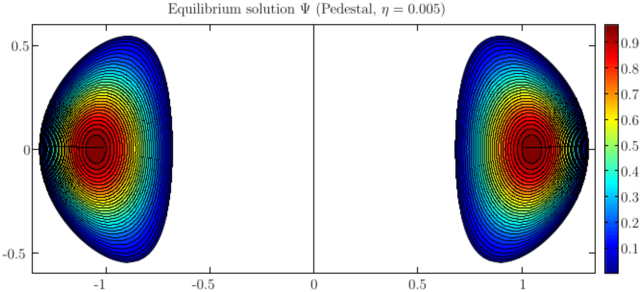
Introduction to MagnetoHydroDynamics (MHD)



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PART I: DESCRIBING A FUSION PLASMA

Method I: Self-consistent particle pushing

- An intuitive idea is to solve for the motion of all the particles iteratively, combining Newton's law with Maxwell equations
- At each time step i, solve

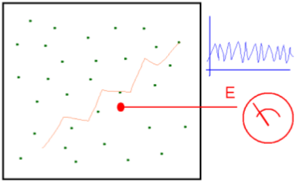
$$m\frac{d^2\mathbf{x}_k^{(i)}}{dt} = q_k\left(\mathbf{E}(\mathbf{x}_k)^{(i-1)} + \frac{d\mathbf{x}_k^{(i)}}{dt}\times\mathbf{B}^{(i-1)}(\mathbf{x}_k)\right) \qquad k = 1,\ldots,N$$

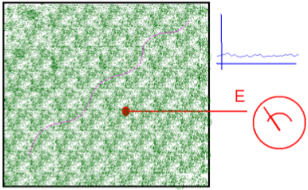
$$\left(\frac{\partial\mathbf{E}}{\partial t}\right)^i = c^2\nabla\times\mathbf{B}^{(i-1)} - \mu_0 c^2\sum_{k=1}^{N} q_k\frac{d\mathbf{x}_k^i}{dt}\delta(\mathbf{x}-\mathbf{x}_k^i)$$

$$\left(\frac{\partial\mathbf{B}}{\partial t}\right)^i = -\nabla\times\mathbf{E}^i$$

- Fast solvers exist for the electromagnetic fields, some relying on a subsidiary mesh, some not needing a mesh
- Even with fast solvers, problem still not tractable even with the most powerful computers when $N\sim 10^{20}-10^{22}$ as in magnetic fusion grade plasmas

METHOD II: COARSE-GRAIN AVERAGE IN PHASE SPACE





(From G. Lapenta's: https://perswww.kuleuven.be/ u0052182/weather/pic.pdf)

- For hot and diffuse systems with a large number of particles, following every single particle is a waste of time and resources
- Replace the discrete particles with smooth distribution function $f(\mathbf{x}, \mathbf{v}, t)$ defined so that

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v}$$

is the expected number of particles in the infinitesimal six-dimensional phase-space volume $d\mathbf{x}d\mathbf{v}$.

Distribution function and Vlasov equation

- Macroscopic (fluid) quantities are velocity moments of f

$$n(\mathbf{x},t) = \iiint f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Density}$$

$$n\mathbf{V}(\mathbf{x},t) = \iiint \mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v} \qquad \text{Mean flow}$$

$$\mathbf{P}(\mathbf{x},t) = m\iiint (\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})fd\mathbf{v} \qquad \text{Pressure tensor}$$

- Conservation of f along the phase-space trajectories of the particles determines the time evolution of f:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla f + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}} f = 0$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

$$\Rightarrow \qquad \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f = 0$$

This is the Vlasov equation

The Boltzmann equation

- In fusion plasmas, we separate, leading to the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

This equation to be combined with Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

- Nonlinear, integro-differential, 6-dimensional PDE
- Describes phenomena on widely varying length ($10^{-5} - 10^{3}$ m) and time ($10^{-12} - 10^{2}$ s) scales
- Still not a piece of cake, and never solved as such for fusion plasmas

Moment approach

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_c$$

- Taking the integrals $\iiint d\mathbf{v}$, $\iiint m\mathbf{v} d\mathbf{v}$ and $\iiint mv^2/2 d\mathbf{v}$ of this equation, we obtain the exact **fluid equations**:

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0 \qquad \text{Continuity}$$

$$mn \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = q_s n_s \left(\mathbf{E} + \mathbf{V}_s \times \mathbf{B} \right) - \nabla \cdot \mathbf{P}_s + \mathbf{R}_s \qquad \text{Momen}$$

$$\frac{d}{dt} \left(\frac{3}{2} p_s \right) + \frac{5}{2} p_s \nabla \cdot \mathbf{V}_s + \boldsymbol{\pi}_s : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = 0 \qquad \text{(Energy)}$$

with $\mathbf{P}_s = p_s \mathbf{I} + \boldsymbol{\pi}_s$.

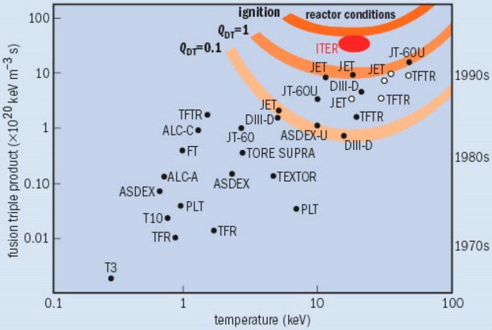
- Closure problem: for each moment, we introduce a new unknown $\Rightarrow$ End up with too many unknowns
- Need to make approximations to close the moment hierarchy

