Introduction to MagnetoHydroDynamics (MHD)

Equilibrium solution $\Psi$ (Pedestal, $\eta = 0.005$)

Antoine Cerfon, Courant Institute, New York University
Email: cerfon@cims.nyu.edu

SULI Introductory Course in Plasma Physics, June 8, 2015
PART I: DESCRIBING A FUSION PLASMA
**Method I: Self-consistent particle pushing**

- An intuitive idea is to solve for the motion of all the particles iteratively, combining Newton’s law with Maxwell equations.
- At each time step $i$, solve

$$m \frac{d^2 \mathbf{x}_k^{(i)}}{dt^2} = q_k \left( \mathbf{E}(\mathbf{x}_k)^{(i-1)} + \frac{d\mathbf{x}_k^{(i)}}{dt} \times \mathbf{B}^{(i-1)}(\mathbf{x}_k) \right) \quad k = 1, \ldots, N$$

$$\left( \frac{\partial \mathbf{E}}{\partial t} \right)^i = c^2 \nabla \times \mathbf{B}^{(i-1)} - \mu_0 c^2 \sum_{k=1}^{N} q_k \frac{d\mathbf{x}_k^i}{dt} \delta(\mathbf{x} - \mathbf{x}_k^i)$$

$$\left( \frac{\partial \mathbf{B}}{\partial t} \right)^i = -\nabla \times \mathbf{E}^i$$

- Fast solvers exist for the electromagnetic fields, some relying on a subsidiary mesh, some not needing a mesh.
- Even with fast solvers, problem still not tractable even with the most powerful computers when $N \sim 10^{20} - 10^{22}$ as in magnetic fusion grade plasmas.
Method II: Coarse-grain average in phase space

(From G. Lapenta’s: https://perswww.kuleuven.be/ u0052182/weather/pic.pdf)

- For hot and diffuse systems with a large number of particles, following every single particle is a waste of time and resources.
- Replace the discrete particles with smooth distribution function $f(x, v, t)$ defined so that

$$f(x, v, t)dx dv$$

is the expected number of particles in the infinitesimal six-dimensional phase-space volume $dx dv$. 
Distribution function and Vlasov equation

- Macroscopic (fluid) quantities are velocity moments of $f$

  $$n(x, t) = \iiint f(x, v, t) dv$$  \hspace{1cm} \text{Density}$$

  $$nV(x, t) = \iiint vf(x, v, t) dv$$  \hspace{1cm} \text{Mean flow}$$

  $$P(x, t) = m \iiint (v - V)(v - V) f dv$$  \hspace{1cm} \text{Pressure tensor}$$

- Conservation of $f$ along the phase-space trajectories of the particles determines the time evolution of $f$:

  $$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt} \cdot \nabla f + \frac{dv}{dt} \cdot \nabla vf = 0$$

  $$\frac{dx}{dt} = v \hspace{1cm} \frac{dv}{dt} = \frac{q}{m} (E + v \times B)$$

  $$\Rightarrow \frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m} (E + v \times B) \cdot \nabla vf = 0$$

This is the Vlasov equation
The Boltzmann Equation

- In fusion plasmas, we separate, leading to the Boltzmann equation:

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \left( \frac{\partial f}{\partial t} \right)_c
\]

This equation to be combined with Maxwell’s equations:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\]

- Nonlinear, integro-differential, 6-dimensional PDE
- Describes phenomena on widely varying length \((10^{-5} – 10^3 \text{ m})\) and time \((10^{-12} – 10^2 \text{ s})\) scales
- Still not a piece of cake, and never solved as such for fusion plasmas
**Moment approach**

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \left( \frac{\partial f}{\partial t} \right)_c
\]

- Taking the integrals \( \iiint d\mathbf{v}, \iiint m\mathbf{v} d\mathbf{v} \) and \( \iiint m\mathbf{v}^2 / 2 d\mathbf{v} \) of this equation, we obtain the exact **fluid equations**:

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0 \quad \text{Continuity}
\]

\[
m n \left( \frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) - \nabla \cdot \mathbf{P}_s + \mathbf{R}_s \quad \text{Momentum}
\]

\[
\frac{d}{dt} \left( \frac{3}{2} p_s \right) + \frac{5}{2} p_s \nabla \cdot \mathbf{V}_s + \mathbf{\pi}_s : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = 0 \quad \text{(Energy)}
\]

with \( \mathbf{P}_s = p_s \mathbf{I} + \mathbf{\pi}_s \).

