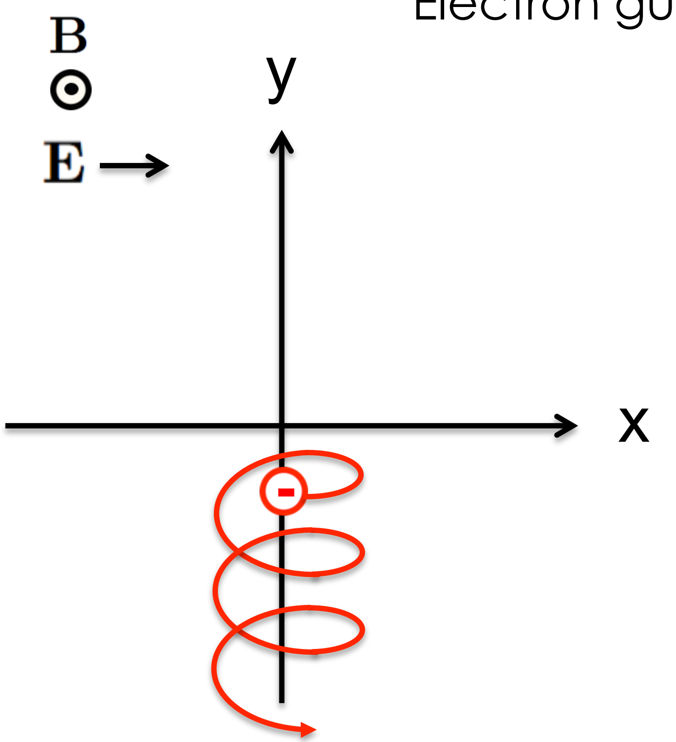
$$v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m} t + \phi_0\right) - \frac{E_x}{B_z}$$

Ion guiding center drifts in the direction $-\hat{\mathbf{y}}$

Guiding Center Drift Due to E x B

$$v_y = \mp v_\perp \sin\left(\frac{|q|B_z}{m}t + \phi_0\right) - \frac{E_x}{B_z}$$

Electron guiding center also drifts in the direction $-\hat{\mathbf{y}}$



The ExB drift can be written more generally as

$$\mathbf{v_E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- **ExB drift is independent of charge and mass**
- **Both electrons and ions move together**

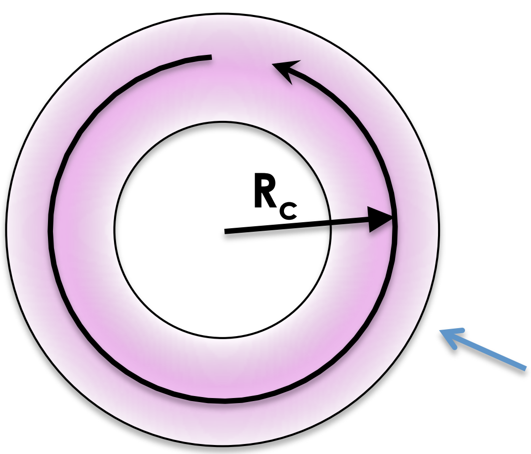
Other Forces Can Cause Guiding Center Drift

- **Any force perpendicular to B can cause particles to drift**

Drift due to force: $$\mathbf{v_d} = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2}$$

Examples of forces: $$\mathbf{F_g} = m\mathbf{g}$$ gravity

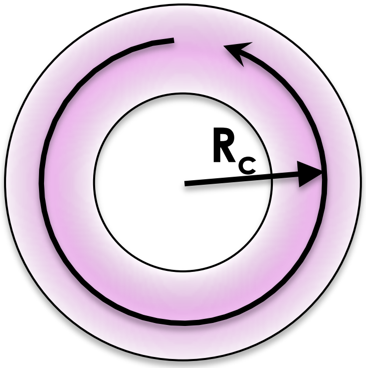
$$\mathbf{F_{cf}} = \frac{mv_{\parallel}^2}{R_c}\hat{\mathbf{r}}$$ centrifugal



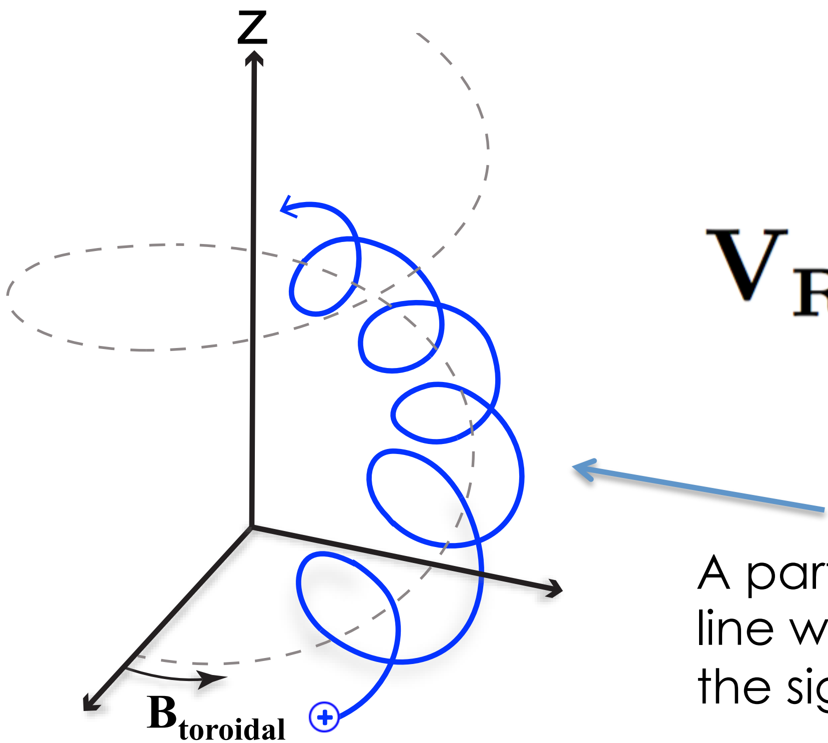
- Bend the magnetic field into a donut shape
- No end losses because the field lines go around and close on themselves
- BUT a particle following a toroidal magnetic field would experience $\mathbf{F_{cf}}$