Kinetic models vs Fluid models

- For some fusion applications/plasma regimes (heating and current drive, transport), kinetic treatment cannot be avoided
- Simplify and reduce dimensionality of the Vlasov equation with approximations:
  - Strong magnetization : Gyrokinetic equation
  - Small gyroradius compared to relevant length scales : Drift kinetic equation
  - Vanishing gyroradius : Kinetic MHD
- In contrast, fluid models are based on approximate expressions for higher order moments (off-diagonal entries in pressure tensor, heat flux) in terms of lower order quantities(density, velocity, diagonal entries in pressure tensor)
- We will now focus on the relevant regime and the approximations made to derive a widely used fluid model: the ideal MHD model

PART II: THE IDEAL MHD MODEL

Lawson Criterion and MHD



Condition for ignition: $\boxed{p\tau_E \geq 8\ bar.s}$ $\qquad T_{min} \sim 15\text{keV}$

- The maximum p is limited by the stability properties
  Job of MHD
- The maximum τ_E is determined by the confinement properties
  Job of kinetic models

Philosophy

- The purpose of ideal MHD is to study the macroscopic behavior of the plasma
- Use ideal MHD to design machines that avoid large scale instabilities
- Regime of interest
  - Typical length scale: the minor radius of the device $a \sim 1m$
    Wave number k of waves and instabilitities considered: $k \sim 1/a$
  - Typical velocities: Ion thermal velocity speed $v_T \sim 500km/s$
  - Typical time scale: $\tau_{MHD} \sim a/v_T \sim 2\mu s$
    Frequency ω_{MHD} of associated waves/instabilities $\omega_{MHD} \sim 500kHz$

EXAMPLE: VERTICAL INSTABILITY

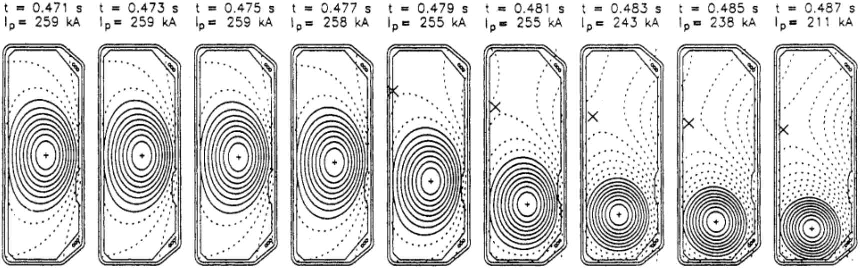


FIG. 2. Loss of vertical position control following a feedback cut at 0.470 s.

Figure from F. Hofmann *et al.*, Nuclear Fusion **37** 681 (1997)

Ideal MHD - Maxwell's equations

- $a \gg \lambda_D$, the distance over which charge separation can take place in a plasma
  $\Rightarrow$ On the MHD length scale, the plasma is neutral : $n_i = n_e$

- $\omega_{MHD}/k \ll c$ and $v_{T_i} \ll v_{T_e} \ll c$ so we can neglect the displacement current in Maxwell's equations:

$$n_i = n_e$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Ideal MHD - Momentum equation

- $a \gg \lambda_D$ and $a \gg r_{Le}$ (electron Larmor radius)
- $\omega_{MHD} \ll \omega_{pe}, \omega_{MHD} \ll \omega_{ce}$
- The ideal MHD model assumes that on the time and length scales of interest, the electrons have an infinitely fast response time to changes in the plasma
- Mathematically, this can be done by taking the limit $m_e \rightarrow 0$
- Adding the ion and electron momentum equation, we then get

$$\rho \frac{d\mathbf{V}}{dt} - \mathbf{J} \times \mathbf{B} + \nabla p = -\nabla \cdot (\boldsymbol{\pi}_i + \boldsymbol{\pi}_e)$$

  where $\rho = m_i n$ and $\mathbf{V}$ is the ion fluid velocity
- If the condition $v_{Ti}\tau_{ii}/a \ll 1$ is satisfied in the plasma

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \qquad \text{(Ideal MHD momentum equation)}$$

Ideal MHD - Electrons

- In the limit $m_e \to 0$, the electron momentum equation can be written as

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{1}{en} \left(\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e + \mathbf{R}_e \right)$$

- This is called the generalized Ohm's law
- Different MHD models (resitive MHD, Hall MHD) keep different terms in this equation
- If $r_{Li}/a \ll 1$, $v_{Ti}\tau_{ii}/a \ll 1$, and $(m_e/m_i)^{1/2}(r_{Li}/a)^2(a/v_{Ti}\tau_{ii}) \ll 1$, the momentum equation becomes the ideal Ohm's law

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$$

- The ideal MHD plasma behaves like a perfectly conducting fluid

Energy equation

- Define the total plasma pressure $p = p_i + p_e$
- Add electron and ion energy equations
- Under the conditions $r_{Li}/a \ll 1$ and $v_{Ti}\tau_{ii}/a \ll 1$, this simplifies as

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0$$

- Equation reminiscent of $pV^\gamma = Cst$: the ideal MHD plasma behaves like a monoatomic ideal gas undergoing a reversible adiabatic process

Ideal MHD - Summary

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{5/3}}\right) = 0$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Valid under the conditions

$$\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{v_i \tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2 \left(\frac{m_e}{m_i}\right)^{1/2} \frac{a}{v_{Ti}\tau_{ii}} \ll 1$$

VALIDITY OF THE IDEAL MHD MODEL (I)

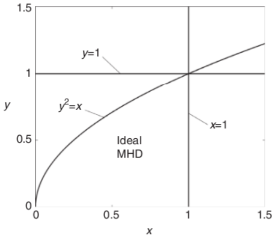
- Are the conditions for the validity of ideal MHD

$$\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{v_i\tau_{ii}}{a}\right) \ll 1 \qquad \frac{r_{Li}}{a} \ll 1 \qquad \left(\frac{r_{Li}}{a}\right)^2\left(\frac{m_e}{m_i}\right)^{1/2}\frac{a}{v_{Ti}\tau_{ii}} \ll 1$$

  mutually compatible?
- Define $x = (m_i/m_e)^{1/2}(v_{Ti}\tau_{ii}/a)$, $y = r_{Li}/a$.