- **Closure problem**: for each moment, we introduce a new unknown \( \Rightarrow \) End up with too many unknowns
- **Need to make approximations** to close the moment hierarchy
**Kinetic models vs Fluid models**

- For some fusion applications/plasma regimes (heating and current drive, transport), kinetic treatment cannot be avoided.

- Simplify and reduce dimensionality of the Vlasov equation with approximations:
  - Strong magnetization: Gyrokinetic equation
  - Small gyroradius compared to relevant length scales: Drift kinetic equation
  - Vanishing gyroradius: Kinetic MHD

- In contrast, fluid models are based on approximate expressions for higher order moments (off-diagonal entries in pressure tensor, heat flux) in terms of lower order quantities (density, velocity, diagonal entries in pressure tensor).

- We will now focus on the relevant regime and the approximations made to derive a widely used fluid model: the ideal MHD model.
PART II: THE IDEAL MHD MODEL
**Lawson Criterion and MHD**

Condition for ignition: \[ p \tau_E \geq 8 \text{ bar.s} \quad T_{min} \sim 15 \text{keV} \]

- The maximum \( p \) is limited by the **stability** properties
  **Job of MHD**
- The maximum \( \tau_E \) is determined by the **confinement** properties
  **Job of kinetic models**
PHILOSOPHY

- The purpose of ideal MHD is to study the macroscopic behavior of the plasma

- Use ideal MHD to design machines that avoid large scale instabilities

- Regime of interest
  - Typical length scale: the minor radius of the device $a \sim 1m$
    Wave number $k$ of waves and instabilities considered: $k \sim 1/a$
  
  - Typical velocities: Ion thermal velocity speed $v_T \sim 500km/s$
  
  - Typical time scale: $\tau_{MHD} \sim a/v_T \sim 2\mu s$
    Frequency $\omega_{MHD}$ of associated waves/instabilities $\omega_{MHD} \sim 500kHz$
**EXAMPLE: VERTICAL INSTABILITY**

FIG. 2. Loss of vertical position control following a feedback cut at 0.470 s.

Figure from F. Hofmann *et al.*, Nuclear Fusion 37 681 (1997)
**Ideal MHD - Maxwell’s equations**

- $a \gg \lambda_D$, the distance over which charge separation can take place in a plasma
  - On the MHD length scale, the plasma is neutral: $n_i = n_e$

- $\omega_{\text{MHD}}/k \ll c$ and $v_{Ti} \ll v_{Te} \ll c$ so we can neglect the displacement current in Maxwell’s equations:

\[
\begin{align*}
  n_i &= n_e \\
  \nabla \cdot \mathbf{B} &= 0 \\
  \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
  \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{align*}
\]
**Ideal MHD - Momentum Equation**

- $a \gg \lambda_D$ and $a \gg r_{Le}$ (electron Larmor radius)
- $\omega_{MHD} \ll \omega_{pe}, \omega_{MHD} \ll \omega_{ce}$
- The ideal MHD model assumes that on the time and length scales of interest, the electrons have an infinitely fast response time to changes in the plasma
- Mathematically, this can be done by taking the limit $m_e \to 0$
- Adding the ion and electron momentum equation, we then get

\[
\rho \frac{dV}{dt} - J \times B + \nabla p = -\nabla \cdot (\pi_i + \pi_e)
\]

where $\rho = m_i n$ and $V$ is the ion fluid velocity

- If the condition $v_{Ti} \tau_{ii}/a \ll 1$ is satisfied in the plasma

\[
\rho \frac{dV}{dt} = J \times B - \nabla p \quad \text{(Ideal MHD momentum equation)}
\]
**Ideal MHD - Electrons**

- In the limit $m_e \to 0$, the electron momentum equation can be written as

\[
E + V \times B = \frac{1}{en} (J \times B - \nabla p_e - \nabla \cdot \pi_e + R_e)
\]

- This is called the **generalized Ohm’s law**

- Different MHD models (resitive MHD, Hall MHD) keep different terms in this equation

