AIP | Physics of Fluids

Plasma equilibrium with rational magnetic surfaces

Allen H. Boozer

Citation: Physics of Fluids **24**, 1999 (1981); doi: 10.1063/1.863297
View online: http://dx.doi.org/10.1063/1.863297
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/24/11?ver=pdfcov
Published by the AIP Publishing

Plasma equilibrium with rational magnetic surfaces

Allen H. Boozer

Princeton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544
(Received 5 February 1981; accepted 14 August 1981)

The self-consistent classical plasma equilibrium with diffusion is studied in a toroidal geometry having a sheared magnetic field. Near each rational surface it is found that the pressure gradient is zero unless the Fourier component of $1/B^2$, which resonates with that surface, vanishes. Despite the resonances, the overall plasma confinement is, in practice, only slightly modified by the rational surfaces.

I. INTRODUCTION

Magnetic surfaces on which the rotational transform ι is a rational number have long been known to be associated with singularities in the plasma equilibrium. Clear discussions of this problem have been given by Grad[1] and Solov'ev and Shafranov.[2] On surfaces with irrational ι, the magnetic field lines ergodically cover the entire surface and the pressure is constant on the surface. On rational surfaces, the magnetic field lines close on themselves. The condition for plasma equilibrium with closed field lines is that the integral $\oint dl/B$ be constant on a pressure surface. In systems in which ι depends on radius, the pressure surfaces defined by the irrational surfaces do not have $\oint dl/B$ constant on rational surfaces except for special cases such as toroidal symmetry.

In this paper we will examine the problem of self-consistent plasma equilibrium in a toroidal system with shear. We find that the plasma equilibrium is controlled by the Fourier transform of $1/B^2$ in the appropriate toroidal and poloidal angles ϕ and θ. Let δ_{nm} be proportional to the nth toroidal harmonic and the mth poloidal harmonic of $1/B^2$; then, near a rational surface $\iota(\psi_r)=n/m$

$$(j_\parallel)_P \propto \frac{\delta_{nm}}{(\psi-\psi_r)}\frac{dP}{d\psi}\cos(n\phi-m\theta),$$

$$\Gamma \propto -\eta_\parallel \frac{|\delta_{nm}|^2}{(\psi-\psi_r)^2}\frac{dP}{d\psi},$$

$$\Phi-\Phi_0(\psi) \propto \eta_\parallel \frac{\delta_{nm}}{(\psi-\psi_r)^2}\frac{dP}{d\psi}\sin(n\phi-m\theta),$$

with $(j_\parallel)_P$ the pressure driven part of the parallel current, P the pressure, Γ the total flux of particles crossing a surface, $\eta_\parallel$ the parallel resistivity, and $\Phi-\Phi_0(\psi)$ the variation of the electrostatic potential in the surface. The smoothness of the particle flux implies $dP/d\psi \propto |\delta_{nm}|^2(\psi-\psi_r)^2$ near a rational surface. Unless $|\delta_{nm}|$ vanishes, which is equivalent to $\oint dl/B$ being constant on the rational surface, $dP/d\psi \propto (\psi-\psi_r)^2$. Assuming $|\delta_{nm}|\neq 0$, we then find $(j_\parallel)_P \propto (\psi-\psi_r)$, that is, it vanishes at $\psi=\psi_r$, while the potential variation remains finite. Since $(j_\parallel)_P$ vanishes everywhere as the plasma pressure goes to zero, any vacuum field configuration with magnetic surfaces and shear gives a plasma equilibrium at low enough plasma pressure. The discussion of toroidal equilibrium by Solov'ev and Shafranov[2] dismissed the possibility that $dP/d\psi$ might be zero at resonant rational surfaces as unphysical. Ohkawa[3] found a resonant enhancement of the diffusion coefficient in the presence of helical perturbations, but the general problem of toroidal confinement was not considered.