  $x \ll 1$ (High collisionality) $y \ll 1$ (Small ion Larmor radius)

  $y^2/x \ll 1$ (Small resistivity)

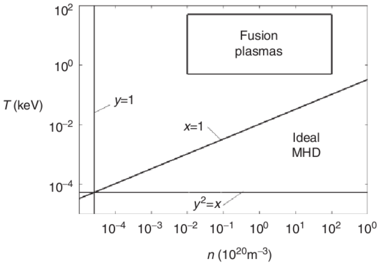


There exists a regime for which ideal MHD is justified (Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

Is that the regime of magnetic confinement fusion?

VALIDITY OF THE IDEAL MHD MODEL (II)

- Express three conditions in terms of n, T, a and β, with β the ratio of plasma pressure and magnetic pressure
- For $\beta = 5\%$ and $a = 1m$ (realistic fusion parameters), we find



The regime of validity of ideal MHD does NOT coincide with the fusion plasma regime (Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

The collisionality of fusion plasmas is too low for the ideal MHD model to be valid.

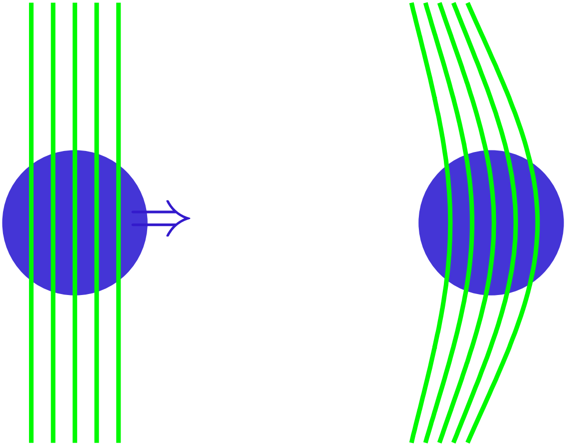
Is that a problem?

Validity of the ideal MHD model (III)

- It turns out that ideal MHD often does a very good job at predicting stability limits for macroscopic instabilities
- This is not due to luck but to subtle physical reasons
- One can show that collisionless kinetic models for macroscopic instabilities are more optimistic than ideal MHD
- This is because ideal MHD is accurate for dynamics perpendicular to the fields lines
- Designs based on ideal MHD calculations are conservative designs

FROZEN IN LAW (I)

- $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0}$: in the frame moving with the plasma, the electric field is zero
- The plasma behaves like a perfect conductor
- The magnetic field lines are "frozen" into the plasma motion



Frozen in law (II): Proof

- $\partial \mathbf{B}/\partial t = \nabla \times \mathbf{E}$, $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{0} \Rightarrow \qquad \partial \mathbf{B}/\partial t = \nabla \times (\mathbf{V} \times \mathbf{B})$
- Calculate the change in the flux $\Phi = \iint_{S(t)} \mathbf{B} \cdot \mathbf{n} dS$ across a moving surface with velocity $\mathbf{u}_\perp$

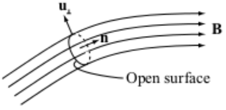


Image from *Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas* by J.P. Goedbloed and S. Poedts, Cambridge University Press (2004)

$$\begin{aligned}
\frac{d\Phi}{dt} &= \iint_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dS - \oint_{\partial S(t)} \mathbf{u}_\perp \times \mathbf{B} \cdot \mathbf{dl} \\
&= \iint \nabla \times (\mathbf{V} \times \mathbf{B}) \cdot \mathbf{n} dS - \oint_{\partial S(t)} \mathbf{u}_\perp \times \mathbf{B} \cdot \mathbf{dl} \\
&= \oint_{\partial S(t)} (\mathbf{V} - \mathbf{u}_\perp) \cdot \mathbf{dl} \\
&= 0 \qquad \text{if } \mathbf{u}_\perp = \mathbf{V}
\end{aligned}$$

i.e. the plasma is tied to the field lines

Magnetic reconnection

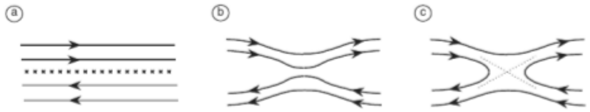


Image from *Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas* by J.P. Goedbloed and S. Poedts, Cambridge University Press (2004)

- Magnetic reconnection: a key phenomenon in astrophysical, space, and fusion plasmas
- Cannot happen according to ideal MHD
- Need to add additional terms in Ohm's law to allow reconnection: resistivity, off-diagonal pressure tensor terms, electron inertia, …
- Associated instabilities take place on longer time scales than τ_{MHD}

PART III: MHD EQUILIBRIUM

Equilibrium state

- By equilibrium, we mean steady-state: $\partial/\partial t = 0$
- Often, for simplicity and/or physical reasons, we focus on static equilibria: $\mathbf{V} = \mathbf{0}$

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\mathbf{J} \times \mathbf{B} = \nabla p$$

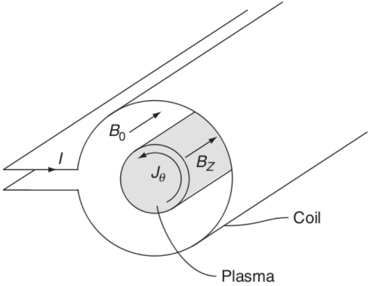
A more condensed form is

$$\nabla \cdot \mathbf{B} = 0 \qquad (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$$

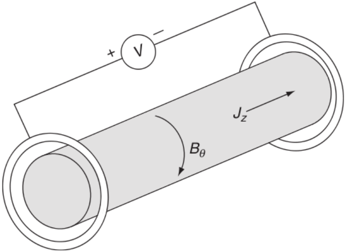
Note that the density profile does not appear

1D EQUILIBRIA (I)

θ-pinch



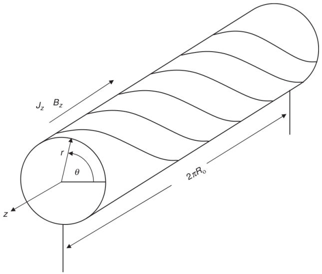
Z-pinch



Combine the two to get....