- If $r_{Li}/a \ll 1$, $\nu_{Ti}\tau_{ii}/a \ll 1$, and \((m_e/m_i)^{1/2}(r_{Li}/a)^2(a/\nu_{Ti}\tau_{ii}) \ll 1\), the momentum equation becomes the **ideal Ohm’s law**

\[
E + V \times B = 0
\]

- The ideal MHD plasma behaves like a **perfectly conducting fluid**
ENERGY EQUATION

▶ Define the total plasma pressure \( p = p_i + p_e \)

▶ Add electron and ion energy equations

▶ Under the conditions \( r_{Li}/a \ll 1 \) and \( v_{Ti} \tau_{ii}/a \ll 1 \), this simplifies as

\[
\frac{d}{dt} \left( \frac{p}{\rho^{5/3}} \right) = 0
\]

▶ Equation reminiscent of \( pV^\gamma = Cst \): the ideal MHD plasma behaves like a monoatomic ideal gas undergoing a reversible adiabatic process
**Ideal MHD - Summary**

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0
\]

\[
\rho \frac{d \mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p
\]

\[
\frac{d}{dt} \left( \frac{p}{\rho^{5/3}} \right) = 0
\]

\[
\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

Valid under the conditions

\[
\left( \frac{m_i}{m_e} \right)^{1/2} \left( \frac{v_i \tau_{ii}}{a} \right) \ll 1 \quad \frac{r_{Li}}{a} \ll 1 \quad \left( \frac{r_{Li}}{a} \right)^2 \left( \frac{m_e}{m_i} \right)^{1/2} \frac{a}{\nu_{Ti} \tau_{ii}} \ll 1
\]
Validity of the Ideal MHD Model (I)

- Are the conditions for the validity of ideal MHD mutually compatible?

\[
\left( \frac{m_i}{m_e} \right)^{1/2} \left( \frac{v_i \tau_{ii}}{a} \right) \ll 1 \quad \frac{r_{Li}}{a} \ll 1 \quad \left( \frac{r_{Li}}{a} \right)^2 \left( \frac{m_e}{m_i} \right)^{1/2} \frac{a}{v_{Ti} \tau_{ii}} \ll 1
\]

- Define \( x = (m_i/m_e)^{1/2}(v_{Ti} \tau_{ii}/a) \), \( y = r_{Li}/a \).

\( x \ll 1 \) (High collisionality) \( y \ll 1 \) (Small ion Larmor radius)

\( y^2/x \ll 1 \) (Small resistivity)

There exists a regime for which ideal MHD is justified (Figure from Ideal MHD by J.P. Freidberg, CUP, 2014)

Is that the regime of magnetic confinement fusion?
Validity of the ideal MHD model (II)

- Express three conditions in terms of $n$, $T$, $a$ and $\beta$, with $\beta$ the ratio of plasma pressure and magnetic pressure.

- For $\beta = 5\%$ and $a = 1m$ (realistic fusion parameters), we find

The regime of validity of ideal MHD does NOT coincide with the fusion plasma regime (Figure from Ideal MHD by J.P. Freidberg, CUP, 2014)

The collisionality of fusion plasmas is too low for the ideal MHD model to be valid.

Is that a problem?
Validity of the ideal MHD model (III)

- It turns out that ideal MHD often does a very good job at predicting stability limits for macroscopic instabilities.

- This is not due to luck but to subtle physical reasons.

- One can show that collisionless kinetic models for macroscopic instabilities are more optimistic than ideal MHD.

- This is because ideal MHD is accurate for dynamics perpendicular to the fields lines.

- Designs based on ideal MHD calculations are conservative designs.
FROZEN IN LAW (I)

- $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$: in the frame moving with the plasma, the electric field is zero
- The plasma behaves like a perfect conductor
- The magnetic field lines are “frozen” into the plasma motion
**Frozen in Law (II): Proof**