Curvature Drift Due to Bending Field Lines

The outward centrifugal force causes curvature drift



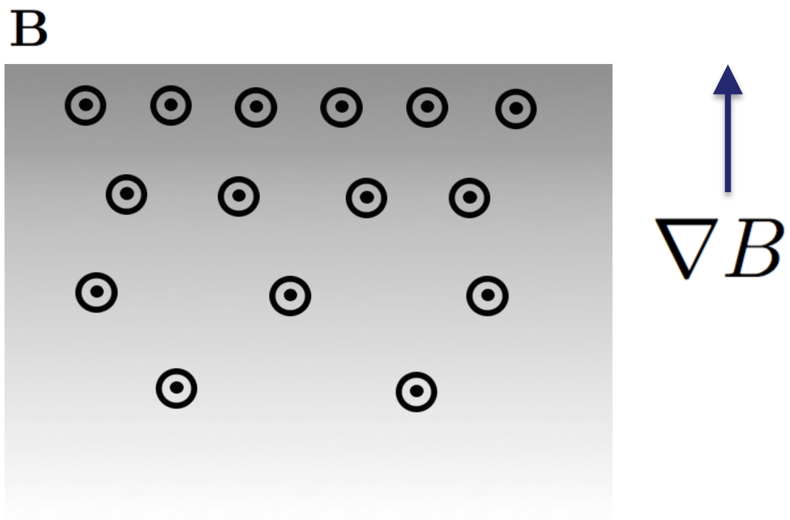
$$\mathbf{F_{cf}} = \frac{mv_\parallel^2}{R_c}\hat{\mathbf{r}}$$

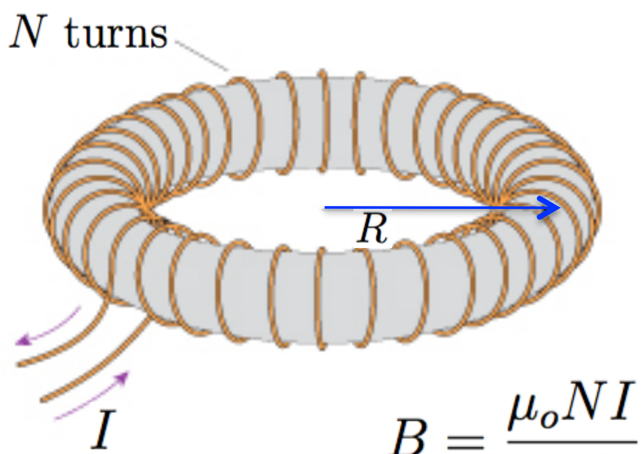


$$\mathbf{V_R} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R_c}\times\mathbf{B}}{R_c^2}$$

A particle moving along a curved field line will drift up or down, depending on the sign of the charge

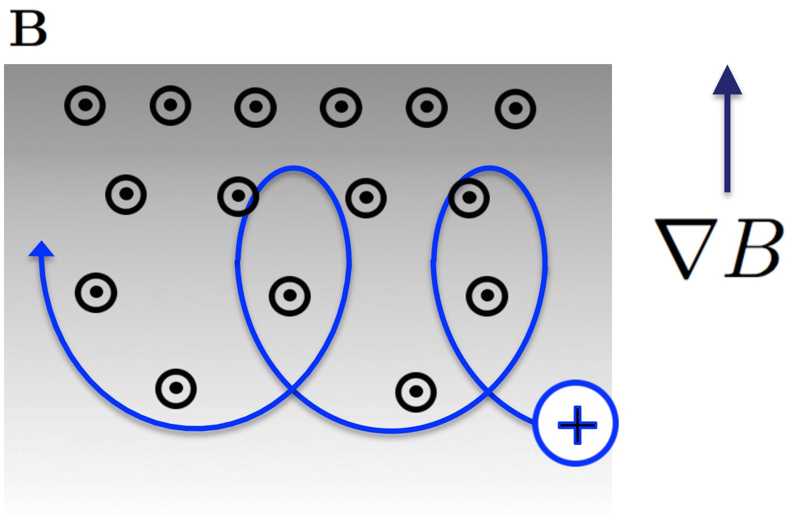
Spatially Varying Magnetic Field Strength Also Causes Drift