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

1D EQUILIBRIA (II)



Screw pinch

(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

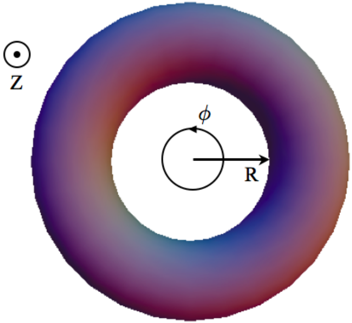
- Equilibrium quantities only depend on r
- Plug into $\nabla \cdot \mathbf{B} = 0$, $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ to find:

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

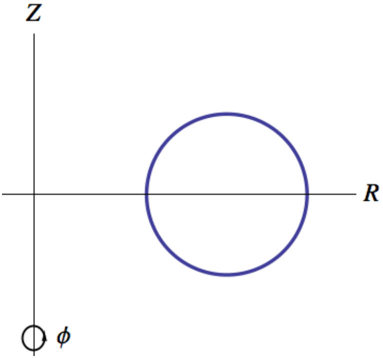
  Balance between plasma pressure, magnetic pressure, and magnetic tension
- Two free functions define equilibrium: e.g. B_z and p, or B_θ and B_z

2D Equilibria: Geometry

Top view



Cross section



Toroidal axisymmetry: $\partial/\partial\phi \equiv 0$

Toroidally axisymmetric equilibria

Step 1:

$$\mathbf{B} = B_\phi(R,Z)\mathbf{e}_\phi + \mathbf{B}_p(R,Z) \text{ and } \nabla\cdot\mathbf{B} = 0 \quad\Rightarrow\quad \nabla\cdot\mathbf{B}_p = 0$$

$$\Rightarrow \mathbf{B} = B_\phi \mathbf{e}_\phi + \frac{1}{R}\nabla\Psi\times\mathbf{e}_\phi$$

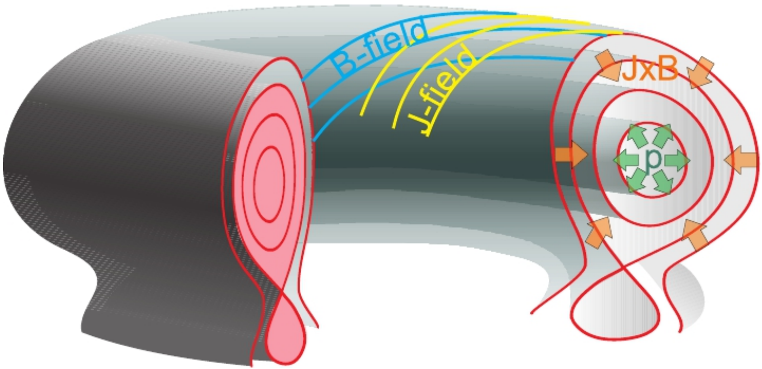
$\Psi = RA_\phi$, with $\mathbf{A}$ vector potential: $\nabla\times\mathbf{A} = \mathbf{B}$.

Step 2:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \Rightarrow \begin{cases} \mu_0 J_\phi = -\frac{1}{R}\left[R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) + \frac{\partial^2\Psi}{\partial Z^2}\right] = -\frac{1}{R}\Delta^*\Psi \\ \mu_0\mathbf{J}_p = \frac{1}{R}\nabla(RB_\phi\times\mathbf{e}_\phi) \end{cases}$$

Step 3:

$$\mathbf{J}\times\mathbf{B} = \nabla p \begin{cases} \cdot\mathbf{B} \quad\Rightarrow\quad \nabla\Psi\times\nabla p = \mathbf{0} \quad\Rightarrow\quad p = p(\Psi) \\ \cdot\mathbf{J} = 0 \quad\Rightarrow\quad \nabla(RB_\phi)\times\nabla\Psi = \mathbf{0} \quad\Rightarrow\quad RB_\phi = F(\Psi) \end{cases}$$



- The regions of constant pressure are nested toroidal surfaces
- Magnetic fields and currents lie on these nested surfaces

Grad-Shafranov equation

Last step: $[\mathbf{J} \times \mathbf{B} = \nabla p] \cdot \nabla \Psi$ gives the Grad-Shafranov equation (GSE):

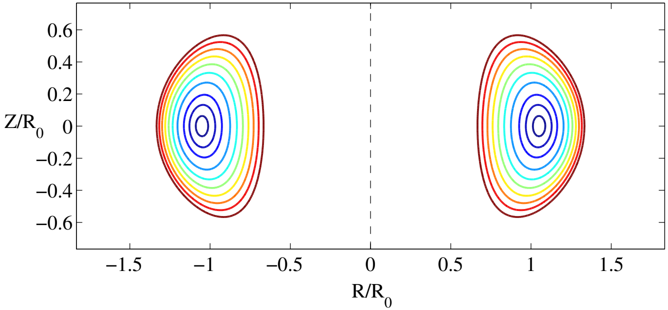
$$R \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \Psi}{\partial R}\right) + \frac{\partial^2 \Psi}{\partial Z^2} = -\mu_0 R^2 \frac{dp}{d\Psi} - F\frac{dF}{d\Psi}$$