- \( \frac{\partial B}{\partial t} = \nabla \times E \), \( E + V \times B = 0 \) \( \Rightarrow \) \( \frac{\partial B}{\partial t} = \nabla \times (V \times B) \)
- Calculate the change in the flux \( \Phi = \int \int_{S(t)} B \cdot ndS \) across a moving surface with velocity \( u_\perp \)

\[
\frac{d\Phi}{dt} = \int \int_{S(t)} \frac{\partial B}{\partial t} \cdot ndS - \int_{\partial S(t)} u_\perp \times B \cdot dl
\]

\[
= \int \int \nabla \times (V \times B) \cdot ndS - \int_{\partial S(t)} u_\perp \times B \cdot dl
\]

\[
= \int_{\partial S(t)} (V - u_\perp) \cdot dl
\]

\[
= 0 \quad \text{if} \quad u_\perp = V
\]

i.e. the plasma is tied to the field lines
Magnetic reconnection: a key phenomenon in astrophysical, space, and fusion plasmas

- Cannot happen according to ideal MHD
- Need to add additional terms in Ohm’s law to allow reconnection: resistivity, off-diagonal pressure tensor terms, electron inertia, ...
- Associated instabilities take place on longer time scales than $\tau_{MHD}$
PART III: MHD EQUILIBRIUM
By equilibrium, we mean steady-state: $\partial / \partial t = 0$

Often, for simplicity and/or physical reasons, we focus on static equilibria: $\mathbf{V} = 0$

\[
\nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\
\mathbf{J} \times \mathbf{B} = \nabla p
\]

A more condensed form is

\[
\nabla \cdot \mathbf{B} = 0 \quad (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p
\]

Note that the density profile does not appear
1D EQUILIBRIA (I)

Combine the two to get....

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)
1D EQUILIBRIA (II)

Screw pinch

(Figure from Ideal MHD by J.P. Freidberg, CUP, 2014)

- Equilibrium quantities only depend on $r$
- Plug into $\nabla \cdot \mathbf{B} = 0$, $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ to find:

$$
\frac{d}{dr} \left( p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0
$$

Balance between plasma pressure, magnetic pressure, and magnetic tension
- Two free functions define equilibrium: e.g. $B_z$ and $p$, or $B_\theta$ and $B_z$
2D EQUILIBRIA: GEOMETRY

Top view

Cross section

Toroidal axisymmetry: $\partial / \partial \phi \equiv 0$
**TOROIDALLY AXISYMMETRIC EQUILIBRIA**

**Step 1:**

\[ \mathbf{B} = B_\phi(R, Z)\mathbf{e}_\phi + \mathbf{B}_p(R, Z) \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0 \implies \nabla \cdot \mathbf{B}_p = 0 \]

\[ \implies \mathbf{B} = B_\phi \mathbf{e}_\phi + \frac{1}{R} \nabla \Psi \times \mathbf{e}_\phi \]

\( \Psi = RA_\phi \), with \( \mathbf{A} \) vector potential: \( \nabla \times \mathbf{A} = \mathbf{B} \).

**Step 2:**

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \implies \begin{cases} 
\mu_0 J_\phi = -\frac{1}{R} \left[ R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \Psi}{\partial R} \right) + \frac{\partial^2 \Psi}{\partial Z^2} \right] = -\frac{1}{R} \Delta^* \Psi \\
\mu_0 J_p = \frac{1}{R} \nabla (RB_\phi \times \mathbf{e}_\phi) 
\end{cases} \]

**Step 3:**

\( \mathbf{J} \times \mathbf{B} = \nabla p \begin{cases} 
\cdot \mathbf{B} \implies \nabla \psi \times \nabla p = 0 \implies p = p(\Psi) \\
\cdot \mathbf{J} = 0 \implies \nabla (RB_\phi) \times \nabla \psi = 0 \implies RB_\phi = F(\Psi) 
\end{cases} \)
- The regions of **constant pressure** are nested toroidal surfaces

- Magnetic fields and currents **lie on these nested surfaces**
**Grad-Shafranov Equation**

Last step: \([\mathbf{J} \times \mathbf{B} = \nabla p] \cdot \nabla \Psi\) gives the Grad-Shafranov equation (GSE):

\[
R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \Psi}{\partial R} \right) + \frac{\partial^2 \Psi}{\partial Z^2} = -\mu_0 R^2 \frac{dp}{d\Psi} - F \frac{dF}{d\Psi}
\]

- Second-order, nonlinear, elliptic PDE. Derived independently by H. Grad\(^1\) and V.D. Shafranov\(^2\).