Other papers which are related to the work reported here are by Kruskal and Kulsrud,[4] Hamada,[5] Greene and Johnson,[6] and Grad.[7]

In Sec. II of the paper the appropriate coordinate system will be established, in Sec. III the equation for the parallel plasma current is derived and solved, in Sec. IV the consequences of Ohm's law are explored, and the conclusions are given in Sec. V.

II. MAGNETIC COORDINATES

Solenoidal vectors such as the magnetic field can always be written in the so-called Clebsch representation

$$\mathbf{B}=\nabla\psi\times\nabla\theta_0 \tag{1}$$

with a field line defined by constant ψ and θ_0. Since we are assuming a scalar pressure with

$$\nabla P=(1/c)\mathbf{j}\times\mathbf{B}, \tag{2}$$

the Clebsch coordinate ψ can be chosen as a function of P alone. The systems we are considering have topologically toroidal pressure surfaces so the function $\psi(P)$ can be chosen with $2\pi\psi$ equal to the magnetic flux inside a pressure surface (i.e., the toroidal flux). This choice of ψ makes θ_0 an angle-like variable.

In addition to the Clebsch or contravariant representation, a magnetic field with a scalar pressure can be written in the covariant form[8]

$$\mathbf{B}=\nabla\chi+\beta\nabla\psi \tag{3}$$

with ψ, θ_0, χ as coordinates. An important role is played by the arbitrariness in the ψ, θ_0, χ, β representation of $\mathbf{B}$. Since ψ is defined, this arbitrariness occurs only in θ_0, χ, and β. It is easy to show that if θ_0, χ, and β represent $\mathbf{B}$, then $\bar\theta_0$, $\bar\chi$, and $\bar\beta$ give a representation if, and only if,

$$\bar\theta_0=\theta_0+\theta_*(\psi),\quad \bar\chi=\chi+\chi_*(\psi),\quad \bar\beta=\beta-\frac{d\chi_*}{d\psi}. \tag{4}$$

The functions θ_* and χ_* are arbitrary functions of ψ.

Although many fundamental properties of the plasma equations are easily illustrated using θ_0 and χ as coordinates, they do obscure the toroidal and the poloidal periodicities of the torus. Angular coordinates θ and ϕ linearly related to θ_0 and χ make this periodicity manifest. Suppose we circuit the torus once toroidally

 0031-9171/81/111999-05$00.90

and come back to the same physical point. In general, χ and θ_0 will not return to their original values $\chi(0)$ and $\theta_0(0)$. Rather after a toroidal circuit

$$\chi = \chi(0) + 2\pi g, \quad \theta_0 = \theta_0(0) - 2\pi x. \tag{5}$$

Both $(\chi(0), \theta_0(0))$ and (χ, θ_0) are representations of the field at the same physical location so g and x must be functions of ψ alone. In one poloidal circuit

$$\chi = \chi(0) + 2\pi I, \quad \theta_0 = \theta_0(0) + 2\pi\sigma. \tag{6}$$

Again, I and σ must be functions of ψ alone. The function σ will be shown to equal the number one. The periodicities can be simply given by defining the poloidal angle θ and the toroidal angle ϕ so that

$$\theta_0 = \sigma\theta - x\phi, \tag{7}$$

$$\chi = g\phi + I\theta. \tag{8}$$

The coordinates of the paper will be ψ, θ, ϕ.

To show that σ is unity, remember that $2\pi\psi$ equals the magnetic flux inside a pressure surface or

$$2\pi\psi = \int \mathbf{B} \cdot d\mathbf{S}_t. \tag{9}$$

The element of surface area in ψ, θ, ϕ coordinates is

$$d\mathbf{S}_t = \frac{\nabla\phi}{\nabla\phi \cdot (\nabla\psi \times \nabla\theta)} d\theta\, d\psi. \tag{10}$$