- Second-order, nonlinear, elliptic PDE. Derived independently by H. Grad[1] and V.D. Shafranov[2].
- The free functions p and F determine the nature of the equilibrium
- In general, the GSE has to be solved numerically

[1] *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy*, Vol. 31, p.190

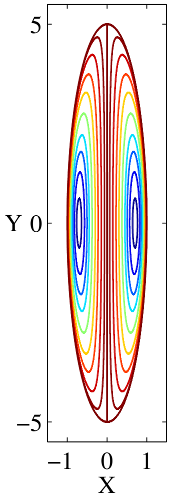
[2] *Sov. Phys. JETP* **6**, 545 (1958)

EXAMPLES (I)



Grad-Shafranov equilibrium for JET tokamak

Examples (II)



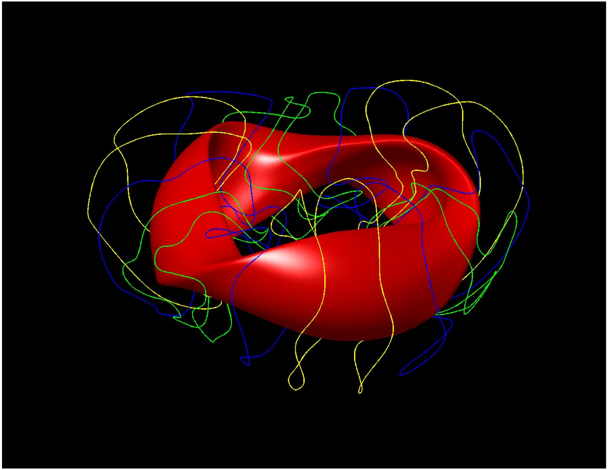
Grad-Shafranov equilibrium for Field Reversed Configuration

Numerical solution to the Grad-Shafranov equation

- Magnetic equilibrium serves as input to stability, wave and transport codes ⇒ important to develop fast and accurate solvers
- Many, many solvers available, from very simple to very advanced (FD, FEM, Integral equations, inverse solvers, ...)
- Free boundary equilibria more challenging than fixed boundary equilibria
- Equilibria with purely toroidal flow are determined by a close variant of the Grad-Shafranov equation ⇒ many Grad-Shafranov codes can compute such equilibria

  Equilibria with both toroidal and poloidal flow can be much more challenging; only a handful of codes available

3D EQUILIBRIA (I)

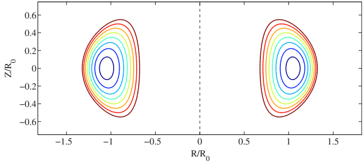


$$\partial/\partial\phi \neq 0$$

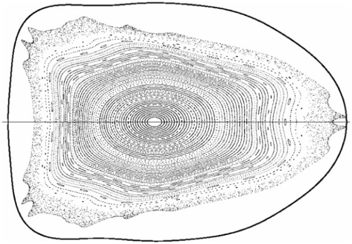
3D EQUILIBRIA (II)

- Equilibrium equations $\nabla \cdot \mathbf{B} = 0\ ,\ (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \nabla p$ still hold
- Existence of nested toroidal surfaces not guaranteed anymore

Tokamak



Stellarator (3D)



- Computing 3D equilibria fast and accurately still a challenge
- Several existing codes, based on different assumptions/approximations and used to design and study stellarators: VMEC, PIES, SPEC, HINT, NSTAB

PART IV: MHD STABILITY

What do we mean by MHD stability?

- That the plasma is initially in equilibrium does not mean it is going to remain there
- The plasma is constantly subject to perturbations, small and large
- The purpose of stability studies is to find out how the plasma will react to these perturbations
  - Will it try to return to the initial steady-state?
  - Will it find a new acceptable steady-state?
  - Will it collapse?

A MECHANICAL ANALOG

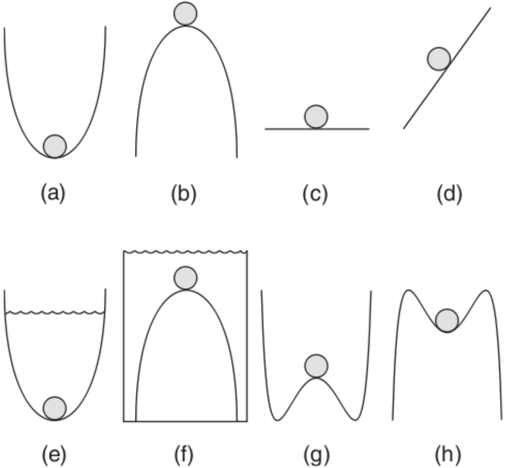


Figure from J.P. Freidberg, *Ideal MHD*, Cambridge University Press (2014)

Solving full nonlinear MHD equations

- Here is an idea to study stability of a magnetically confined plasma:
  - Choose a satisfying plasma equilibrium
  - Perturb it
  - Solve the full MHD equations with a computer
  - Analyze results
- Such an approach provides knowledge of the entire plasma dynamics
- There exist several numerical codes that can do that, for various MHD models (not only ideal): M3D, M3D-C1, NIMROD
- Computationally intensive
- Get more information than one needs?