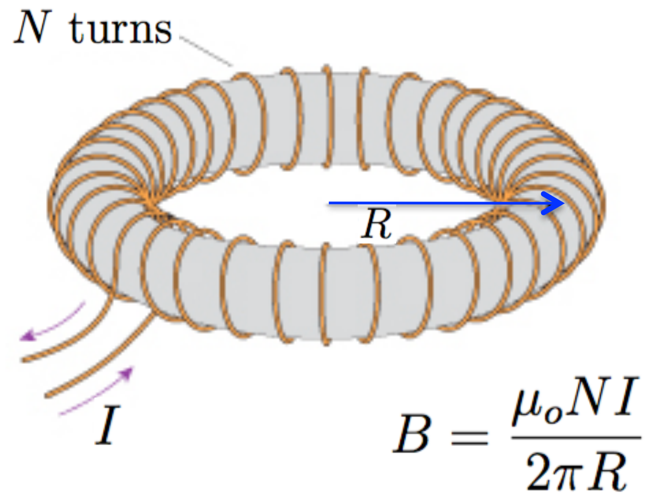




$$B = \frac{\mu_o N I}{2\pi R}$$

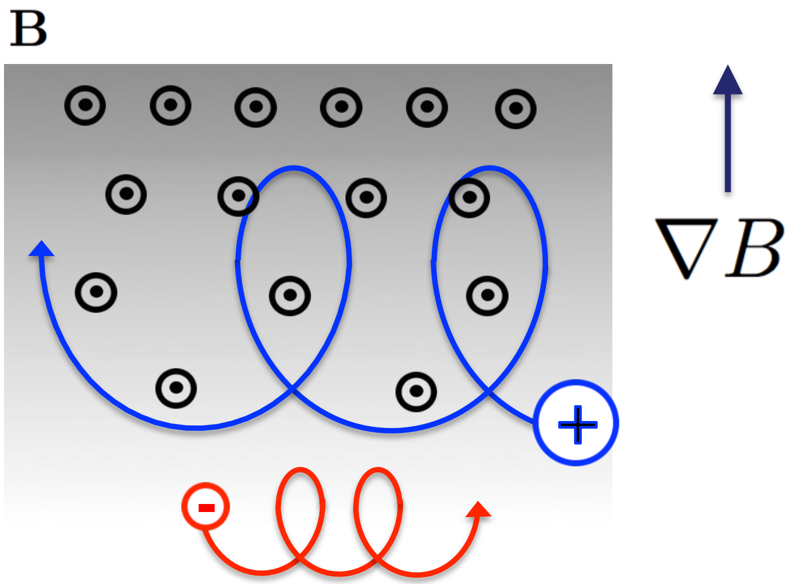
Spatially Varying Magnetic Field Strength Also Causes Drift

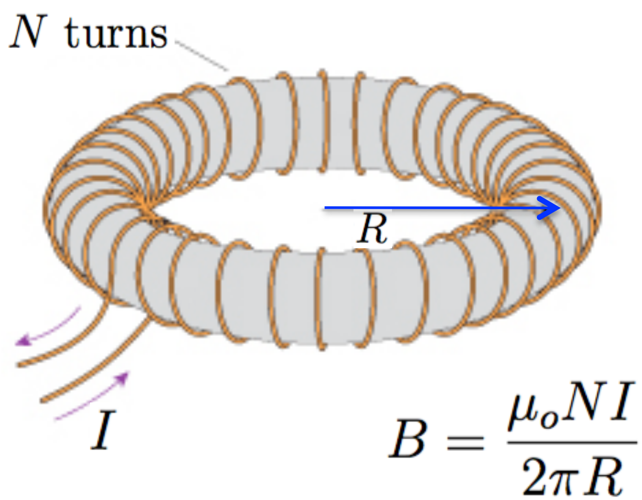




- **The gyro-radius will be larger where the field is weaker and smaller where the field is stronger**

Spatially Varying Magnetic Field Strength Also Causes Drift





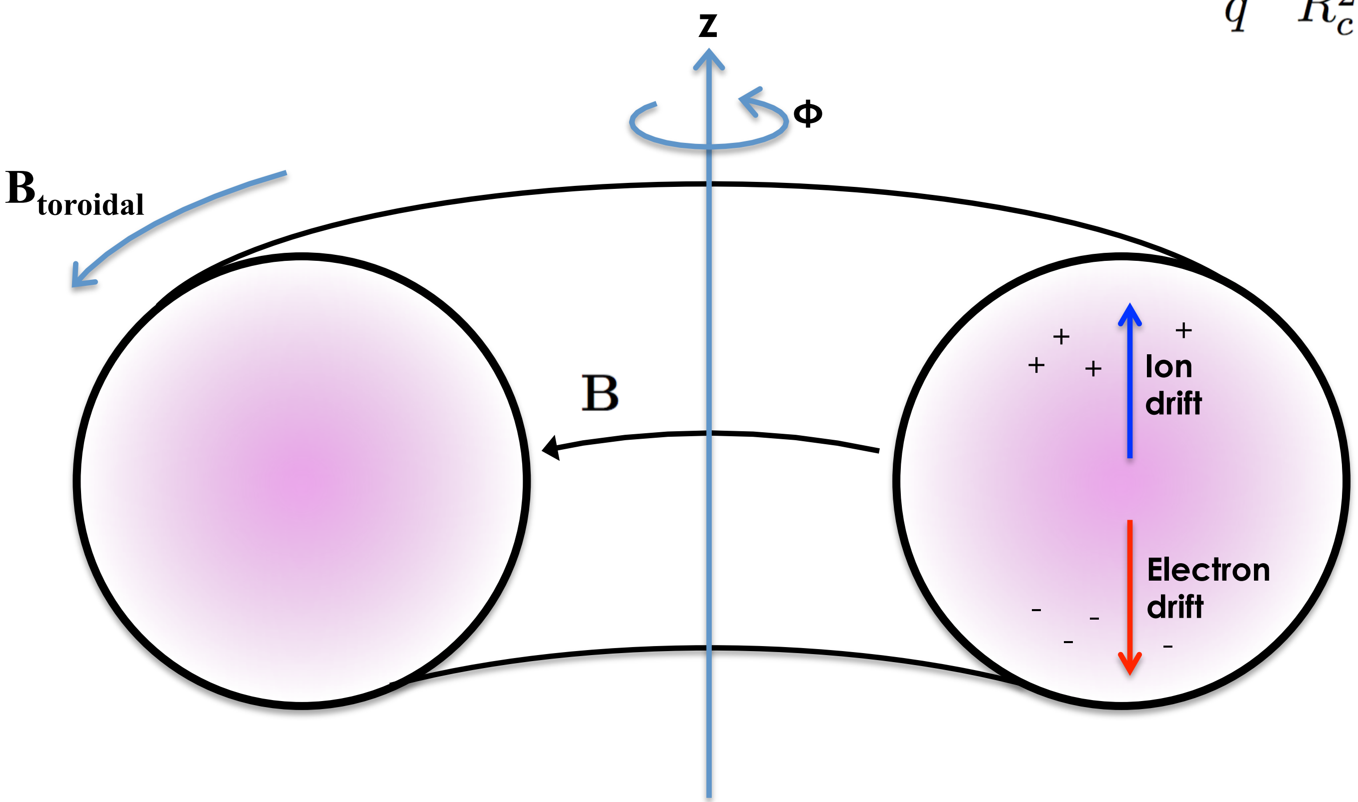
- The gyro-radius will be larger where the field is weaker and smaller where the field is stronger
- The resulting drift velocity is described by:

$$\mathbf{V}_{\nabla \mathbf{B}} = \frac{m v_\perp^2}{2qB} \frac{\mathbf{B} \times \nabla B}{B^2}$$

What Happens To Charged Particles In A Purely Toroidal Magnetic Field?