- The free functions \(p\) and \(F\) determine the nature of the equilibrium

- In general, the GSE has to be solved **numerically**

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\(^2\)Sov. Phys. JETP 6, 545 (1958)
Grad-Shafranov equilibrium for JET tokamak
EXAMPLES (II)

Grad-Shafranov equilibrium for Field Reversed Configuration
Numerical solution to the Grad-Shafranov equation

- Magnetic equilibrium serves as input to stability, wave and transport codes ⇒ important to develop fast and accurate solvers

- Many, many solvers available, from very simple to very advanced (FD, FEM, Integral equations, inverse solvers, …)

- Free boundary equilibria more challenging than fixed boundary equilibria

- Equilibria with purely toroidal flow are determined by a close variant of the Grad-Shafranov equation ⇒ many Grad-Shafranov codes can compute such equilibria

Equilibria with both toroidal and poloidal flow can be much more challenging; only a handful of codes available
3D EQUILIBRIA (I)

\[ \frac{\partial}{\partial \phi} \neq 0 \]
3D EQUILIBRIA (II)

- Equilibrium equations $\nabla \cdot \mathbf{B} = 0$, $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ still hold
- Existence of nested toroidal surfaces not guaranteed anymore

- Computing 3D equilibria fast and accurately still a challenge
- Several existing codes, based on different assumptions/approximations and used to design and study stellarators: VMEC, PIES, SPEC, HINT, NSTAB
PART IV: MHD STABILITY
What do we mean by MHD stability?

- That the plasma is initially in equilibrium does not mean it is going to remain there.

- The plasma is constantly subject to perturbations, small and large.

- The purpose of stability studies is to find out how the plasma will react to these perturbations:
  - Will it try to return to the initial steady-state?
  - Will it find a new acceptable steady-state?
  - Will it collapse?
A mechanical analog

Figure from J.P. Freidberg, *Ideal MHD*, Cambridge University Press (2014)
SOLVING FULL NONLINEAR MHD EQUATIONS

- Here is an idea to study stability of a magnetically confined plasma:
  - Choose a satisfying plasma equilibrium
  - Perturb it
  - Solve the full MHD equations with a computer
  - Analyze results

- Such an approach provides knowledge of the entire plasma dynamics

- There exist several numerical codes that can do that, for various MHD models (not only ideal): M3D, M3D-C1, NIMROD

- Computationally intensive

- Get more information than one needs?
Figure from R. Paccagnella et al., Nuclear Fusion 49 035003 (2009)
**Linear Stability (I)**

- Ideal MHD dynamics can be so fast and detrimental that one may often require linear stability for the equilibrium

- This can simplify the mathematical analysis tremendously

- Start with an MHD equilibrium:
  \[ \nabla \cdot \mathbf{B}_0 = 0, \quad (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \mu_0 \nabla p_0 \]

- Take full ideal MHD equations, and write \( Q = Q_0(\mathbf{r}) + Q_1(\mathbf{r}, t) \) for each physical quantity, where \( Q_1 \) is considered very small compared to \( Q_0 \)

- Drop all the terms that are quadratic or higher orders in the quantities \( Q_1 \) (linearization)
By design, the system is now **linear** in the unknown quantities $\rho_1, v_1, J_1, B_1, p_1$

- Much easier to solve in a computer
- There’s a trick that makes life even easier
LINEAR STABILITY (III)

- Introduce the plasma displacement vector \( \xi \) defined such that

\[
\mathbf{v}_1 = \frac{\partial \xi}{\partial t}, \quad \mathbf{v}_1(r, 0) = \frac{\partial \xi}{\partial t}(r, 0), \quad \xi(r, 0) = 0
\]

- Linearized ideal MHD equations reduce to

\[
\rho \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F}(\xi) \quad \text{with}
\]

\[
\mathbf{F}(\xi) = \frac{1}{\mu_0} \left\{ \nabla \times \left[ \nabla \times (\xi \times \mathbf{B}_0) \right] \right\} \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \left[ \nabla \times (\xi \times \mathbf{B}_0) \right]
\]

\[
+ \nabla \left( \xi \cdot \nabla \rho_0 + \gamma \rho_0 \nabla \cdot \xi \right)
\]

- \( \mathbf{F} \) is called the ideal MHD linear force operator

- The problem of linear stability is reduced to an initial value problem with three linear equations and three unknowns: the components of \( \xi \)
LINEAR STABILITY (IV): NORMAL MODE ANALYSIS