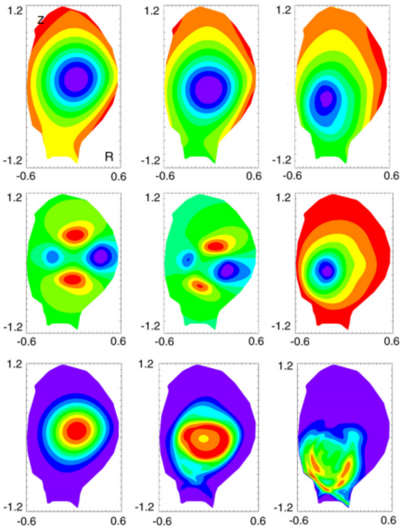
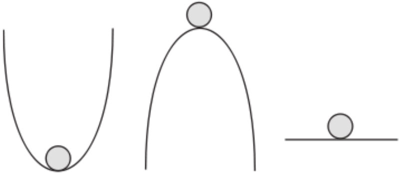


Figure from R. Paccagnella *et al.*, Nuclear Fusion **49** 035003 (2009)

Linear stability (I)

- Ideal MHD dynamics can be so fast and detrimental that one may often require linear stability for the equilibrium



- This can simplify the mathematical analysis tremendously
- Start with an MHD equilibrium:
  $\nabla \cdot \mathbf{B}_0 = 0\ ,\ (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \mu_0 \nabla p_0$
- Take full ideal MHD equations, and write $Q = Q_0(\mathbf{r}) + Q_1(\mathbf{r}, t)$ for each physical quantity, where Q_1 is considered very small compared to Q_0
- Drop all the terms that are quadratic or higher orders in the quantities Q_1 (linearization)

Linear stability (II)

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{J}_1 \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}_1 - \nabla p_1$$

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}_1 = 0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$$

$$\nabla \cdot \mathbf{B}_1 = 0$$

$$\mu_0 \mathbf{J}_1 = \nabla \times \mathbf{B}_1$$

- By design, the system is now linear in the unknown quantities $\rho_1, \mathbf{v}_1, \mathbf{J}_1, \mathbf{B}_1, p_1$
- Much easier to solve in a computer
- There's a trick that makes life even easier

Linear stability (III)

- Introduce the plasma displacement vector $\boldsymbol{\xi}$ defined such that

$$\mathbf{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t} \qquad \mathbf{v}_1(\mathbf{r},0) = \frac{\partial \boldsymbol{\xi}}{\partial t}(\mathbf{r},0) \qquad \boldsymbol{\xi}(\mathbf{r},0) = \mathbf{0}$$

- Linearized ideal MHD equations reduce to

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}\left(\boldsymbol{\xi}\right) \qquad \text{with}$$

$$\mathbf{F}\left(\boldsymbol{\xi}\right) = \frac{1}{\mu_0}\left\{\nabla \times \left[\nabla \times \left(\boldsymbol{\xi} \times \mathbf{B}_0\right)\right]\right\} \times \mathbf{B}_0 + \left(\nabla \times \mathbf{B}_0\right) \times \left[\nabla \times \left(\boldsymbol{\xi} \times \mathbf{B}_0\right)\right]$$

$$+ \nabla\left(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}\right)$$

- $\mathbf{F}$ is called the ideal MHD linear force operator
- The problem of linear stability is reduced to an initial value problem with three linear equations and three unknowns: the components of $\boldsymbol{\xi}$

Linear stability (IV): normal mode analysis

- Even the IVP in the previous slide may give more information than we need
- Sometimes, we just want to know if the equilibrium is stable or not
- A normal mode analysis provides the desired framework for this
- Write $\boldsymbol{\xi}(\mathbf{r},t) = \hat{\boldsymbol{\xi}}(\mathbf{r})\, e^{-i\omega t}$.
  $\omega_I > 0$ corresponds to exponential growth.
- The linearized momentum equation takes the form

$$-\rho\omega^2\hat{\boldsymbol{\xi}} = \mathbf{F}(\hat{\boldsymbol{\xi}})$$

- ω^2 is an an eigenvalue of the linear operator $-\mathbf{F}(\hat{\boldsymbol{\xi}})/\rho$
- It can be showed (some lines of algebra...) that $\mathbf{F}$ is a self-adjoint operator
- In ideal MHD, ω^2 is a purely real quantity
- $\omega^2 \geq 0$ means the mode is stable; $\omega^2 \leq 0$ means the mode is unstable

Eigenvalues in ideal MHD

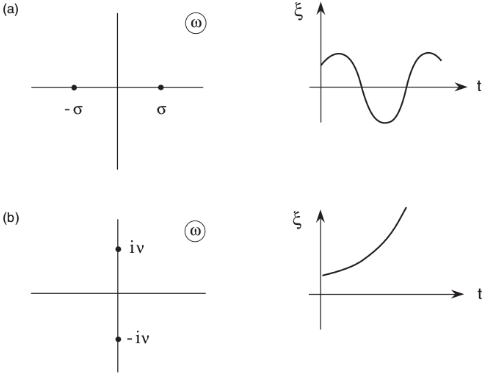


Fig. 6.6. (a) Stable waves and (b) instabilities in ideal MHD.

Figure from *Principles of Magnetohydrodynamics With Applications to Laboratory and Astrophysical Plasmas* by J.P. Goedbloed and S. Poedts, Cambridge University Press (2004)

Illustration: Waves in ideal MHD (I)

- Consider the stability of an infinite, homogeneous plasma:

$$\mathbf{B} = B_0 \overrightarrow{e_z}$$
$$\mathbf{J} = \overrightarrow{0}$$
$$p = p_0$$
$$\rho = \rho_0$$
$$\mathbf{v} = \mathbf{0}$$

- Given geometry, expand $\hat{\boldsymbol{\xi}}(\mathbf{r})$ as

$$\hat{\boldsymbol{\xi}}(\mathbf{r}) = \tilde{\boldsymbol{\xi}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

- Dynamics is anisotropic because of the magnetic field: $\mathbf{k}_\perp + k_\| \mathbf{e_z}$
- Without loss of generality, $\overrightarrow{k} = k_\perp \overrightarrow{e_y} + k_\| \overrightarrow{e_z}$

ILLUSTRATION: WAVES IN IDEAL MHD (II)