- Charged particles in a curved magnetic field will experience both ∇B and curvature drift: these effects add

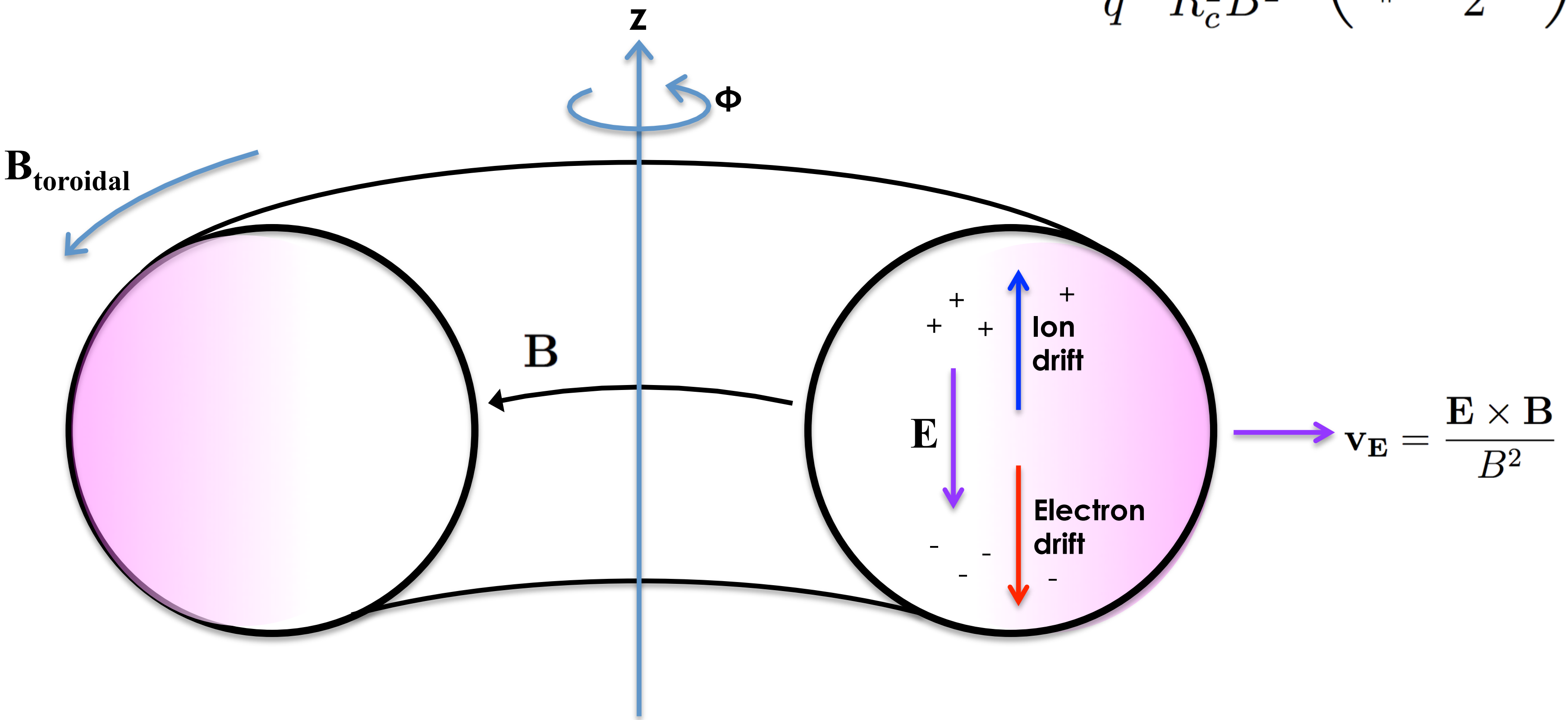
$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$



Charged Particles Will Drift Outward

- Charged particles in a curved magnetic field will experience both ∇B and curvature drift