Using Eqs. (1), (7), and (8), $\mathbf{B} = \sigma\nabla\psi \times \nabla\theta + x\nabla\phi \times \nabla\psi$, which implies

$$\psi = \int \sigma\, d\psi. \tag{11}$$

In ψ, θ, ϕ coordinates the contravariant form of the magmagnetic field is then

$$\mathbf{B} = \nabla\psi \times \nabla\theta + x(\psi)\nabla\phi \times \nabla\psi, \tag{12}$$

while the covariant form is

$$\mathbf{B} = g(\psi)\nabla\phi + I(\psi)\nabla\theta + \beta_*\nabla\psi, \quad \beta_* = \beta + \left(\frac{dg}{d\psi}\phi + \frac{dI}{d\psi}\theta\right). \tag{13}$$

The total toroidal current inside a flux surface is

$$\int \mathbf{j} \cdot d\mathbf{S}_t = \frac{c}{4\pi}\int \mathbf{B} \cdot d\mathbf{l}_p = \frac{c}{4\pi}\int \mathbf{B} \cdot \frac{\nabla\phi \times \nabla\psi}{\nabla\phi \cdot (\nabla\psi \times \nabla\theta)} d\theta = \frac{c}{2} I. \tag{14}$$

The total poloidal current outside a flux surface can similarly be shown to be $cg(\psi)/2$.

III. THE CURRENT DENSITY

The covariant representation of B, Eq. (13), gives a simple expression for the current density

$$\mathbf{j} = \frac{c}{4\pi}\nabla \times \mathbf{B} = \frac{c}{4\pi}\left((\nabla\phi \times \nabla\psi)\frac{\partial\beta}{\partial\phi} - (\nabla\psi \times \nabla\theta)\frac{\partial\beta}{\partial\theta}\right). \tag{15}$$

The cross product of this expression with the contravariant representation of B, Eq. (12), gives an equation for β

$$\nabla P = \frac{1}{c}\mathbf{j} \times \mathbf{B} = \frac{1}{4\pi}\nabla\phi \cdot (\nabla\psi \times \nabla\theta)\left(\frac{\partial\beta}{\partial\phi} + x\frac{\partial\beta}{\partial\theta}\right)\nabla\psi. \tag{16}$$

The inverse of the ψ, θ, ϕ Jacobian can be found by dotting together the covariant, Eq. (13), and contravariant, Eq. (12), representations of B

$$\nabla\phi \cdot (\nabla\psi \times \nabla\theta) = \frac{B^2}{g + xI}. \tag{17}$$

The equation for β is then

$$\frac{\partial\beta}{\partial\phi} + x\frac{\partial\beta}{\partial\theta} = \frac{4\pi}{B^2}(g + xI)\frac{dP}{d\psi}. \tag{18}$$

The parallel component of j can be found by dotting the covariant expression for B, Eq. (13), into j, Eq. (15), to obtain

$$\frac{4\pi}{c}\frac{j_\parallel}{B} = \frac{1}{g + xI}\left(I\frac{\partial\beta}{\partial\phi} - g\frac{\partial\beta}{\partial\theta}\right). \tag{19}$$

Let us now solve the equations for β and $j_\parallel/B$. The function β need not be periodic in θ and ϕ; however, $j_\parallel/B$ must be. One finds a homogeneous solution for β

$$\beta_h = \mu_h(\psi)(x\phi - \theta), \tag{20}$$

with μ_h an arbitrary function of ψ. To this solution the inhomogeneous solution to Eq. (18) must be added. To find the inhomogeneous solution let

$$\frac{1}{B^2} = \frac{1}{B_0^2}\left(1 + {\sum_{n,m}}' \delta_{nm}\cos(n\phi - m\theta + \lambda_{nm})\right), \tag{21}$$

with the prime on the sum implying that the term $n = 0$, $m = 0$ is eliminated. That is, we assume that the field strength is each magnetic surface is known and that it can be appropriately expanded in a Fourier series. Then, one finds

$$\beta = \mu_h(\psi)(x\phi - \theta) + \frac{4\pi}{B_0^2}(g + xI)\frac{dP}{d\psi}\phi + \beta_*, \tag{22}$$

with β_*, which we will presently show is the β_* of Eq. (13), equal to

$$\beta_* = \frac{4\pi}{B_0^2}\frac{dP}{d\psi}{\sum_{n,m}}' \frac{g + xI}{n - xm}\delta_{nm}\sin(n\phi - m\theta + \lambda_{nm}). \tag{23}$$