- $-\rho\omega^2\hat{\boldsymbol{\xi}} = \mathbf{F}(\hat{\boldsymbol{\xi}})$ can be written as

$$\omega^2\rho_0\tilde{\boldsymbol{\xi}} = \frac{B_0^2}{\mu_0}\left\{\mathbf{k}\times\left[\mathbf{k}\times\left(\tilde{\boldsymbol{\xi}}\times\mathbf{e}_z\right)\right]\right\}\times\mathbf{e}_z + \gamma p_0\mathbf{k}\mathbf{k}\cdot\tilde{\boldsymbol{\xi}}$$

- Writing the expression for each component, we get the system

$$\begin{bmatrix} \omega^2 - k_\parallel^2 v_A^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - k_\perp^2 v_S^2 & -k_\parallel k_\perp v_S^2 \\ 0 & -k_\perp k_\parallel v_S^2 & \omega^2 - k_\parallel^2 v_S^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Two key velocities appear:

$$v_A = \sqrt{\frac{B_0^2}{\mu_0\rho_0}} \qquad v_S = \sqrt{\gamma\frac{p_0}{\rho_0}}$$

v_A is called the *Alfvén velocity*, in honor of Hannes Alfvén, the Swedish scientist who first described MHD waves, and v_s is the adiabatic sound speed

Illustration: Waves in ideal MHD (III)

$$\begin{bmatrix} \omega^2 - k_{\|}^2 v_A^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_A^2 - k_{\perp}^2 v_S^2 & -k_{\|} k_{\perp} v_S^2 \\ 0 & -k_{\perp} k_{\|} v_S^2 & \omega^2 - k_{\|}^2 v_S^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \\ \hat{\xi}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

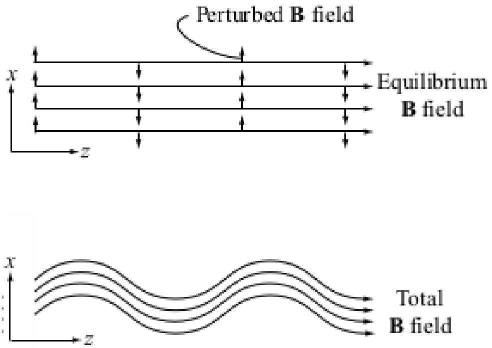
- For nontrivial solutions, determinant of the matrix should be 0
- This leads to the following three possibilities for ω^2:

$$\omega^2 = k_{\|}^2 v_A^2 \ , \ \omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 \pm \sqrt{1 - 4\frac{k_{\|}^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

- One can see that $\omega^2 \geq 0$
- The infinite homogeneous magnetized plasma is always MHD stable
- Some of the modes above become unstable in magnetic fusion configurations, because of gradients and field line curvature

Shear Alfvén wave

- Branch $\omega^2 = k_{\|}^2 v_A^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

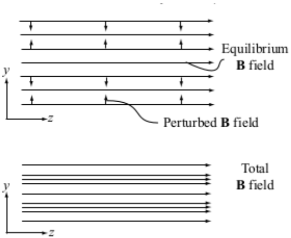
- Transverse wave
- Balance between plasma inertia and field line tension
- Incompressible ⇒ often the most unstable MHD mode in fusion devices

Fast magnetosonic wave

- Fast magnetosonic wave given by

$$\omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 + \sqrt{1 - 4\frac{k_\parallel^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

- Physics simplifies in the limit $v_S^2 \ll v_A^2$: it is then called the compressional Alfvén wave, with dispersion relation $\omega^2 = k^2 v_A^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

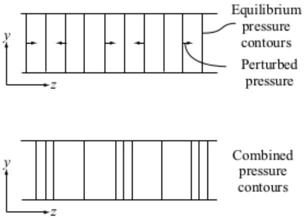
- Plasma motion perpendicular to field lines, compressible
- Oscillation between plasma kinetic energy and magnetic compressional energy

Slow magnetosonic wave

- Slow magnetosonic wave given by

$$\omega^2 = \frac{k^2}{2}\left(v_A^2 + v_S^2\right)\left[1 - \sqrt{1 - 4\frac{k_\parallel^2}{k^2}\frac{v_A^2 v_S^2}{\left(v_S^2 + v_A^2\right)^2}}\right]$$

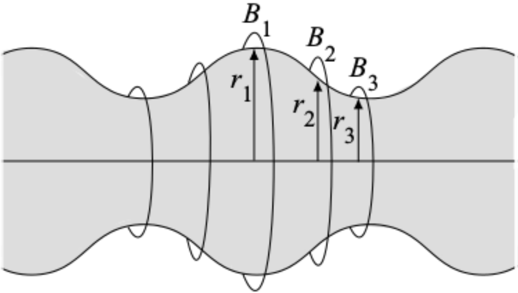
- Physics simplifies in the limit $v_S^2 \ll v_A^2$: it is then called the sound wave, with dispersion relation $\omega^2 = k^2 v_S^2$



(Figure from *Ideal MHD* by J.P. Freidberg, CUP, 2014)

- Plasma motion parallel to field lines, compressible
- Oscillation between plasma kinetic energy and plasma internal energy (plasma pressure)

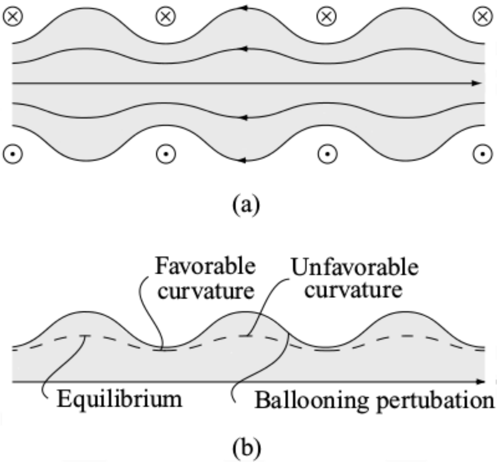
Common MHD instabilities (I)