- Even the IVP in the previous slide may give more information than we need
- Sometimes, we just want to know if the equilibrium is stable or not
- A normal mode analysis provides the desired framework for this
- Write $\xi(\mathbf{r}, t) = \hat{\xi}(\mathbf{r}) e^{-i\omega t}$.
  $\omega_I > 0$ corresponds to exponential growth.
- The linearized momentum equation takes the form
  
  $$-\rho \omega^2 \hat{\xi} = F(\hat{\xi})$$

- $\omega^2$ is an an eigenvalue of the linear operator $-F(\hat{\xi})/\rho$
- It can be showed (some lines of algebra...) that $F$ is a self-adjoint operator
- In ideal MHD, $\omega^2$ is a purely real quantity
- $\omega^2 \geq 0$ means the mode is stable; $\omega^2 \leq 0$ means the mode is unstable
EIGENVALUES IN IDEAL MHD

Fig. 6.6. (a) Stable waves and (b) instabilities in ideal MHD.

Figure from *Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas* by J.P. Goedbloed and S. Poedts, Cambridge University Press (2004)
ILLUSTRATION: WAVES IN IDEAL MHD (I)

- Consider the stability of an infinite, homogeneous plasma:
  \[ \mathbf{B} = B_0 \hat{e}_z \]
  \[ \mathbf{J} = \mathbf{0} \]
  \[ p = p_0 \]
  \[ \rho = \rho_0 \]
  \[ \mathbf{v} = \mathbf{0} \]

- Given geometry, expand \( \hat{\xi}(\mathbf{r}) \) as
  \[ \hat{\xi}(\mathbf{r}) = \tilde{\xi} e^{i \mathbf{k} \cdot \mathbf{r}} \]

- Dynamics is anisotropic because of the magnetic field: \( \mathbf{k}_\perp + k_\parallel \hat{e}_z \)

- Without loss of generality, \( \mathbf{k} = k_\perp \hat{e}_y + k_\parallel \hat{e}_z \)
ILLUSTRATION: WAVES IN IDEAL MHD (II)

- $-\rho \omega^2 \hat{\xi} = F(\hat{\xi})$ can be written as

$$\omega^2 \rho_0 \tilde{\xi} = \frac{B_0^2}{\mu_0} \left\{ \textbf{k} \times \left[ \textbf{k} \times (\tilde{\xi} \times \textbf{e}_z) \right] \right\} \times \textbf{e}_z + \gamma p_0 \textbf{k} \cdot \tilde{\xi}$$

- Writing the expression for each component, we get the system

$$\begin{bmatrix}
\omega^2 - k^2 v_A^2 & 0 & 0 \\
0 & \omega^2 - k^2 v_A^2 - k^2 v_S^2 & -k^2 v_S^2 \\
0 & -k^2 v_S^2 & \omega^2 - k^2 v_S^2 \\
\end{bmatrix} \begin{bmatrix}
\hat{\xi}_x \\
\hat{\xi}_y \\
\hat{\xi}_z \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}$$

- Two key velocities appear:

$$v_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \quad v_S = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

$v_A$ is called the Alfvén velocity, in honor of Hannes Alfvén, the Swedish scientist who first described MHD waves, and $v_S$ is the adiabatic sound speed.
ILLUSTRATION: WAVES IN IDEAL MHD (III)

\[
\begin{bmatrix}
\omega^2 - k_{||}^2 v_A^2 & 0 & 0 \\
0 & \omega^2 - k_{||}^2 v_A^2 - k_{\perp}^2 v_S^2 & -k_{||} k_{\perp} v_S^2 \\
0 & -k_{\perp} k_{||} v_S^2 & \omega^2 - k_{\perp}^2 v_S^2
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}_x \\
\hat{\xi}_y \\
\hat{\xi}_z
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

- For nontrivial solutions, determinant of the matrix should be 0
- This leads to the following three possibilities for $\omega^2$:

\[
\omega^2 = k_{||}^2 v_A^2 , \quad \omega^2 = \frac{k^2}{2} \left( v_A^2 + v_S^2 \right) \left[ 1 \pm \sqrt{1 - 4 \frac{k_{||}^2 v_A^2 v_S^2}{k^2 (v_S^2 + v_A^2)^2}} \right]
\]