The equations are simpler if the force-free current part of β is singled out by defining

$$\mu = \mu_h + \frac{4\pi}{B_0^2}\frac{dP}{d\psi}I; \tag{24}$$

then

$$\beta = \mu(\psi)(x\phi - \theta) + \frac{4\pi}{B_0^2}\frac{dP}{d\psi}(g\phi + I\theta) + \beta_*. \tag{25}$$

The parallel current is given by Eq. (19)

$$\frac{4\pi}{c}\frac{j_\parallel}{B} = \mu(\psi) + \frac{4\pi}{B_0^2}\frac{dP}{d\psi}{\sum_{n,m}}' \frac{nI + mg}{n - xm}\delta_{nm}\cos(n\phi - m\theta + \lambda_{nm}). \tag{26}$$

The first term on the right side of this equation represents the force-free current and the second term the Pfirsch–Schlüter current. The poloidal and toroidal currents can be evaluated using Eqs. (15) and (25),

$$\frac{dI}{d\psi} = \mu - \left(\frac{4\pi}{B_0^2}\frac{dP}{d\psi}\right)I, \tag{27}$$

$$\frac{dg}{d\psi} = -x\mu - \left(\frac{4\pi}{B_0^2}\frac{dP}{d\psi}\right)g. \tag{28}$$

Consequently, β of Eq. (25) can be rewritten as

$$\beta=-\frac{dg}{d\psi}\phi-\frac{dI}{d\psi}\theta+\beta_*, \tag{29}$$

which identifies the β_* of Eq. (23) with that of Eq. (13).

It is of interest to note that in force-free magnetic fields the plasma is minimum average B stable if B_0 increases away from the magnetic axis by the V^{**} criterion of Johnson and Greene.[9] It can be easily shown that B_0^2 is the volume average of B^2 on a flux surface. Consequently, V^{**} and minimum average B stability are the same.

IV. CONSEQUENCES OF OHM'S LAW

The Ohm's law of magnetohydrodynamics,

$$\mathbf{E}+(1/c)\mathbf{v}\times\mathbf{B}=\boldsymbol{\eta}\cdot\mathbf{j}, \tag{30}$$

allows us to evaluate a plasma diffusion coefficient. A potential part of the electric field can be separated. Let

$$\mathbf{E}=-\nabla\Phi+\boldsymbol{\epsilon}. \tag{31}$$

The perpendicular part of ϵ can be written in terms of a velocity $\mathbf{u}$

$$\epsilon_\perp+\frac{1}{c}\mathbf{u}\times\mathbf{B}=0. \tag{32}$$

The velocity $\mathbf{u}$ represents an overall pinching of the field and the plasma. Ohm's law can be rewritten as

$$-\nabla\Phi+\epsilon_\parallel+(1/c)(\mathbf{v}-\mathbf{u})\times\mathbf{B}=\boldsymbol{\eta}\cdot\mathbf{j}. \tag{33}$$

The parallel component of this equation gives

$$\frac{\partial\Phi}{\partial\phi}+x\frac{\partial\Phi}{\partial\theta}=(g+xI)\left(\frac{\epsilon_\parallel}{B}-\eta_\parallel\frac{j_\parallel}{B}\right). \tag{34}$$

This equation with the expression for $j_\parallel/B$, Eq. (26), implies the choice

$$\epsilon_\parallel=\frac{\eta_\parallel c}{4\pi}\mu(\psi)\mathbf{B}, \tag{35}$$

$$\Phi=\Phi_0(\psi)-\eta_\parallel c\frac{(g+xI)}{B_0^2}\frac{dP}{d\psi}\sum_{n,m}{}'\frac{nI+mg}{(n-xm)^2}\delta_{nm}\times\sin(n\phi-m\theta+\lambda_{nm}), \tag{36}$$

with Φ_0 an arbitrary function of ψ.