Interchange instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

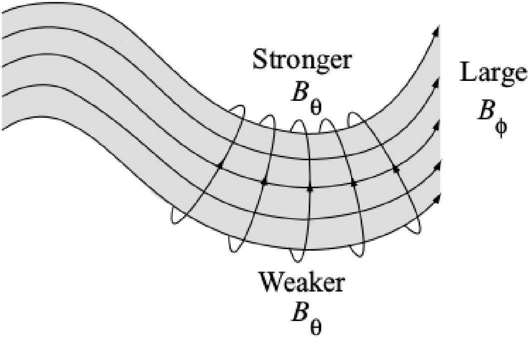
Common MHD instabilities (II)



Ballooning instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

COMMON MHD INSTABILITIES (III)

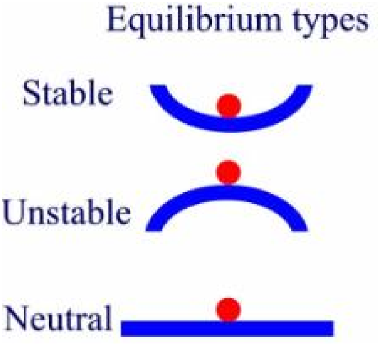


Kink instability

(Figure from *Plasma Physics and Fusion Energy* by J.P. Freidberg, CUP, 2008)

Linear stability: Energy approach

- Change in potential energy: $\delta W = \overrightarrow{F} \cdot \overrightarrow{dl}$
- Static equilibrium condition at x_0: $\delta \mathbf{W}\Big|_{\mathbf{x}=\mathbf{x_0}} = \mathbf{0}$
- Stability



$$\delta^2 \mathbf{W}\Big|_{\mathbf{x}=\mathbf{x_0}} > \mathbf{0}$$

$$\delta^2 \mathbf{W}\Big|_{\mathbf{x}=\mathbf{x_0}} < \mathbf{0}$$

$$\delta^2 \mathbf{W}\Big|_{\mathbf{x}=\mathbf{x_0}} = \mathbf{0}$$

IDEAL MHD ENERGY PRINCIPLE (I)

- For historical reasons, the second variation is called δW in the plasma physics jargon
- A useful variational principle can be derived in ideal MHD, called the energy principle , which takes the following form:

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi})}{K(\boldsymbol{\xi})}$$

where

$$\delta W(\boldsymbol{\xi}) = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})dV$$

$$\mathbf{F}(\boldsymbol{\xi}) = \frac{1}{\mu_0}\{\nabla\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]\}\times\mathbf{B}_0 + (\nabla\times\mathbf{B}_0)\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)]$$

$$+\nabla(\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0\nabla\cdot\boldsymbol{\xi})$$

$$K(\boldsymbol{\xi}) = \frac{1}{2}\int \rho|\boldsymbol{\xi}|^2 dV$$

Ideal MHD Energy Principle (II)

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi})}{K(\boldsymbol{\xi})} \qquad \delta W(\boldsymbol{\xi}) = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})dV$$

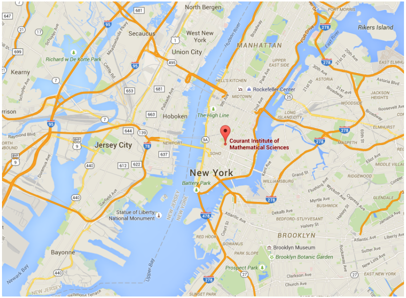
- **Energy Principle**: An ideal MHD equilibrium is stable if and only if $\delta W(\boldsymbol{\xi} > 0$ for all bounded $\boldsymbol{\xi}$ satisfying the boundary conditions

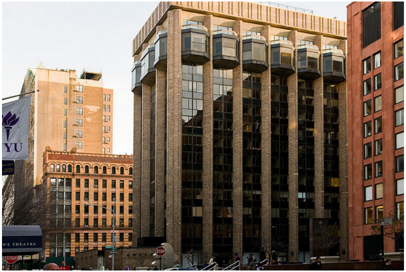
- Energy principle very useful to prove instability of an equilibrium by coming up with a good guess for $\boldsymbol{\xi}$ that makes δW negative

- Formula also very helpful to calculate the ω^2 numerically with high accuracy

POST SCRIPTUM: THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES (CIMS) AT NYU

CIMS IN MANHATTAN





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#5 Tie	University of Minnesota—Twin Cities Minneapolis, MN

- Abel prize in 2005, 2007, 2009 and 2015
- 18 members of the National Academy of Sciences

  5 members of the National Academy of Engineering

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- Particular emphasis on Partial Differential Equations
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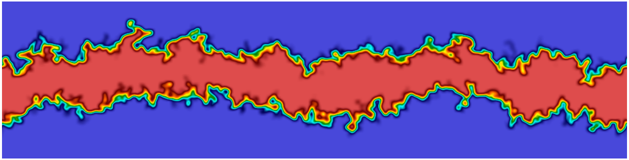
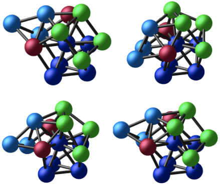
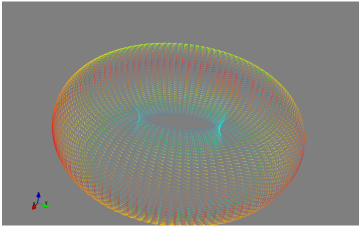
- ~ 60 faculty
  ~ 100 PhD students







MFD division at CIMS

- Founded by Harold Grad in 1954
- 3 faculty, 3 post-docs, 2 PhD students
- Work on MHD, wave propagation, kinetic models
  Analytic "pen and paper" work
  Development of new numerical solvers
- Collaboration with colleagues specialized in scientific computing, computational fluid dynamics, stochastic calculus, etc.
- Funding currently available for PhD students
- Feel free to contact me if you have any questions