- One can see that $\omega^2 \geq 0$
- The infinite homogeneous magnetized plasma is always MHD stable
- Some of the modes above become unstable in magnetic fusion configurations, because of gradients and field line curvature
**Shear Alfvén Wave**

- Branch $\omega^2 = k^2 v_A^2$

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Transverse wave

- Balance between plasma inertia and field line tension

- Incompressible ⇒ often the most unstable MHD mode in fusion devices
Fast magnetosonic wave

Fast magnetosonic wave given by

\[
\omega^2 = \frac{k^2}{2} \left( v_A^2 + v_S^2 \right) \left[ 1 + \sqrt{1 - 4 \frac{k^2}{k^2} \frac{v_A^2 v_S^2}{v_S^2 + v_A^2} (v_S^2 + v_A^2)^2} \right]
\]

Physics simplifies in the limit \( v_S^2 \ll v_A^2 \): it is then called the compressional Alfvén wave, with dispersion relation \( \omega^2 = k^2 v_A^2 \)

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

Plasma motion perpendicular to field lines, compressible

Oscillation between plasma kinetic energy and magnetic compressional energy
**Slow Magnetosonic Wave**

- Slow magnetosonic wave given by

\[
\omega^2 = \frac{k^2}{2} (v_A^2 + v_S^2) \left[ 1 - \sqrt{1 - 4 \frac{k^2}{k^2} \frac{v_A^2 v_S^2}{(v_S^2 + v_A^2)^2}} \right]
\]

- Physics simplifies in the limit \(v_S^2 \ll v_A^2\): it is then called the sound wave, with dispersion relation \(\omega^2 = k^2 v_S^2\)

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Plasma motion parallel to field lines, compressible
- Oscillation between plasma kinetic energy and plasma internal energy (plasma pressure)
Common MHD instabilities (I)

Interchange instability

(Figure from Plasma Physics and Fusion Energy by J.P. Freidberg, CUP, 2008)
COMMON MHD INSTABILITIES (II)

Ballooning instability

(Figure from Plasma Physics and Fusion Energy by J.P. Freidberg, CUP, 2008)
Common MHD instabilities (III)

Kink instability

(Figure from Plasma Physics and Fusion Energy by J.P. Freidberg, CUP, 2008)
LINEAR STABILITY: ENERGY APPROACH

- Change in potential energy: \( \delta W = \vec{F} \cdot \vec{dl} \)

- Static equilibrium condition at \( x_0 \): \( \delta W \bigg|_{x=x_0} = 0 \)

- Stability

\[ \delta^2 W \bigg|_{x=x_0} > 0 \]
\[ \delta^2 W \bigg|_{x=x_0} < 0 \]
\[ \delta^2 W \bigg|_{x=x_0} = 0 \]
IDEAL MHD ENERGY PRINCIPLE (I)

- For historical reasons, the second variation is called \( \delta W \) in the plasma physics jargon
- A useful variational principle can be derived in ideal MHD, called the energy principle, which takes the following form:

\[
\omega^2 = \frac{\delta W(\xi)}{K(\xi)}
\]

where

\[
\delta W(\xi) = -\frac{1}{2} \int \xi \cdot F(\xi) dV
\]

\[
F(\xi) = \frac{1}{\mu_0} \left\{ \nabla \times [\nabla \times (\xi \times B_0)] \right\} \times B_0 + (\nabla \times B_0) \times [\nabla \times (\xi \times B_0)]
+ \nabla (\xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi)
\]

\[
K(\xi) = \frac{1}{2} \int \rho |\xi|^2 dV
\]
Ideal MHD Energy Principle (II)

\[ \omega^2 = \frac{\delta W(\xi)}{K(\xi)} \quad \delta W(\xi) = -\frac{1}{2} \int \xi \cdot F(\xi) dV \]

- **Energy Principle**: An ideal MHD equilibrium is stable if and only if \( \delta W(\xi) > 0 \) for all bounded \( \xi \) satisfying the boundary conditions.

- Energy principle very useful to prove instability of an equilibrium by coming up with a good guess for \( \xi \) that makes \( \delta W \) negative.

- Formula also very helpful to calculate the \( \omega^2 \) numerically with high accuracy.
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