To understand the velocity $\mathbf{u}$ we will consider Faraday's law

$$\frac{\partial\mathbf{B}}{\partial t}=-c\nabla\times\mathbf{E}=\nabla\times(\mathbf{u}\times\mathbf{B})-c\nabla\times\epsilon_\parallel. \tag{37}$$

Evaluating $\mathbf{u}\times\mathbf{B}$ using the contravariant expression for $\mathbf{B}$, Eq. (12), one finds

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t}+\mathbf{u}\cdot\nabla\psi=-\left(\frac{\eta_\parallel c^2}{4\pi}\mu\right)I, \tag{38}$$

$$\frac{\partial\psi_p(x,t)}{\partial t}+\mathbf{u}\cdot\nabla\psi_p=\left(\frac{\eta_\parallel c^2}{4\pi}\mu\right)g, \tag{39}$$

with $d\psi_p/d\psi=x$. One can easily show that $2\pi\psi_p$ is the poloidal flux within an additive function of time. One can define the plasma loop voltage $V(\psi,t)$ by

$$V=\frac{\partial}{\partial\psi}\left(\frac{1}{2\pi}\int\epsilon\cdot\mathbf{B}\,d^3x\right) \tag{40}$$

in the usual approximations of a tokamak $V\simeq 2\pi RE_\phi$. Evaluating Eq. (40) using Eq. (35) for $\epsilon_\parallel$, one finds

$$\mu=\frac{2}{c}\frac{V}{(g+xI)\eta_\parallel}. \tag{41}$$

This equation plus Eqs. (38) and (39) imply

$$\frac{\partial x(\psi,t)}{\partial t}=\frac{c}{2\pi}\frac{\partial V}{\partial\psi}. \tag{42}$$

In steady state, one must clearly have the loop voltage a constant V_0. One then has

$$\mathbf{u}\cdot\nabla\psi=-\frac{c}{2\pi}\frac{I}{g+xI}V_0, \tag{43}$$

and the total flux of particles due to $\mathbf{u}$ with $\rho(\psi)$ the density is

$$\Gamma_p=\int\rho\mathbf{u}\cdot d\mathbf{S}_\psi=-2\pi c\frac{\rho I}{B_0^2}V_0. \tag{44}$$

This is the classical pinch effect.

The particle diffusive flux can be evaluated using Ohm's law, Eq. (33). This equation can be solved for the perpendicular part of $\mathbf{v}-\mathbf{u}$ and hence $(\mathbf{v}-\mathbf{u})\cdot\nabla\psi$. One finds, using Eq. (36),

$$\begin{aligned}(\mathbf{v}-\mathbf{u})\cdot\nabla\psi&=\frac{c}{g+xI}\left(I\frac{\partial\Phi}{\partial\phi}-g\frac{\partial\Phi}{\partial\theta}\right)-\eta_\perp c^2\frac{\nabla P}{B^2}\cdot\nabla\psi\\&=-\frac{\eta_\parallel c^2}{B_0^2}\frac{dP}{d\psi}\sum_{n,m}{}'\left(\frac{nI+mg}{n-xm}\right)^2\delta_{nm}\cos(n\phi-m\theta+\lambda_{nm})\\&\quad-\eta_\perp c^2\frac{dP}{d\psi}\frac{|\nabla\psi|^2}{B^2}.\end{aligned} \tag{45}$$