$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$

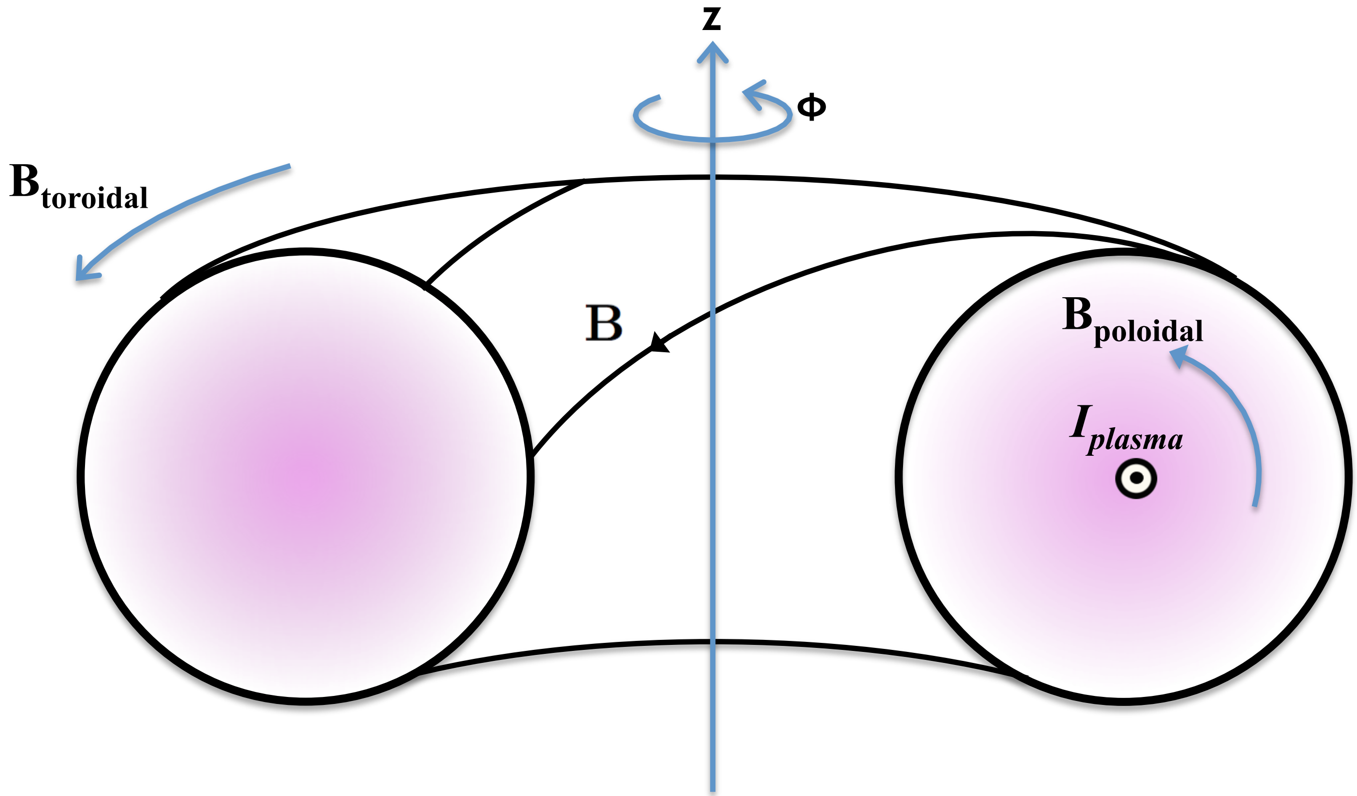


- This means that no matter what, particles in a torus with a purely toroidal field will drift radially out and hit the walls.

Tokamak Solution: Add Poloidal Magnetic Field

Toroidal: long way around
Poloidal: short way around

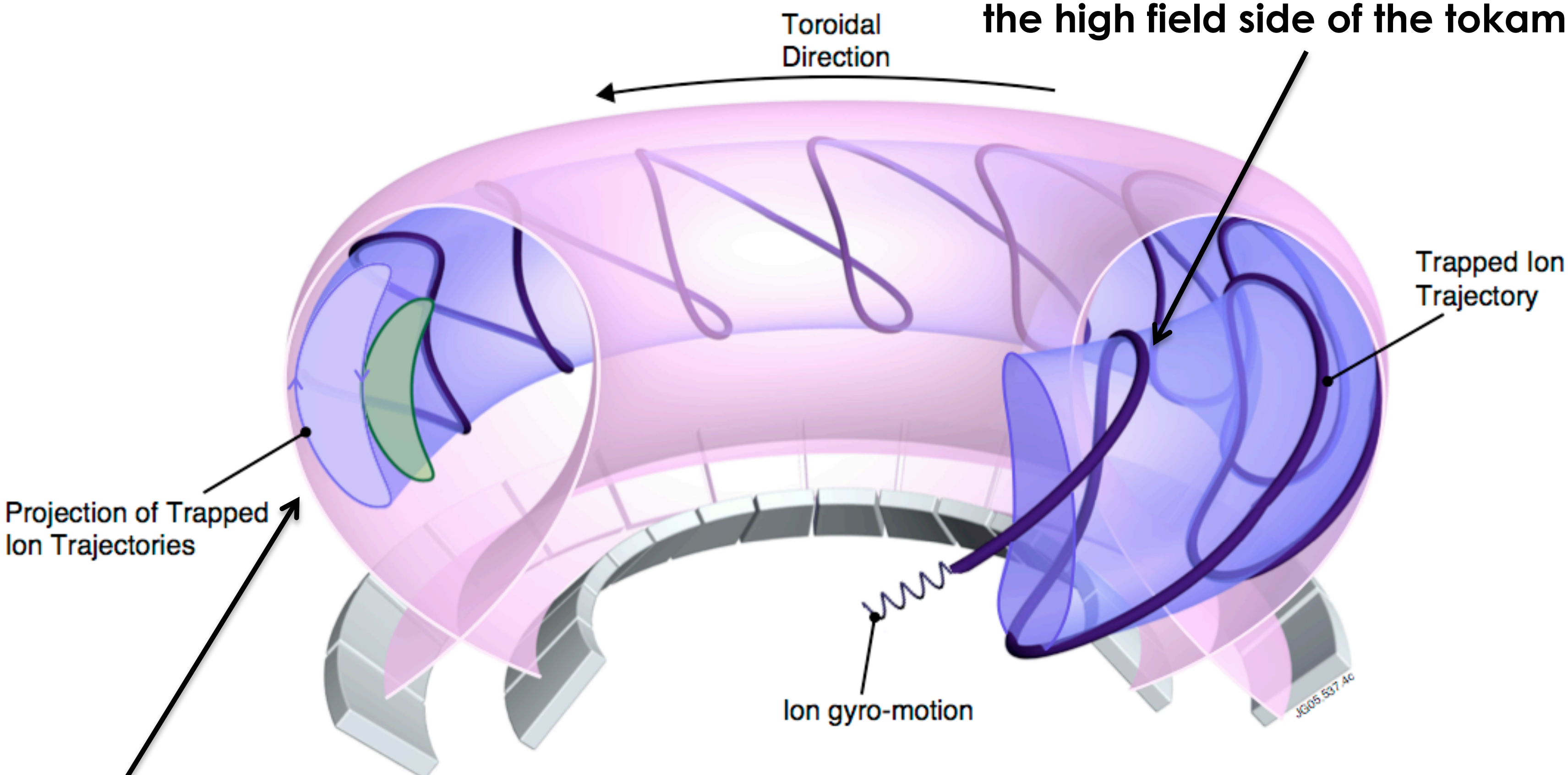
1. **Use external coils to apply a toroidal magnetic field**
2. **Drive toroidal current in the plasma to generate a poloidal magnetic field**