The total diffusive particle flux crossing a magnetic surface is

$$\begin{aligned}\Gamma_d&=\int\rho(\mathbf{v}-\mathbf{u})\cdot\frac{\nabla\psi}{\nabla\phi\cdot(\nabla\psi\times\nabla\theta)}d\theta\,d\phi\\&=(g+xI)\rho\int(\mathbf{v}-\mathbf{u})\cdot\nabla\psi\frac{d\theta\,d\phi}{B^2}.\end{aligned} \tag{46}$$

The expression for the total diffusive flux can be rewritten as

$$\Gamma_d=-(D_\parallel+D_\perp)\frac{dP}{d\psi}, \tag{47}$$

with

$$D_\parallel=2\pi^2\eta_\parallel\frac{c^2(g+xI)}{B_0^4}\sum_{n,m}{}'\left(\frac{nI+mg}{n-xm}\right)^2\delta_{nm}^2, \tag{48}$$

$$D_\perp=\eta_\perp c^2\int\frac{\nabla\psi}{B^2}\cdot d\mathbf{S}_\psi,\quad dS_\psi=\frac{\nabla\psi}{\nabla\phi\cdot(\nabla\psi\times\nabla\theta)}d\theta\,d\phi; \tag{49}$$

that is, $d\mathbf{S}_\psi$ is the area element of a flux surface.

Let us assume that the plasma is in a steady state. Then, particle conservation implies

$$\nabla\cdot n\mathbf{v}=s, \tag{50}$$

with s the source of particles per unit volume. The total flux of particles $\Gamma=\Gamma_p+\Gamma_d$ obeys

$$\frac{d\Gamma}{d\psi} = \int s \frac{d\theta d\phi}{\nabla\phi\cdot(\nabla\psi\times\nabla\theta)} = S(\psi), \tag{51}$$

with $S(\psi)\,d\psi$ the number of particles added between two differentially separated flux surfaces. Clearly, Γ must be a smoothly varying function of ψ.

V. DISCUSSION

In steady-state the total flux of particles across a magnetic surface is

$$\Gamma = -D(\psi)\frac{dP}{d\psi} + \Gamma_p, \tag{52}$$

with $D(\psi) = D_\parallel + D_\perp$ given by Eqs. (48) and (49) and Γ_p given by Eq. (44). The parallel current driven diffusion coefficient $D_\parallel$ is singular at each rational surface; that is, near a rational surface

$$D_\parallel \propto [\delta_{nm}{}^2/(\psi-\psi_{nm})^2], \tag{53}$$

with ψ_{nm} the value of ψ when $n = xm$. The total particle flux $\Gamma(\psi)$ must be slowly varying in ψ so we find near a rational surface

$$\frac{dP}{d\psi} \propto \frac{(\psi-\psi_{nm})^2}{\delta_{nm}^2}; \tag{54}$$

consequently, unless $|\delta_{nm}| = 0$, at $\psi = \psi_{nm}$, $dP/d\psi$ vanishes at rational surfaces. The Pfirsch–Schlüter or pressure driven part of the parallel current near a rational surface [see Eq. (26)] is

$$(j_\parallel)_P \propto \frac{\delta_{nm}}{\psi-\psi_{nm}}\frac{dP}{d\psi}\cos(n\phi - m\theta) \propto (\psi-\psi_{nm}). \tag{55}$$

Consequently, the Pfirsch–Schlüter part of $j_\parallel$ vanishes at each rational surface rather than being singular and the pressure driven part of $j_\parallel$ goes to zero everywhere as the plasma pressure goes to zero. The electrostatic potential, interestingly, retains a finite variation on rational magnetic surfaces even though $dP/d\psi$ is zero on these surfaces.