- **The resulting helical magnetic field is much better at confining charged particles.**
- **The challenge: how to drive current in plasma in steady state while keeping the plasma stable and free of disruptions?**

Banana Orbits

Particles that don't have enough $v_{||}$ are reflected by the mirror force at the high field side of the tokamak

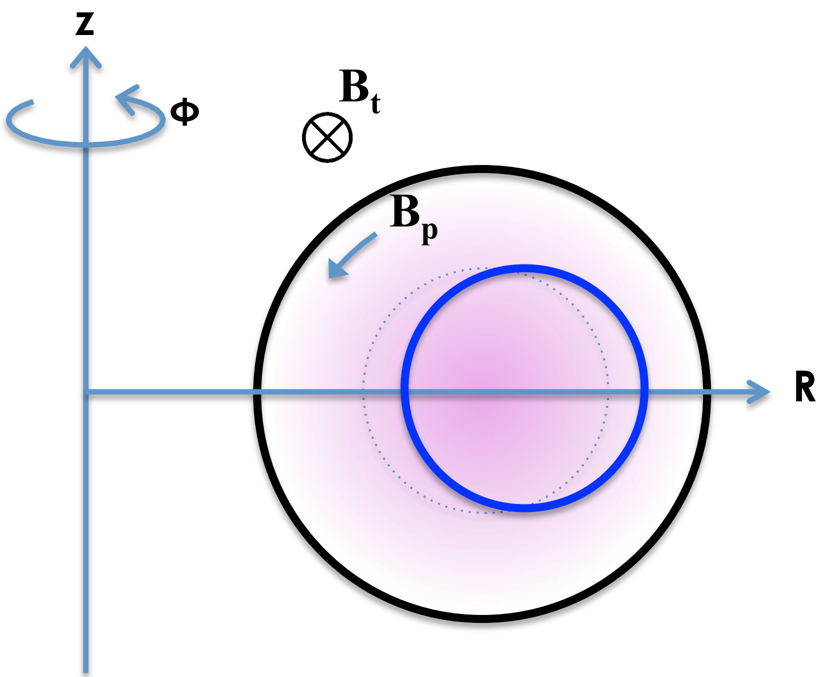


Trapped particles won't hit the wall if the banana orbit width Δr is small enough

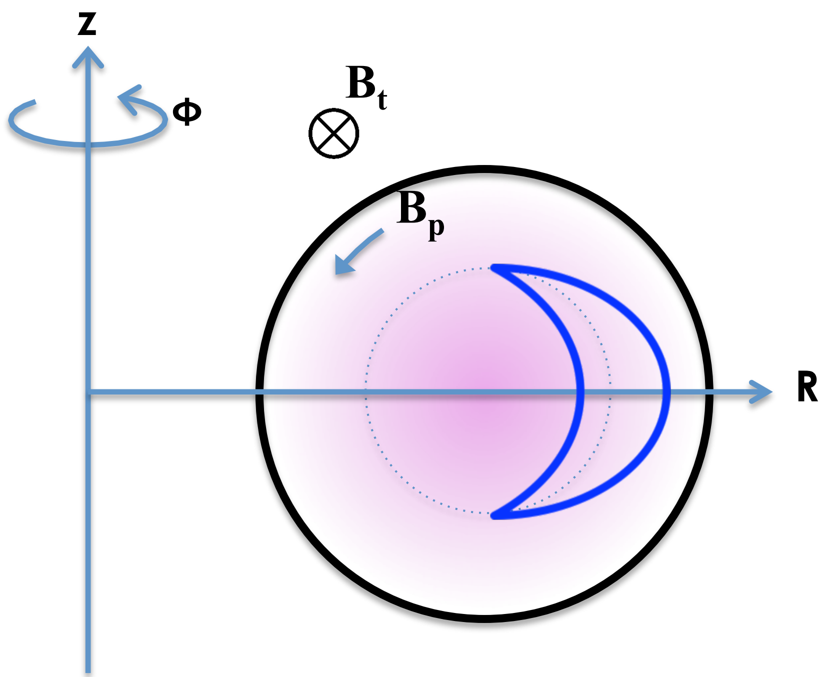
Image credit: euro-fusion.org

There Are Two Main Classes of Particle Orbits In Tokamaks

$$\mathbf{V_R} + \mathbf{V_{\nabla B}} = \frac{m}{q}\frac{\mathbf{R_c} \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$



Passing



Trapped

Particles with sufficient $\mathbf{v}_{||}$ will follow the helical magnetic field around the torus

Particles with lower $\mathbf{v}_{||}$ are reflected as they encounter stronger B and therefore execute "banana" orbits as they precess around the torus B