The singularity of $D(\psi)$ at each rational surface is not as important as it might first appear. Consider a region of the plasma with no sources so $d\Gamma/d\psi = 0$. Define a smoothing function $\Delta(\psi)$ such that $\Delta(\psi)$ goes to zero for $|\psi|$ small but finite and which has a unit integral over ψ. Then let

$$\bar{P}(\psi) \equiv \int \Delta(\psi-\psi_1)P(\psi_1)d\psi_1, \tag{56}$$

$$\begin{aligned}\frac{d\bar{P}}{d\psi} &= -\int \frac{d\Delta}{d\psi_1}P(\psi_1)d\psi_1 = +\int \Delta\frac{dP}{d\psi_1}d\psi_1\\ &= -\Gamma\int\frac{\Delta(\psi-\psi_1)}{D(\psi_1)}d\psi_1.\end{aligned} \tag{57}$$

Defining

$$\frac{1}{\bar{D}(\psi)} \equiv \int \frac{\Delta(\psi-\psi_1)}{D(\psi_1)}d\psi_1, \tag{58}$$

one has

$$\Gamma = -\bar{D}(\psi)\frac{d\bar{P}}{d\psi}. \tag{59}$$

No matter how narrow the region over which Δ is different from zero, as long as the region is finite, the function $\bar{D}(\psi)$ is finite everywhere. This follows from the fact that the Fourier transform of a smooth function vanishes exponentially for high n or m; hence, high-order rational surfaces have an exponentially small effect on the integral leading to $\bar{D}$.

An interesting application of the expression for $D_\parallel$ is to derive the well-known Pfirsch–Schlüter diffusion coefficient for a stellarator. This is done by assuming that the field strength has the obvious form

$$1/B^2 \simeq (1/B_0^2)[1 - 2\epsilon\cos\theta - 2\delta\cos(N\phi - l\theta)]. \tag{60}$$

The only terms in δ_{nm} are $\delta_{01} = 2\epsilon$ and $\delta_{Nl} = 2\delta$. We assume that the plasma has negligible net toroidal current, $I = 0$, and the toroidal field dominates so $g = RB_0$. The diffusion coefficient one is used to seeing, $D_\parallel^*$, is $D_\parallel$ divided by the area of the magnetic surface $(2\pi r)(2\pi R)$ and is also divided by $d\psi/dr = rB_0$, since the usual $D_\parallel^*$ multiplies dP/dr rather than $dP/d\psi$. Equation (48) implies, with $\epsilon = r/R$,

$$D_\parallel^* = \frac{2}{x^2}\frac{\eta_\parallel c^2}{B_0^2}\sum_{n,m}{}' \left(\frac{\delta_{nm}}{2\epsilon}\right)^2\left(\frac{x}{x-(n/m)}\right)^2. \tag{61}$$

The resonance $n = 0$, $m = 1$, with $\delta_{01} = 2\epsilon$ gives a contribution

$$D_{01}^* = \frac{2}{x^2}\frac{\eta_\parallel c^2}{B_0^2}, \tag{62}$$

the Pfirsch–Schlüter coefficient. The resonance at $n = N$, $m = l$, $\delta_{Nl} = 2\delta$ gives

$$D_{Nl}^* = \frac{2}{x^2}\frac{\eta_\parallel c^2}{B_0^2}\left(\frac{\delta}{\epsilon}\right)^2\left(\frac{x}{x-N/l}\right)^2. \tag{63}$$

Customary stellarator designs have $\delta \sim \epsilon$ but $N/l \gg x$ so the Pfirsch–Schlüter coefficient gives an accurate approximation, $D_{01}^* \gg D_{Nl}^*$.

The relationship between the magnetic coordinates of this paper and other toroidal magnetic coordinates is of interest. Let ψ_m, θ_m, ϕ_m be any set of toroidal magnetic coordinates. By magnetic coordinates we mean that

$$\mathbf{B}\cdot\nabla\psi_m = 0, \quad \mathbf{B}\cdot\nabla(\theta_m - x\phi_m) = 0. \tag{64}$$

Naturally, θ_m and ϕ_m must preserve the periodicities of the torus. The general expressions for the alternative magnetic coordinates are ψ_m a function of ψ alone,

$$\theta_m = \theta + xp(\psi,\theta,\phi), \quad \phi_m = \phi + p(\psi,\theta,\phi), \tag{65}$$

with p periodic in θ and ϕ but otherwise an arbitrary function. A major characteristic of a coordinate system is its Jacobian J. Now

$$\frac{1}{J_m} \equiv (\nabla\psi_m\times\nabla\theta_m)\cdot\nabla\phi_m = \frac{d\psi_m}{d\psi}\left(1 + \frac{\partial p}{\partial\phi} + x\frac{\partial p}{\partial\theta}\right)\frac{B^2}{g + xI}. \tag{66}$$

There is clearly considerable freedom in the choice of the Jacobian.