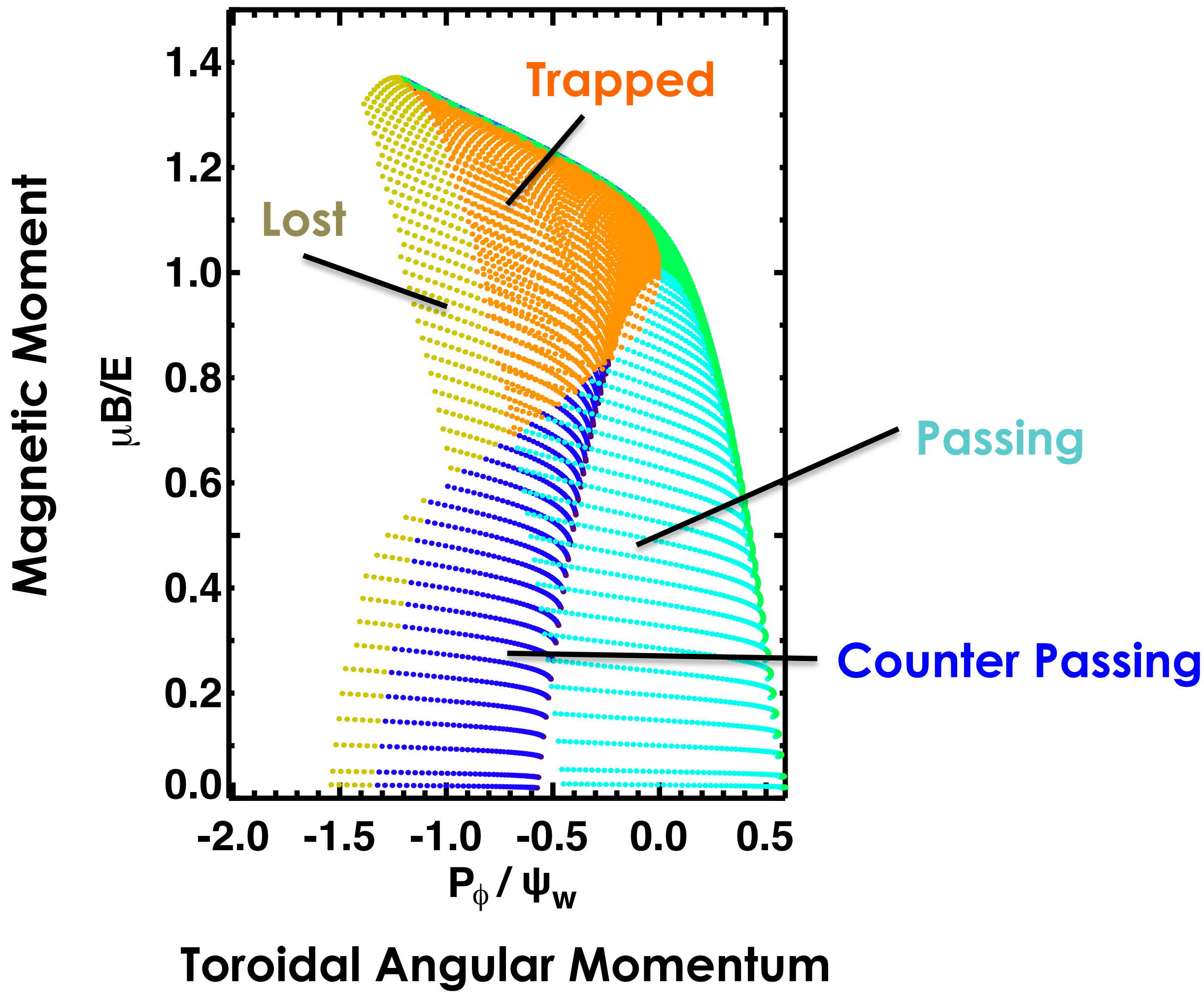
Conclusions

- **Charged particles undergo gyromotion about magnetic fields, and are free to move along the magnetic field line.**

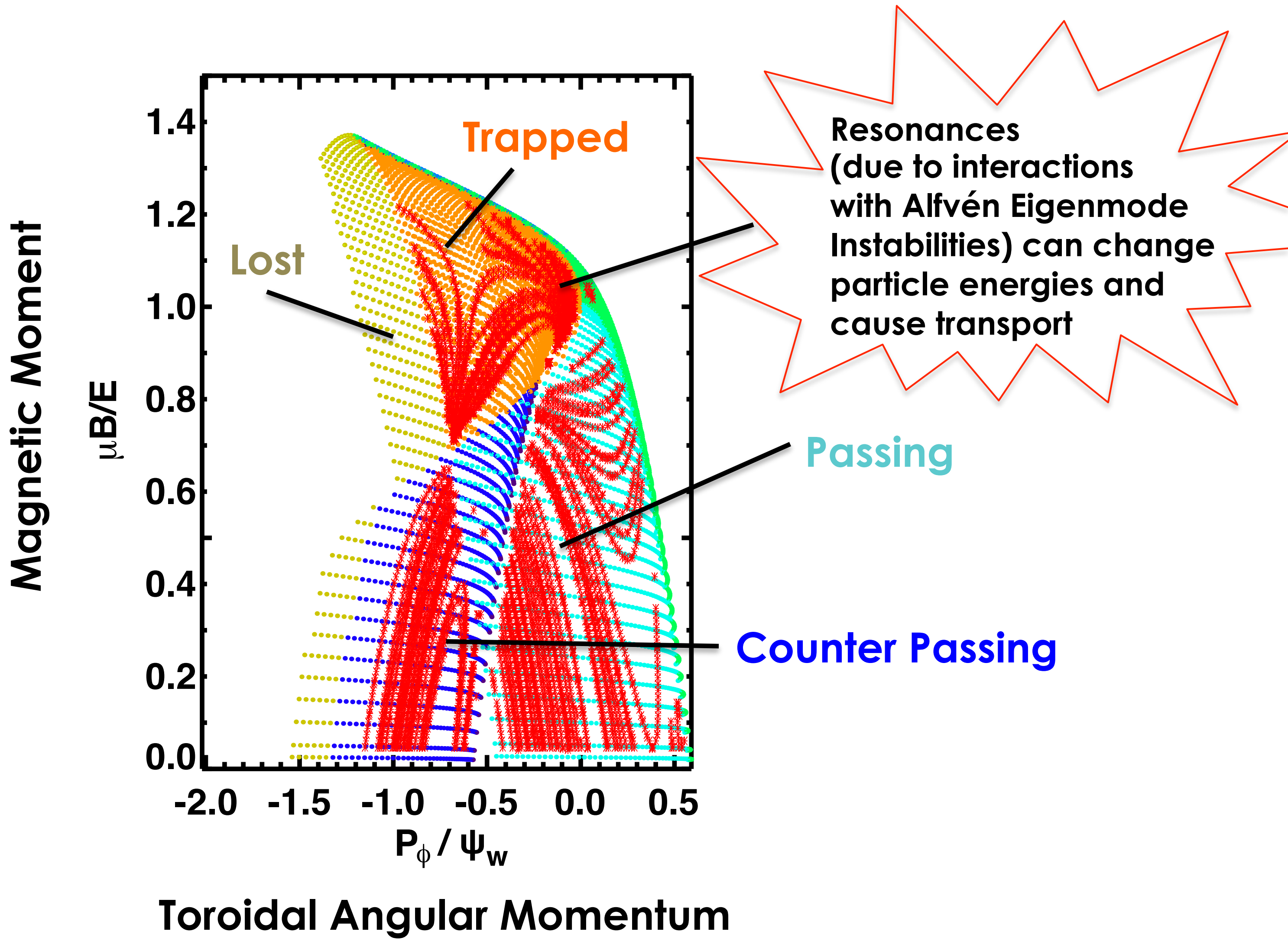
- **Depending on magnetic field geometry or the presence of other forces like electric fields, particles can drift across field lines (and even leave the system→hit the walls).**