The best known set of magnetic coordinates is the Hamada set,[5,6] which will be denoted by a subscript H. In these coordinates the Jacobian is a constant or more generally a function of ψ_m alone. For Hamada coordinates

$$\frac{\partial p_H}{\partial \psi} + x\frac{\partial p_H}{\partial \theta} = \frac{1}{J_H(\psi_H)}\frac{d\psi}{d\psi_H}\frac{g+xI}{B^2} - 1\,. \tag{67}$$

If p_H is to be periodic, $\psi_H(\psi)$ must be chosen so that

$$\frac{d\psi_H}{d\psi} = \frac{1}{J_H}\frac{g+xI}{B_0^2}\,; \tag{68}$$

then,

$$p_H = \sum_{n,m}{}' \frac{\delta nm}{n-xm}\sin(n\phi - m\theta + \lambda_{nm})\,, \tag{69}$$

using Eq. (21) for $1/B^2$. Hamada coordinates make the current lines straight as well as the field lines. Equations (23) and (69) imply

$$\beta_* = (g+xI)\frac{4\pi}{B_0^2}\frac{dp}{d\psi}p_H\,. \tag{70}$$

Using Eqs. (25), (27), and (28) and the expressions for the Hamada angles, $\theta_H = \theta + xp_H$ and $\phi_H = \phi + p_H$, one finds

$$\beta = -\frac{dg}{d\psi}\phi_H - \frac{dI}{d\psi}\theta_H\,. \tag{71}$$

This equation proves the straightness of the field lines since β and ψ are the Clebsch coordinates of the current; that is,

$$(4\pi/c)\mathbf{j} = \nabla\beta\times\nabla\psi \tag{72}$$

using Eq. (23) and Amphere's law.

The expression for p_H, Eq. (69), is singular if $x=n/m$ and $\delta_{nm}\neq 0$. Clearly, a resonant term, $\delta_{nm}\neq 0$ with $x=n/m$, cannot be removed from the Fourier decomposition of the Jacobian with a nonsingular p and, hence, a nonsingular coordinate transformation. Consequently, Solov'ev and Shafranov[2] were able to find an expression for $j_\parallel$ which has the same singular form as Eq. (25) but is expressed in terms of the Fourier coefficients of Jacobian of the general, nonsingular, magnetic coordinates.

ACKNOWLEDGMENT

This work was supported by U.S. Department of Energy Contract No. DE-AC02-76-CH03073.

[1]H. Grad, Phys. Fluids 10, 137 (1967) (see esp. Sec. IV).
[2]L. S. Solov'ev and V. D. Shafranov, in *Reviews of Plasma Physics* (Consultants Bureau, New York, 1970), Vol. 5, p. 1.
[3]T. Ohkawa, Phys. Lett. A 38, 21 (1972).
[4]M. D. Kruskal and R. M. Kulsrud, Phys. Fluids 1, 265 (1958).
[5]S. Hamada, Nucl. Fusion 2, 23 (1962).
[6]J. M. Greene and J. L. Johnson, Phys. Fluids 5, 510 (1962).
[7]H. Grad, Ann. N. Y. Acad. of Sci. 357, 223 (1980).
[8]A. H. Boozer, Phys. Fluids 22, 904 (1980).
[9]J. L. Johnson and J. M. Greene, Plasma Phys. 9, 611 (1967).