- **The challenge for fusion energy research is to confine enough charged particles that are energetic enough for long enough lengths of time to achieve sustained fusion.**

Classifying Particle Orbits In Tokamaks Is Important in Understanding Basic Physics Mechanisms Like Wave-Particle Interactions



Classifying Particle Orbits In Tokamaks Is Important in Understanding Basic Physics Mechanisms Like Wave-Particle Interactions



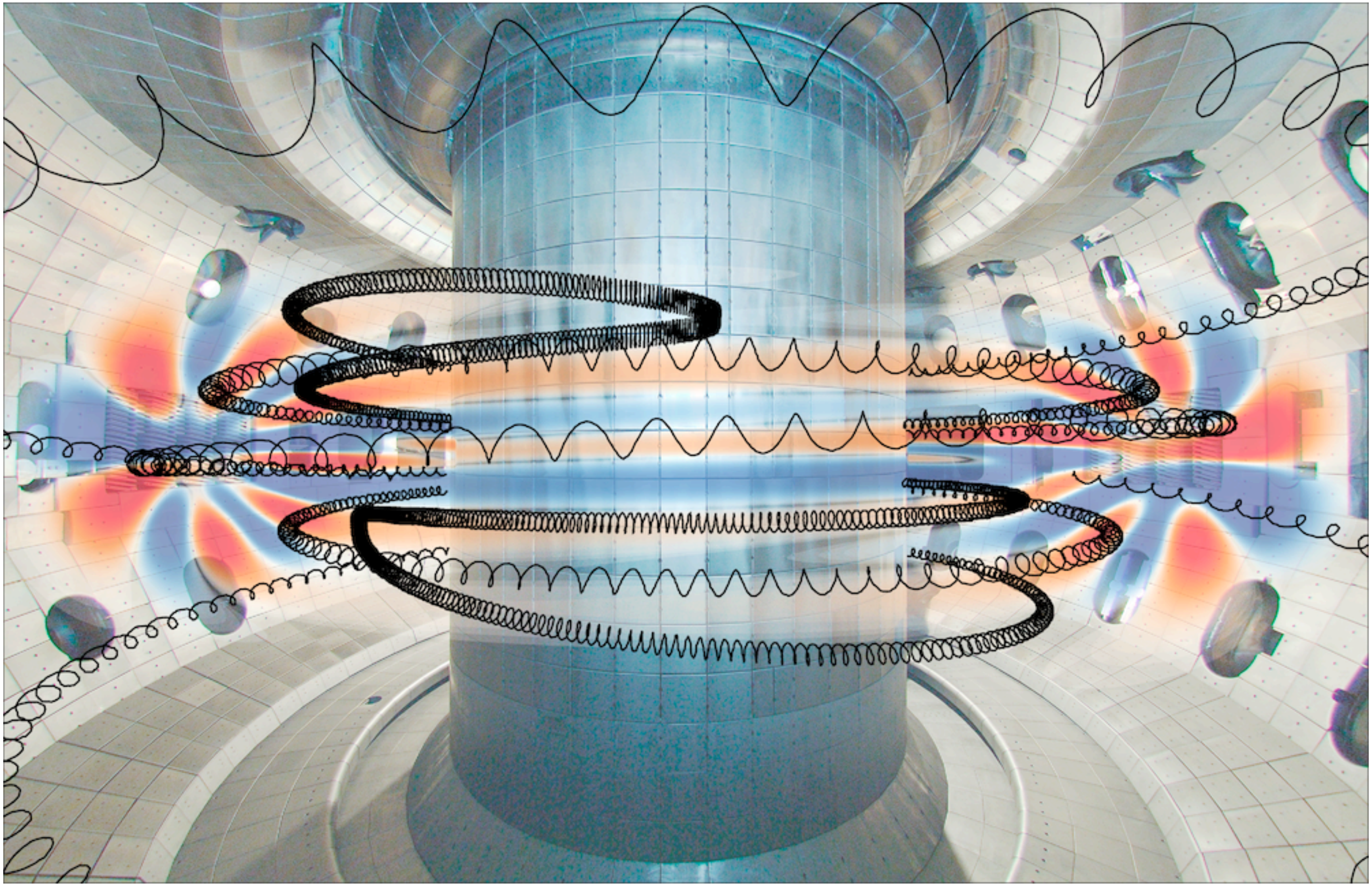


Image credit: Pace et. al.,*Physics Today* (2015)